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The Impact of Consumption and Liquidity Constraints on Optimal Consumption and Investment Decisions

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ABSTRACT

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Both individual and institutional investors face a number of constraints in their consumption and investment decisions. We look at well-motivated constraints on the consumption process as well as liquidity constraints and study their impact on optimal consumption and investment policies under a dynamic discrete time setting.

Acknowledgements

As I sit down to write this, I know it will never be perfect. Just like my dissertation thesis. One can not compress thirty-one years of being at the receiving end of love, friendship, benevolence, advice and sacrifices into a few words, however carefully chosen. Yet, here is an attempt at this monumental task.

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CHAPTER 1

Introduction

The dynamic consumption and investment optimization problem has been well studied. Merton [1969] and Samuelson [1969] laid the foundations with their seminal papers that dealt with this problem in the continuous-time and discrete-time settings respectively. This thesis looks at the impact of consumption and liquidity restrictions on the optimal consumption and investment policies.

Chapter 2 provides a solution to the discrete time dynamic consumption and investment problem where the agent's consumption is constrained to be non-decreasing. The continuous time version of this problem was solved by Dybvig [1995]. Motivation for such a constraint comes from a couple of different sources. They are examined below, with a focus on two factors in particular, viz. consumption requirements and trading frequencies.

- First, we can look at the case of a very long-term institutional investor such as a university endowment, that might require quarterly or annual payouts from its portfolio to maintain its operations and pay its employees. Typically, a university endowment faces a large cost for decreases in these payouts, both in monetary forms such as severance pay as well as in social terms such as loss of goodwill.

  Moreover, it is now the norm for institutional investors such as university endowments or pension funds to maintain investments in alternative assets such as hedge funds since they offer the prospects of attractive returns (at least the ones that survive) vis-à-vis the market and also allow these institutional investors to further diversify their holdings. However, investments in hedge funds bring their own baggage - complications such as lock-out periods, advance notice requirements for withdrawals, quarterly trade dates and the like. Even if the institutional investor has the resources and the manpower to execute multiple trades on a daily or even hourly basis, the very nature of these alternative assets do not allow for frequent changes in such a position.

- We can also consider the case of a long-term individual investor with an intolerance for a decline in standard of living. Most analytical consumption and investment optimization models ignore the undesirability of fluctuations in consumption patterns, especially downturns. If the individual investor is using such a model to optimize the benefits of her wealth, clearly there is something amiss when the model sends her on a lavish spending spree one month, only to preach thrift for the next six. Duesenberry [1949] equates the investor's standard of living to the consumption level, and discusses the preference for a rising standard of living as one of the primary societal goals.

A more theoretical argument can be built from *prospect theory* as proposed by Kahneman and Tversky [1979]. Once an agent has experienced a particular consumption level, any drop in consumption will be perceived as a loss and the marginal cost of this loss is high. Consequently, the agent would have a strong preference for non-decreasing consumption - her current consumption level being the *reference point* for the agent's Kahneman-Tversky value function. The non-decreasing consumption formulation represents an extreme case of prospect theory, where the value function below the reference point is $-\infty$, or equivalently, at each decision step, the utility of consumption lower than that at the previous step is $-\infty$.

Another theoretical conceptualization of this constraint is as an extreme version of habit formation.

In terms of the frequency with which the investment decisions are implemented, most individual investors are unlikely to trade on an daily or even weekly basis - a quarterly or even annual readjustment of the investor's portfolio seems more probable.

Both the above settings motivate a solution to this consumption-constrained problem in a discrete time setting. In addition, it is also important to examine the impact of the time interval between decision epochs on the optimal consumption and investment policy.

In recent years, institutional investors' holdings in hedge funds have increased significantly. Chapter 3 discusses the case of an investor with CRRA utility, who can allocate wealth to a bond, a stock and a hedge fund. If the investor chooses to trade continuously, she will optimize allocations as per the Merton model, whereas if she chooses to rebalance her portfolio at a lower frequency (e.g. monthly, quarterly or annually), the investor will use the Samuelson model to obtain optimal allocations.

However, as mentioned earlier, hedge funds typically do not allow for continuous trading, and usually have specific lock-out periods or advance notice requirements. Given this liquidity constraint on the hedge fund, what is the investor's optimal investment strategy if she wants to allocate some of her wealth to hedge funds (since they promise attractive returns and diversification benefits), and also wants to take advantage of her ability to trade in and out of the stock and the bond on a continuous basis? What is the impact of the length of the lock-out period? One would expect that the allocations to the hedge fund would decrease as the length of the time interval increases. How does this impact vary when the problem parameters such as the stock-hedge fund correlation, the expected growth rates, the volatilities, the risk-free rate of return, etc. change?

This lockup requirement represents illiquidity of a very specific form. Longstaff [2001] studies the impact of illiquidity on optimal portfolio choice and the price of the asset. However, the setting in Longstaff [2001] is one with limited liquidity whereas in our situation the trading opportunities for one out of the two risky assets are restricted to specific points in time.

As part of this analysis, Chapter 3 examines the impact of including consumption on the allocation to the hedge fund. The annual or quarterly payouts at institutional investors such as pension funds and university endowments can be cast as consumption streams. For proprietary trading desks or fund of funds, on the other hand, consumption is not a consideration.

Chapter 3 also lays out a methodology to compute the premium associated with the lockup restriction, termed the *lockup premium* - this allows potential investors to check whether the lockup requirement is justified for a particular hedge fund. Aragon [2007] demonstrates that the excess returns for funds with lockup restrictions are significantly higher than those for funds without these restrictions. Derman [2007] and Derman et al. [2007] represent recent attempts at computing this premium, but as discussed in Chapter 3, there are several shortcomings in their approach. The analysis in Chapter 3 however, does not consider survivorship among hedge funds, something that Derman [2007] and Derman et al. [2007] both do.

The value of hedge fund share price information is important in analyzing the impact of the typical secrecy that surrounds hedge fund performance reporting. Based on a comparison of the case where the hedge fund price is available only just before a decision has to be made on adjusting the position in the hedge fund to the default situation where the hedge fund share price is always known, Chapter 3 describes a method to compute the value of this information in the form of an *information premium*.

Chapter 4 brings together the results obtained in Chapters 2 and 3 by addressing the consumption and investment optimization problem for an agent facing a non-decreasing consumption constraint, and having the ability to continuously trade in a bond and a stock, and trade at pre-specified regular intervals in a hedge fund.

CHAPTER 2

Discrete-Time Optimization of Consumption and Investment Decisions given Intolerance for a Decline in Standard of Living

Abstract

We extend Samuelson's (1969) discrete-time dynamic consumption and investment optimization problem to the case where the investor is intolerant of any decline in her standard of living. This constraint represents a strong form of habit formation such that the consumption rate is non-decreasing over time. To achieve this objective, the investor first guarantees a consumption perpetuity at the current consumption rate and then allocates the remaining wealth under a state-dependent, adjusted coefficient of relative risk aversion. We study the impact of the length of the time interval on the optimal consumption and investment policies. This effect has implications for investors considering investments in assets, such as hedge funds and private equity, that have restrictions on trading intervals.

2.1. Introduction

The lifetime consumption and investment optimization problem is well studied, both in single- and multi-period settings. In their classic papers on lifetime portfolio selection given continuous- and discrete-time settings respectively, both Merton [1969] and Samuelson [1969] show that under a constant relative risk aversion (CRRA) utility assumption, optimal consumption (*rate* for continuous-time, *amount* for discrete-time) at any instant is of constant proportion to the then current wealth. Consequently, the consumption pattern can vary drastically, and a significant loss of utility may result from losses on investments in risky assets. This is equivalent to allowing the agent's standard of living to fluctuate considerably, which is not desirable.

Duesenberry [1949] makes the case for equating standard of living to consumption levels, which is a reasonably accurate quantification. The net result is a characterization of the problem as a search for optimal consumption and investment policies under the constraint that the consumption rate is non-decreasing. Another argument in favor of this constraint can be built from *prospect theory* as proposed by Kahneman and Tversky [1979]. Once an agent has experienced a particular consumption level, any drop in consumption will be perceived as a *loss* - even though the agent did not actually possess an equivalent consumption perpetuity. The marginal cost of this loss is high, and consequently, the agent would have a strong preference for avoiding a decrease in consumption - the current consumption level being the *reference point* for the agent's Kahneman-Tversky value function. The non-decreasing consumption constraint is an extreme application of prospect theory, where the value function below the reference point is $-\infty$, or equivalently, at each decision step, the utility of a consumption rate lower than that at the previous step is $-\infty$.

Constantinides [1990] formalizes the inclusion of habit persistence by introducing time dependence in the utility function to reflect a reference value depending on past consumption. As Constantinides observes, this model provides a resolution of the equity premium puzzle that empirical consumption levels appear too smooth relative to equity market volatility. The model here can be viewed as an alternative representation of habit formation that is developed over discrete intervals of time.

The focus of our paper is to study the *discrete-time* optimization of consumption and investment decisions given intolerance for a decline in standard of living, i.e., under a non-decreasing consumption constraint. This constraint might seem very restrictive but we show that it does not actually create significant utility losses relative to the *no habit formation* case. Therefore, it serves as a useful alternative to other forms of habit formation utilities.

Dybvig [1995] derives an analytical solution for the *continuous-time* version of this problem. As noted by Dybvig, Black and Perold [1992] arrive at a solution similar to that of Dybvig in a special case of their *constant proportions portfolio insurance* (CPPI) strategy where consumption is restricted to be above some fixed minimum level and the utility function is partly linear and partly in the more traditional CRRA form. The structure of the utility function is derived based on the minimum subsistence level of consumption; however, this derivation is not well motivated in economic terms but rather comes across as a consequence of the (CPPI) strategy.

The non-decreasing consumption requirement can be observed in the following settings:

- A long-term individual investor with an intolerance for a decline in standard of living, as measured by her expenditures;
- A very long-term institutional investor such as a university endowment that might require quarterly or annual payouts from its portfolio to maintain its operations and pay its employees and which faces a large monetary or *social* cost for decreases in these payouts (e.g., severance pay, loss of goodwill).

In either of the above settings, how practical is the continuous-time setting? The typical individual investor is unlikely to rebalance her portfolio or consumption targets every day, or even every week. A typical institutional investor such as the university endowment, or say a pension fund, invariably has investments in alternative assets such as hedge funds. These usually give rise to complications such as lock-out periods, advance notice requirements for withdrawals, quarterly trade dates and the like. So even if the institutional investor has the resources and the manpower to execute multiple trades on a daily or even an hourly basis, the regulations attached to these alternative assets would not allow for this to happen.

One approach for the discrete-time setting is to use the continuous-time solution as an approximation, but how good is this approximation? Do the optimal consumption and allocation policies have exactly the same structure in both continuous-time and discrete-time frameworks? What is the impact of the length of the time interval on these optimal policies? To answer these research questions, we need a solution method for the discrete-time consumption and investment optimization problem with a non-decreasing consumption rate. Since discrete-time solutions are not analytically available as they are in the continuous-time case, the complete structure of these solutions is unknown. In this paper, we present a solution for the discrete-time problem in terms of the structure of the optimal consumption and investment policies and a convergent numerical algorithm.

At each decision epoch, the agent must first decide whether or not to increase the consumption rate. Then, a consumption perpetuity at the current consumption rate must be guaranteed by investing an appropriate amount in the risk-free asset. The agent now has to allocate the remaining wealth between the risky and risk-free assets. However, this allocation must now be done under a pseudo-coefficient of relative risk aversion to account for the amount already put into the perpetuity. In the continuous-time case, the pseudo-coefficient is a constant, independent of the state variables. In the discrete-time case, however, there is an added complication. As we will see, the agent derives value from two sources - first from guaranteeing herself a steady consumption perpetuity, and second from the chance that her wealth grows to a level where it is optimal for her to increase the level of this steady consumption. Both the continuous-time and discrete-time solutions call for an increase in consumption as soon as the wealth level hits a new maximum. In the continuous-time case, it is possible to constantly monitor the wealth level and change the consumption rate in an appropriate fashion exactly when the new wealth maximum is achieved, so that optimal amounts of value are derived from both the above sources. In the discrete-time case, however, the monitoring of wealth takes place at discrete decision epochs and some of the value associated with an increase in consumption may be lost due to a delayed implementation of this increase. Since there is now a *non-zero* probability that we might not be able to take *immediate* advantage of an increase in the consumption level, we have to incorporate this into our analysis. We show that in the discrete-time framework, the pseudo-coefficient is a non-constant function of the ratio of two state variables - the previous consumption rate and the current wealth.

The solution is analytical to a large extent, but requires a (convergent) numerical iteration scheme to determine the above function over the feasible state space. Because the numerical scheme is fairly flexible, it is possible to extend this methodology to similar problems with transaction costs or tax considerations.

It is also worth pointing out that although the continuous-time problem has an analytical solution, the assumed geometric Brownian motion of prices implies a lognormal distribution for the price of the risky asset; consequently, because of continuous monitoring, the wealth at any future time is lognormally distributed. In the discrete-time case, even with a normality assumption on the returns, the wealth at a future decision epoch will not be lognormally distributed, since the portfolio growth factor is now a weighted sum of two lognormally distributed growth factors, and hence is not lognormal. Under this scenario, Ohlson and Ziemba [1976] have demonstrated the use of a lognormal approximation to the weighted sum of lognormally distributed growth factors which could be utilized; however, even normality of individual asset returns is not clear (e.g., Brooks and Kat [2001] shows that hedge fund returns are not normally distributed and exhibit both negative skewness and leptokurtosis).

Based on these observations, in solving the discrete-time case, we take advantage of the versatility of the numerical scheme and avoid placing normality restrictions on the distribution model for the risky asset's returns, as done by Dybvig [1995] in analytically solving the continuous-time case, and utilize numerical optimization techniques instead of seeking analytical optima. The continuous-time solution only serves as an approximate result when considering risky assets that have non-normal return distributions.

We conduct an analysis of the impact of the length of the time interval on the optimal consumption and investment policy. A similar analysis is carried out by Rogers [2001] for the Merton [1969] problem with no consumption constraints. Our results imply that the combined effect of limited trading opportunities and consumption constraints or strong habit formation has a greater impact on investor decisions than these two factors have alone.

From a portfolio theory perspective, Table 2.1 shows the gap in research that we fill with this work.

	Continuous Time	**Discrete Time**
Unconstrained	Merton (1969)	Samuelson (1969)
Non-Decreasing Consumption	Dybvig (1995)	?

Table 2.1. Research Gap

2.2. The Discrete Time Problem

First, we state the discrete-time problem in a fashion similar to the continuous-time case. The notation we use is a little different from Dybvig [1995], both by necessity and preference.

2.2.1. Setup for the Discrete Time Problem

Consider an agent who is intolerant of any decline in her standard of living, as measured by the rate c_n at which she consumes between decision epochs n and $n+1$ ($n \geq 0$, *integer*). In other words, c_n is non-decreasing for $n \geq 0$. The decision epochs arrive at an interval of time τ. Conditioning on meeting the constraint of increasing consumption, the von Neumann - Morgenstern utility function is given by

$$\int_{t=0}^{\infty} e^{-\delta t} u(c_t) dt = \sum_{n=0}^{\infty} \left(\frac{1 - e^{-\delta \tau}}{\delta} \right) e^{-\delta n \tau} u(c_n) \tag{2.1}$$

where $u(\cdot)$ is the felicity function and the constant $\delta > 0$ is the pure rate of time preference. We can interpret the increasing consumption requirement as a modification of the utility that introduces strong habit formation and relaxes time separability. Effectively, the assumed utility function follows from replacing u with $\hat{u}$, which is a function of c_t and its time derivative $\dot{c}_t$, such that $\hat{u}(c_t, \dot{c}_t) = u(c_t)\delta(\dot{c}_t \geq 0)$, where δ is an indicator function that is 1 if $\dot{c}_t \geq 0$ and $-\infty$ otherwise.

The agent has an initial endowment of W_0, an inherited consumption rate of c_{0-} and can allocate unconsumed wealth to either a risk-free asset or a risky one. The one-period growth factor for the riskless asset is R (constant), while $\tilde{R}_n$ represents the n^{th} period growth rate (stochastic) for the risky asset. We assume that $\{\tilde{R}_n; n \geq 0\}$ are independent and identically distributed - we will often use $\tilde{R}$ to represent the stochastic return of the risky asset over any single period. We can also define a similar quantity Δ (constant) as the growth factor corresponding to the time preference for utility.

If r is the instantaneous riskless rate and δ is the pure rate of time preference, then we simply have

$$R = e^{r\tau} \tag{2.2}$$

$$\Delta = e^{\delta\tau} \tag{2.3}$$

Note that we have not assumed a particular distribution for the returns on the risky asset. If we operate in the Black-Scholes scheme and the risky asset provides lognormally distributed returns, i.e.,

$$\frac{dS}{S} = \mu dt + \sigma dB \tag{2.4}$$

where μ and $\sigma > 0$ are constants and B is a standard Wiener process, then we have

$$\tilde{R}_n \sim e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma(B_{(n+1)\tau} - B_{n\tau})} \tag{2.5}$$

or equivalently

$$\tilde{R} \sim e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}Z} \tag{2.6}$$

where $Z \sim N(0, 1)$.

As with consumption, the reallocation of the wealth after consumption can occur only at the decision epochs, which are equispaced at τ. For a feasible solution that allows us to guarantee maintaining a non-decreasing consumption rate, we must have

$$W_0 \geq \frac{c_{0-}}{r} \tag{2.7}$$

Operating in the domain of constant relative risk aversion (as implied by the assumed scale independence of preferences), let the Arrow-Pratt coefficient of relative risk aversion for the agent be $1-\gamma$. If $\gamma = 0$, then $u(c) = log(c)$; otherwise, $u(c) = c^{\gamma}/\gamma$ for $\gamma < 1$, $\gamma \neq 0$.

2.2.2. Statement of the Discrete-Time Problem

Given the above setting, what is the optimal course of action for the agent in terms of consumption and investment allocation at each decision epoch? We are looking for the policy that allows us to determine the optimal values of the consumption rate c_n and the portion of the remaining wealth α_n allocated to the risky asset at any decision epoch $n \geq 0$, *integer*. The problem can then be stated as follows:

Problem 1. ***(The Discrete-Time Problem)***

Choose adapted $\{c_n\}_{n=0}^{\infty}$ *and* $\{\alpha_n\}_{n=0}^{\infty}$ *to maximize* $\mathbb{E}\left[\sum_{n=0}^{\infty}\left(\frac{1-e^{-\delta\tau}}{\delta}\right)e^{-\delta n\tau}u(c_n)\right]$ *subject to*

$$c_0 \geq c_{0-} \tag{2.8}$$

$$c_n \geq c_m \qquad \forall\ n > m \geq 0, \quad n,m \textit{ integers} \tag{2.9}$$

$$\textit{and} \qquad W_n \geq 0 \qquad \forall\ n \geq 0 \tag{2.10}$$

where $\{W_n\}_{n=0}^{\infty}$ *is given by*

$$W_{n+1} = W_n \left[(1-\alpha_n)R + \alpha_n \tilde{R}_n \right] - \frac{c_n}{r}(R-1) \tag{2.11}$$

given the initial wealth W_0*. Again, if the instantaneous rate of return on the riskless asset was* r *and the risky asset had lognormally distributed returns as specified by equation (2.4), then equation (2.11) could be written as*

$$W_{n+1} = W_n \left[(1-\alpha_n)e^{r\tau} + \alpha_n e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma(B_{(n+1)\tau} - B_{n\tau})} \right] - \frac{c_n}{r}(R-1) \tag{2.12}$$

It is assumed that $\mu - r$*,* σ *and* δ *are all positive.*

Note that the growth factor in the above equation is not lognormally distributed and this complicates the search for a purely analytical solution. Samuelson [1969] uses the concept of the *util-prob mean* as defined in Samuelson and Merton [1969] whereas Ohlson and Ziemba [1976] provide a lognormal approximation to this growth factor based on a matching of the first two moments. The lognormal approximation gets worse as the time interval between decision epochs increases. Given the ease with which numerical integration can be performed, our approach will be similar to that adopted in Samuelson [1969] but we will avoid the *util-prob* terminology.

2.2.3. Issues in the Solution of the Discrete-Time Problem

In the continuous-time case, the agent is able to regulate the consumption rate level c such that it is bounded below by the optimal minimal consumption level which is a fraction of the wealth level at that instant. Equivalently, the wealth w is bounded above by an optimal (finite) multiple of the consumption rate as a result of this continuous monitoring. The probability that the wealth w at any time will be greater than this optimal multiple of the consumption rate is 0. However, in the discrete-time case, typical risky asset return distributions (such as *normal*) do not limit the upside, and consequently, the probability that the wealth level W increases past any finite level in a discrete interval of time is *non-zero* in such cases. Since this probability is adapted, we have to now account for it in our computations. As we will see, this becomes important in the context of finding the appropriate risk aversion level for the wealth remaining after guaranteeing the consumption perpetuity, and consequently affects the optimal allocation of remaining wealth between the risky and riskless assets.

2.2.4. Proposed Form of Solution

The state variables for our dynamic system are the current wealth W_n and the consumption rate at the previous decision epoch c_{n-1}. Define a new state variable

$$X_n \equiv \frac{c_{n-1}}{W_n} \tag{2.13}$$

which will be of use later. We will refer to this as the *Ratio of Previous Consumption to Current Wealth*, or simply *Ratio*. Clearly, once we know the values of two of these state variables, the value of the third is immediately known.

Note that the consumption rate must be positive. Moreover, the highest value of c_n for which we can guarantee that consumption rate is non-decreasing is rW_n, at which point we must put all our wealth into the risk-free asset and consume the equivalent perpetuity amount. Using some straightforward algebra, we see that $c_n = rW_n$ implies that $c_n = rW_{n+1}$ because of the wealth dynamics in equation (2.11) and consequently, the range of c_{n-1} is $[0, rW_n]$. The advantage of inventing the new state variable X_n is that its range is $[0, r]$ - here the bounds on X_n are independent of the wealth W_n. This allows us to define the state space $\mathcal{S}$ as

$$\mathcal{S} = \mathbb{R}_+ \times [0, r] \tag{2.14}$$

In the analysis conducted in Appendix A, we use both

$$s \equiv (w, x) \in \mathcal{S} \tag{2.15}$$

and

$$y \equiv (w, c_-), \quad where \quad \left(w, \frac{c_-}{w}\right) \in \mathcal{S} \tag{2.16}$$

in addition to the notation (w, x) and (w, c_-) to denote feasible states for our problem. Clarifications on the notation will be made as necessary.

We are trying to find the optimal consumption and investment policies $c_n(W_n, c_{n-1})$ and $\alpha_n(W_n, c_{n-1})$ or equivalently $c_n(W_n, X_n)$ and $\alpha_n(W_n, X_n)$. For the sake of brevity, we refer to these as c_n and α_n respectively. We now propose a particular form for the solution and show that it is a valid form for the optimal solution to Problem 1.

Theorem 1. ***(Form of the solution to the Discrete Time Problem)*** *Let $u(c) = log(c)$ for $\gamma = 0$ and $u(c) = \frac{c^\gamma}{\gamma}$ for $\gamma < 1$, $\gamma \neq 0$. Then Problem 2 is feasible (in the sense that the constraints (2.8), (2.9), and (2.10), can be satisfied) if and only if both $c_{0-} \leq rW_0$ and $r > 0$. Given feasibility, there exists a function $\gamma^*(w, c_-)$ such that for all feasible* *states* (w, c_-) *(i.e. for* $c_- \le rw, w \in \mathbb{R}_+$*),*

$$\gamma^*(w,c_-) \begin{cases} \in (0,1) & if \quad \gamma < 0 \\ \in (\gamma,1) & if \quad \gamma \ge 0 \end{cases} \tag{2.17}$$

which generates the value function V_{γ^*} *that solves the Bellman equation for our problem,*

$$\begin{aligned} V_{\gamma^*}(w,c_-) = & \max_{\substack{0\le\alpha\le 1 \\ c\ge c_-}} \frac{1-e^{-\delta\tau}}{\delta} u(c) \\ + & \Delta^{-1}\mathbb{E}_{\tilde{R}}\left[V_{\gamma^*}\left(w[(1-\alpha)R+\alpha\tilde{R}] - \frac{c}{r}(R-1), c\right)\Big|w,c_-\right] \end{aligned} \tag{2.18}$$

The value function $V_{\gamma^*}(\cdot)$ *for any feasible state* (w, c_-) *is given by*

$$\begin{aligned} V_{\gamma^*}(w,c_-) = & \frac{1}{\delta} u\left(max\left(c_-, r^*(w,c_-)w\right)\right) \\ + & \frac{r}{\delta}\left(\frac{1}{r^*(w,c_-)} - \frac{1}{r}\right)^{1-\gamma^*(w,c_-)} \left(max\left(c_-, r^*(w,c_-)w\right)\right)^{\gamma} \\ & \left(\frac{w}{max(c_-, r^*(w,c_-)w)} - \frac{1}{r}\right)^{\gamma^*(w,c_-)} \end{aligned} \tag{2.19}$$

where

$$r^*(w,c_-) \equiv r\left(\frac{\gamma^*(w,c_-)-\gamma}{1-\gamma}\right) \tag{2.20}$$

The solution to Problem 2 is characterized at each decision epoch by the amount invested in the risky asset,

$$\alpha(W_n, c_{n-1})W_n = \hat{\alpha}(W_n, c_{n-1})(W_n - \frac{c_n}{r}) \tag{2.21}$$

where

$$\hat{\alpha}(w, c_-) = argmax\ \mathbb{E}_{\tilde{R}}\left[V_{\gamma^*}\left(\left(w - \frac{c}{r}\right)[(1-\hat{\alpha})R + \hat{\alpha}\tilde{R}] + \frac{c}{r}, c\right)\Big| w, c_-\right] \tag{2.22}$$

$$c = max(c_-, r^* w) \tag{2.23}$$

and a non-decreasing consumption rate process $\{c_n\}$ *regulated to be no smaller than* $\{r^*(W_n, c_{n-1})W_n\}$*:*

$$(\forall\ n) \quad \sum_{i=0}^{n}(c_i - r^*(W_i, c_{i-1})W_i)^+(c_i - c_{i-1}) = 0 \ \textit{where}\ c_{-1} \equiv c_{0-} \tag{2.24}$$

$$(\forall\ n) \quad c_n \geq r^*(W_n, c_{n-1})W_n \tag{2.25}$$

$$\textit{and} \quad (\forall\ n) \quad \sum_{i=0}^{n}(c_n - c_{n-1})^- = 0 \ \textit{where}\ c_{-1} \equiv c_{0-} \tag{2.26}$$

Here, x^+ *(resp.* x^-*) denotes the positive part (resp. negative part) of* x*.*

Proof: Proving this theorem requires several intermediate results. For the corresponding lemmas, their proofs, and the proof of this theorem, see Appendix A. □

Note that we can combine the original wealth dynamics in equation (2.11) with equation (2.21) and obtain a new variation of the wealth dynamics,

$$W_{n+1} = \left(W_n - \frac{c_n}{r}\right)\left[(1 - \hat{\alpha}(W_n, c_{n-1}))R + \hat{\alpha}(W_n, c_{n-1})\tilde{R}_n\right] + \frac{c_n}{r} \tag{2.27}$$

We will use this version of the wealth dynamics in a subsequent proof. Also, this version provides a more intuitive understanding of the consumption and investment process -

essentially, the agent places an amount $\frac{c_n}{r}$ into the risk-free asset as her *consumption perpetuity*, which leaves her with an excess of $(W_n - \frac{c_n}{r})$ to invest in either the risky or the risk-free asset. $\hat{\alpha}(W_n, c_{n-1})$ is the proportion of this amount that goes into the risky asset, which gives us the growth factor $\left[(1-\hat{\alpha}(W_n, c_{n-1}))R + \hat{\alpha}(W_n, c_{n-1})\tilde{R}_n\right]$ for the excess amount $(W_n - \frac{c_n}{r})$. Finally, over the next period, the consumption perpetuity grows at a rate $\frac{c_n}{r} r = c_n$, which is the same as the agent's consumption rate. Consequently, at the end of the period, the agent has consumed at the rate c_n and still has an amount $\frac{c_n}{r}$ left in the consumption perpetuity. In effect, the agent has guaranteed herself a perpetual consumption rate of c_n.

Over the rest of paper we use a more compact form of notation - γ^*, r^*, α and $\hat{\alpha}$ - these still represent functions of the state variables and are not constants.

2.2.5. Implications of the Proposed Form

The above solution is still incomplete since we do not yet know how to determine some of the parameters. However, before moving on to completing the solution, we try to get a quick understanding of its key elements:

- the amount of wealth invested in the risky asset αW_n
- the consumption rate c_n
- the value function $V_{\gamma^*}(W_n, c_{n-1})$

The parameters, γ^* and r^*, also have economic interpretations which can be derived from our understanding of αW_n.

αW_n as defined in equation (2.21) is not the usual *proportion* of wealth invested in the risky asset, but instead gives the actual amount of the investment. Examining the terms on the right hand side of the equation, we note that $\frac{c_n}{r}$ is the exact amount invested in the risk-free asset that guarantees a consumption perpetuity at the rate of c_n. The remaining amount can be invested either in the risky asset or the risk-free asset. The term $\hat{\alpha}$, based on its definition in equation (2.22), can be interpreted as the optimal proportion of wealth invested in the risky asset when the agent has CRRA utility with a risk aversion of $1-\gamma^*$. This in turn leads us to the conclusion that $1-\gamma^*$ is the risk aversion of the agent for the amount left over from her wealth after she has guaranteed the lifelong consumption stream at the rate of c_n. Clearly, the agent will be less risk averse with this amount, and hence we can expect that $\gamma^* > \gamma$. r^*, as defined in equation (2.20), can be interpreted as a pseudo-interest rate corresponding to γ^*.

Consumption rate $\{c_n\}$ is non-decreasing and bounded below by r^*W_n. It is also bounded above by rW_n. If c_n hits the rW_n level, the investor will put all of her current wealth W_n in the risk-free asset and thereby maintain a constant consumption perpetuity at the rate of c_n. The term $max(c_{n-1}, r^*W_n)$ as seen in the value function definition (equation (2.19)) represents the consumption rate decision at decision epoch n, i.e., c_n. This expression also uniquely solves equations (2.24), (2.25) and (2.26). An informal way of stating the consumption rate is

$$c_n = max(c_{n-1}, r^*W_n) \tag{2.28}$$

or, in stationary form,

$$c = max(c_-, r^*w) \tag{2.29}$$

Examining the value function as defined in equation (2.19), the term

$$u\left(max(c_{n-1}, r^*W_n)\right)$$

is the utility rate corresponding to the consumption rate c_n over the n^{th} time period;

$$\frac{1}{\delta}u\left(max(c_{n-1}, r^*W_n)\right)$$

thus gives us the present value of the perpetual utility stream that has been guaranteed by guaranteeing the consumption rate $c_n = max(c_{n-1}, r^*W_n)$. This term captures the value derived by the agent by guaranteeing herself a perpetual consumption stream of $max(c_{n-1}, r^*W_n)$. The remaining term on the right hand side of equation (2.19) represents the value derived from being able to invest in the risky asset, allowing for the possible growth of the agent's wealth to a level where it is feasible to increase her consumption rate.

2.2.6. Completion of the Solution

Now that we have an understanding of the form of the solution, we return to the determination of the as-yet unknown parameters. We still do not have a clear way of evaluating γ^*, r^*, and $\hat{\alpha}$. However, from equations (2.20) and (2.22), we see that r^* is merely a function of γ^*; $\hat{\alpha}$ can also be determined once we know how to evaluate γ^* for all feasible states (w, c_-). Consequently, the key element that is missing is the function $\gamma^*(\cdot)$ in terms of the state variables. The corresponding parameter in the continuous-time case can be calculated analytically (Dybvig [1995]) - albeit restricted to the case of a lognormal return distribution for the risky asset. Note that we do not assume that γ^* will turn out to be state independent (i.e., a constant) as in the continuous-time case. Unfortunately, an analytical expression for $\gamma^*(\cdot)$ is not easy to obtain in the discrete-time case. However, we can evaluate γ^* from having to satisfy the Bellman equation (2.18). We first show that, for this problem, the Bellman equation can be expressed solely in terms of the new state variable X_n as defined in equation (2.13).

Theorem 2. ***Reduction of the Bellman Equation to One State Variable*** *The Bellman equation (2.18) can be expressed as an equation in a single state variable - the Ratio of Previous Consumption to Current Wealth, x, as follows*

$$
\begin{aligned}
U_{\gamma^*}(x) = {} & \frac{1}{\gamma}\frac{1-e^{-\delta\tau}}{\delta}(max(x,r^*))^{\gamma} + \\
& \Delta^{-1}\mathbb{E}_{\tilde{R}}\left[\left((1-max(x,r^*))\left[(1-\alpha)R+\alpha\tilde{R}\right]\right)^{\gamma}U_{\gamma^*}(x_+)\middle|w,x\right]
\end{aligned}
\tag{2.30}
$$

Also, the dynamics of x are independent of the wealth w and can be written as

$$
x_+ = \frac{1}{\left(\frac{1}{max(x,r^*)}-\frac{1}{r}\right)\left[(1-\hat{\alpha})R+\hat{\alpha}\tilde{R}\right]+\frac{1}{r}}
\tag{2.31}
$$

Proof: See Appendix A. □

Consequently, γ^* purely depends on the value of the state variable X_n. We model γ^* as a function of X_n and use an iterative scheme to obtain the function numerically over the domain $[0, r]$ of X_n ($n \geq 0$, *integer*). This easily leads to a valuation of r^*. Equation (2.22) allows us to evaluate $\hat{\alpha}$ once γ^* is known.

2.2.6.1. Iteration Scheme. Since an analytical solution for $\gamma^*(\cdot)$ is not easily obtained, we use the following iterative scheme to numerically evaluate $\gamma^*(\cdot)$ over the domain $[0, r]$ of $X \equiv \frac{c_-}{W}$:

- Fix wealth level $W = w$ arbitrarily
- Divide the domain $[0, r]$ of $X \equiv \frac{c_-}{W}$ into N intervals
- At each ratio $x = \frac{ir}{N}$, $i = 0..N$, being evaluated:
  - Assign $\gamma^*_{[1]}(x) = 1 - R^*$
  - Set $r^*_{[1]}(x) = r\frac{\gamma^*_{[1]}(x)-\gamma}{1-\gamma}$
  - Set $\hat{\alpha}_{[1]}(x) = argmax\ \mathbb{E}_{W_+}\left[V^{[1]}_{\gamma^*}\left(W_+(w, x, \hat{\alpha}), max(xw, r^*_{[1]}(x)w)\right)\big|w, x\right]$
- Set $j = 2$
- Do until a stopping criterion is met:
  - Loop $i = 0..N$
    - For $x = \frac{ir}{N}$, evaluate $\mathbb{E}_{W_+}\left[V^{[j-1]}_{\gamma^*}\left(W_+(w, x, \hat{\alpha}), max(xw, r^*_{[j-1]}(x)w)\right)\big|w, x\right]$ numerically using the latest values of $\gamma^*(\cdot)$'s and of $\hat{\alpha}(\cdot)$'s at each ratio
    - Find $\gamma^*_{[j]}(x)$ and $r^*_{[j]}(x) = r\frac{\gamma^*_{[j]}(x)-\gamma}{1-\gamma}$ such that

$$V^{[j]}_{\gamma^*}(w, xw) = u\left(max\left(xw, r^*_{[j-1]}(x)w\right)\right) + \Delta^{-1}\mathbb{E}_{W_+}\left[V^{[j-1]}_{\gamma^*}\left(W_+(w, x, \hat{\alpha}), max\left(xw, r^*_{[j-1]}(x)w\right)\right)\big|w, x\right]$$

      where $V^{[j]}_{\gamma^*}(w, xw)$ is evaluated under $\gamma^*_{[j]}(x)$ and $r^*_{[j]}(x)$
    - Set

$$\hat{\alpha}_{[j]}(x) = argmax\ \mathbb{E}_{W_+}\left[V^{[j-1]}_{\gamma^*}\left(W_+(w, x, \hat{\alpha}), max\left(xw, r^*_{[j-1]}(x)w\right)\right)\big|w, x\right]$$

  - Increment j by 1
- Report the latest values of $\gamma^*(\cdot)$'s as result

While the iterative scheme outlined above is fairly intuitive, it should be pointed out that instead of using $\gamma^*_{[1]}(\cdot) = 0$, which would correspond to the traditional starting point of the value iteration algorithm ($V_{\gamma^*} = 0$) as used in Bertsekas and Shreve [1978, reprinted 1996], we start with the continuous-time equivalent of γ^* (a constant) as our initial guess to the solution. This is because we expect the value of the objective function for the discrete-time problem to be reasonably close to its continuous-time counterpart and therefore the above iterative scheme would converge significantly faster. Numerical experiments with the two different starting points bear this out.

2.2.6.2. Iteration Results. The iterative scheme described above has reasonably quick convergence. Of course, the time taken for the iterations to stop not only depends on the number of ratios evaluated, but also on the stopping criterion used. In this section, we discuss the convergence pattern of the γ^* and $\hat{\alpha}$ curves over many iterations.

The following results are based on an implementation of the above algorithm in Matlab. 100 iterations were performed using parameter values $r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, and $\tau = 1$ (annual decision epochs). The number of intervals was set at $N = 100$. For the numerical optimization elements, the tolerance for change in function value was set at 5×10^{-20} and for search algorithms (to find new values of $\gamma^*(\cdot)$ that satisfy the Bellman equation) the tolerance for change in variable value was set at 5×10^{-10}. In doing the following computations, we have used two enhancements to the above algorithm. First, we use the optimal consumption policy $c = max(c_-, r^*(\frac{c_-}{w})w)$ to realize that if $\frac{c_-}{w} < r^*(\frac{c_-}{w})$, then we will immediately raise our consumption c to $r^*(\frac{c_-}{w})w$. Writing this out in terms of $x = \frac{c_-}{w}$, we see that if we are in state x such that $x < r^*(x)$, then the optimal consumption policy will immediately move us to the state $\bar{x}$ such that

$\bar{x} = r^*(\bar{x})$. A direct consequence of this is that the functions we are seeking ($\gamma^*(\cdot)$, $r^*(\cdot)$, $\hat{\alpha}(\cdot)$ and $V_{\gamma^*}(\cdot)$) will all be constant for $x = [0, \bar{x}]$. It is reasonably easy to find $\bar{i}$ such that $\bar{x} \in [\frac{\bar{i}r}{N}, \frac{(\bar{i}+1)r}{N}]$; we can suitably code the algorithm to take advantage of this property of the solution.

Secondly, in trying to find $\hat{\alpha}(x)$, the algorithm is highly sensitive for values of x close to r. The reason for this is straightforward - any value of $\gamma^*(r)$ (remember the interpretation of $1 - \gamma^*$ as a pseudo-coefficient of relative risk aversion) will satisfy the Bellman equation for state $x = r$ because in this state the agent's entire wealth is put into the consumption perpetuity, leaving nothing to be potentially invested in the risky asset. Consequently, the exact value assigned to $\hat{\alpha}(r)$ is also irrelevant, since it is a proportion of a zero residual amount. To compute the expectation of the next-step value function, we use numerical integration. For any ratio x, the value function $V_{\gamma^*}(x)$ is calculated using a $\gamma^*(x)$ value obtained by linear interpolation. Experimenting with the interpolation of the value function between the last few (approximately $5-10$ in number) ratios evaluated, we found that the $\hat{\alpha}(\cdot)$ values vary in either direction from the otherwise flat behavior displayed over the preceding values of x. After having studied several interpolation schemes over the ending evaluated ratios, we decided to force $\hat{\alpha}(\cdot)$ values to stay constant over these ratios and find appropriate $\gamma^*(\cdot)$ values that satisfy the Bellman equation for those states. The effects are negligible - after 10 iterations, the variations in $\gamma^*(\cdot)$ between different schemes are of the order of 10^{-6} and the variations in the value function $V_{\gamma^*}(\cdot)$ between different schemes are of the order of 10^{-4} % - and these variations grow smaller with each iteration.

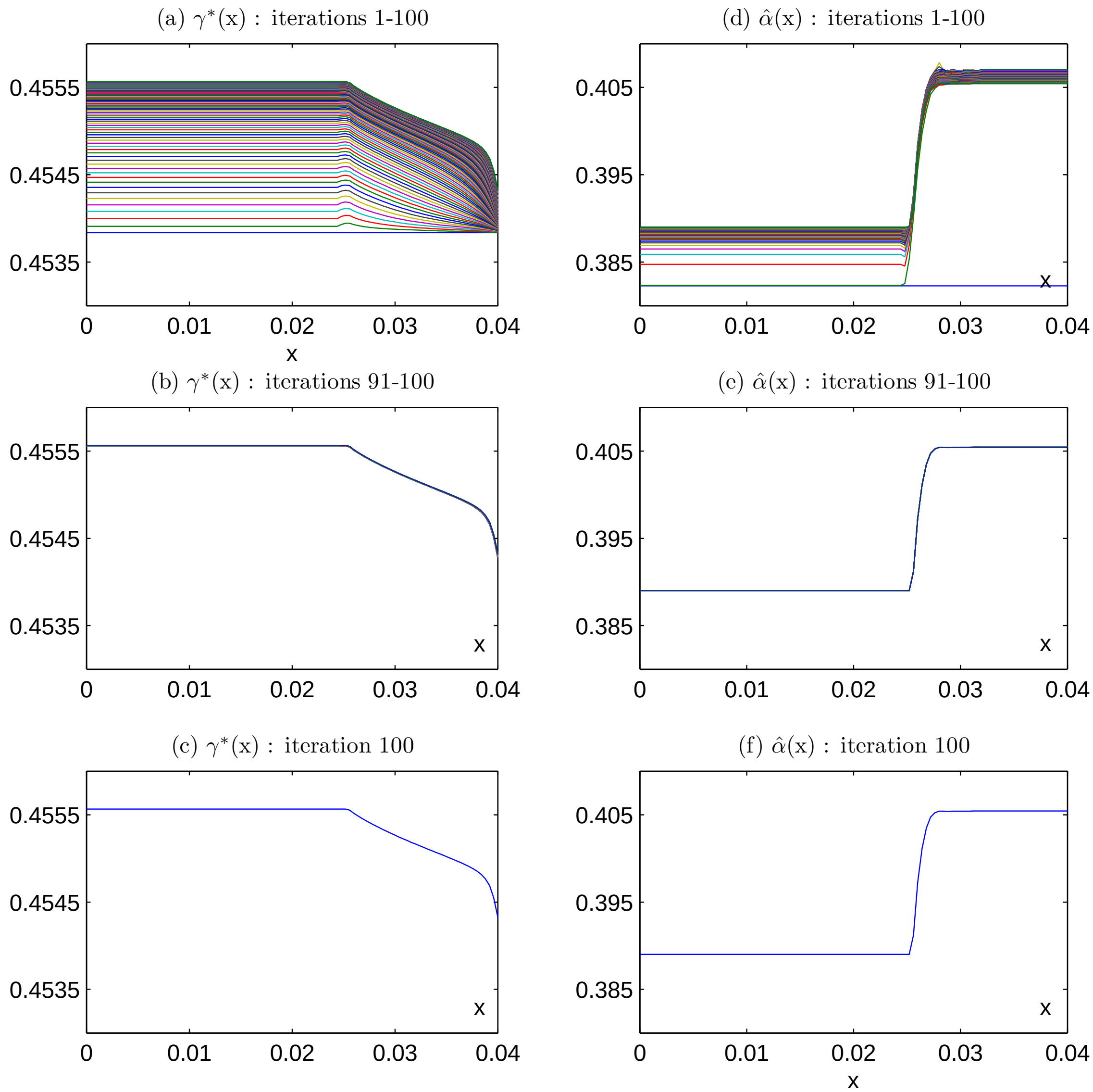

Figure 2.1. Evolution of γ^* and $\hat{\alpha}$ curves
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, $\tau = 1$ yr, $N = 100$

Returning to the results of the iteration scheme as shown in figure (2.1), the first iteration is the initialization of the γ^* curve to its continuous-time counterpart and is represented by the constant line at the bottom of figure (2.1(a)). As we go through the iterations, the γ^* curve rises and takes on a particular shape, which we discuss below. The convergence is fairly quick - by the tenth iteration, the change in the value of γ^* at each evaluated ratio is smaller than 10^{-4}. This corresponds to a change in the value function of the order of 10^{-1} %. Figure (2.1(b)) shows the final ten iterations - at this stage the changes in the γ^* curve at each evaluated ratio are of the order of 10^{-5} with the corresponding change in the value function being of the order of 10^{-3} %. Figure (2.1(c))shows the $\gamma^*(\cdot)$ function after the hundredth iteration is completed. Figures (2.1(d)), (2.1(e)) and (2.1(f)) show the corresponding evolution of the $\hat{\alpha}$ curve over iterations 1 through 100, 91 through 100 and the final iteration respectively. Again, we see a quick convergence rate with the last ten iterations showing very little change.

We will now attempt to explain the shape of the $\gamma^*(\cdot)$ and $\hat{\alpha}(\cdot)$ curves, as seen in figures (2.1(c)) and (2.1(f)) respectively. The explanation is based on the following key factor:

The probability $Q(x)$, when the current state is x, that during the next time interval, the agent's wealth rises to a level where it would be optimal to increase the consumption rate.

The corresponding *intermediate* state $\tilde{x}$ would be lower in value than $r^*(\tilde{x})$. Since this state $\tilde{x}$ would be achieved during the time interval and not at a decision epoch, the agent would be unable to actually implement an increase in the consumption rate. In some sense, this would be a lost opportunity and therefore the agent faces a potential loss in value. This is the fundamental impact of switching to a discrete-time framework from a continuous-time framework and drives all distinctions between the two. Since this probability $Q(x)$ is adapted, it impacts our optimal consumption and investment policy through the functions $\gamma^*(\cdot)$, $r^*(\cdot)$ and $\hat{\alpha}(\cdot)$. The agent can counter this negative influence in two ways - by consuming at a higher rate upfront and by reducing her level of risk aversion for the amount left over once the consumption perpetuity is guaranteed. At $x = r$, the consumption at each decision epoch is exactly rw and all of the agent's wealth is invested in the risk-free asset. Since $\frac{1}{r}$ is the perpetuity factor, the wealth level will remain constant at w for all subsequent decision epochs. Thus, $Q(r) = 0$. Also, as pointed out earlier, the exact values of $\gamma^*(r)$ and $\hat{\alpha}(r)$ are irrelevant since the Bellman equation for $x = r$ will be satisfied by any value of $\gamma^*(r)$, and $\hat{\alpha}(r)$ is a proportion out of a zero residual amount. In the above calculations, we have used a fourth order extrapolation of the $\gamma^*(\cdot)$ function to determine its value at r, and we have assumed that $\hat{\alpha}(\cdot)$ stays constant near r, as indicated by $\hat{\alpha}$ values for preceding ratios. For $\gamma^*(r)$, we could also leave its value to be equal to the corresponding constant in the continuous time case. This would be consistent with the fact that $Q(r) = 0$ - i.e. for $x = r$, the discrete-time framework does not lead to any potential loss in value.

$Q(x)$ depends on three things - the amount left over after guaranteeing the consumption perpetuity, the proportion of this invested in the risky asset ($\hat{\alpha}$) and the ideal fraction of wealth to be consumed (r^*). As we lower the value of x from r, the amount left over after guaranteeing the prevailing consumption rate increases. This increase dominates any change in $\hat{\alpha}(x)$ (which is more or less constant for this range of x values). As we see in figure (2.1(c)), $\gamma^*(x)$ increases as x decreases from r. Since $r^*(x)$ is a positive linear transform of $\gamma^*(x)$, $r^*(x)$ increases as x decreases. The net effect is that $Q(x)$ increases as x decreases from r, and the agent counters the increased potential loss in value by

- increasing the *ideal* fraction $r^*(x)$ of wealth to be consumed;

- taking on a riskier investment policy, i.e. lowering her risk aversion $(1-\gamma^*(x))$ for the wealth remaining once her consumption perpetuity is guaranteed.

The key to understanding the impact of shifting to the discrete-time framework lies in the combined effect of the above actions - although the investment policy is riskier, it is being applied to a lower residual amount after guaranteeing the consumption perpetuity, and the net result is higher upfront consumption and a lower net proportion invested in the risky asset. All else being equal, the agent, therefore, has a more conservative approach to her situation in the discrete-time framework, as one would expect.

For values of x close to r, the ratio of consumption rate to wealth is close to the maximum possible (r). We expect that the *ideal* ratio of consumption rate to wealth $r^*(x)$ for these states will be lower than x since otherwise we would be forced to increase x even further. So even though $r^*(x)$ is increasing, the agent is not able to achieve this ideal consumption-to-wealth ratio right away. That begets the question - when can the agent increase her consumption rate? It is reasonable to expect $r^* > 0$ for all feasible values of x - in fact, it is a direct consequence of equation (2.20) and $\gamma^* > \gamma$ (as established in A-9). Therefore, given feasibility, we must have some $x = \bar{x} > 0$ such that $\bar{x} = r^*(\bar{x})$ - this is exactly the value of x corresponding to the kink in the curves.

We now shift our attention to the portion of the curves where $x < \bar{x}$. For any state $x < \bar{x}$, the previous step consumption c_- is less than $r^*(x)w$; hence, the consumption must immediately be increased to $r^*(x)w$, or alternatively, x must be increased to $r^*(x)$, but x cannot be increased to any point y where $y < r^*(y)$, since the optimal policy would immediately require an increase in x to a value higher than y. Therefore, x must be increased to $\bar{x}$ since $\bar{x} = r^*(\bar{x})$. A direct consequence of this is that

$$r^*(x) = r^*(\bar{x}) = \bar{x} \quad \forall \quad x < \bar{x} \tag{2.32}$$

and similarly

$$\gamma^*(x) = \gamma^*(\bar{x}) \quad \forall \quad x < \bar{x} \tag{2.33}$$

$$\hat{\alpha}(x) = \hat{\alpha}(\bar{x}) \quad \forall \quad x < \bar{x} \tag{2.34}$$

and the corresponding curves are all flat for $x \in [0, \bar{x}]$. The final point of interest is with respect to the $\hat{\alpha}$ curve

Observation 3. *The $\hat{\alpha}$ curve seems to be converging to a constant value for all $x \in (\bar{x}, r]$.*

This is certainly a plausible result under the CRRA utility scheme and bears similarities to the solution in Dybvig [1995], as well as in Merton [1969] and Samuelson [1969]. It might be possible to utilize this form of the $\hat{\alpha}$ curve and add more structure into the solution - this is something that warrants further investigation.

2.3. Computational Results

In this section we discuss the results of simulation runs. We use this to illustrate some of the qualitative differences between three consumption and investment strategies:

- *Discrete Time Unconstrained (DT-U)*: This is the classic Samuelson policy, which is optimal for the discrete-time consumption and investment problem without any constraints on the consumption process;
- *Discrete Time Constrained(DT-C)*: This is the policy derived in this paper, which is optimal for the discrete-time consumption and investment problem when the consumption process is required to be non-decreasing;
- *Continuous Time Constrained (CT-C)*: This is an approximate solution to the discrete-time problem based on the Dybvig policy, which is optimal for the continuous-time version of the problem.

For CT-C, we are applying the continuous-time result in a discrete-time setting. To do this, we assume that the consumption rate for this policy is fixed at each decision epoch for the next time interval. This rate is fixed according to the policy that is optimal for the continuous-time version of the problem. Clearly, CT-C is a non-optimal solution to our problem. Moreover, the problem addressed in this paper can be thought of as the Samuelson problem with an additional constraint. Consequently, we would expect that the objective function values are (in an *expected value* or *mean* sense) the highest for DT-U, smaller for DT-C and the lowest for CT-C when these policies are applied to a discrete-time setting. We would also expect a strong correlation between the results for the DT-C and CT-C strategies.

2.3.1. Results from Multiple Simulation Runs

Figure (2.2) shows a comparative histogram of 10,000 simulation runs. For each run, the stock price is simulated over a period of 100 years, and each of the three strategies is applied over this time horizon. The histogram represents the distribution of the discounted sum of utility - a proxy for the objective function.

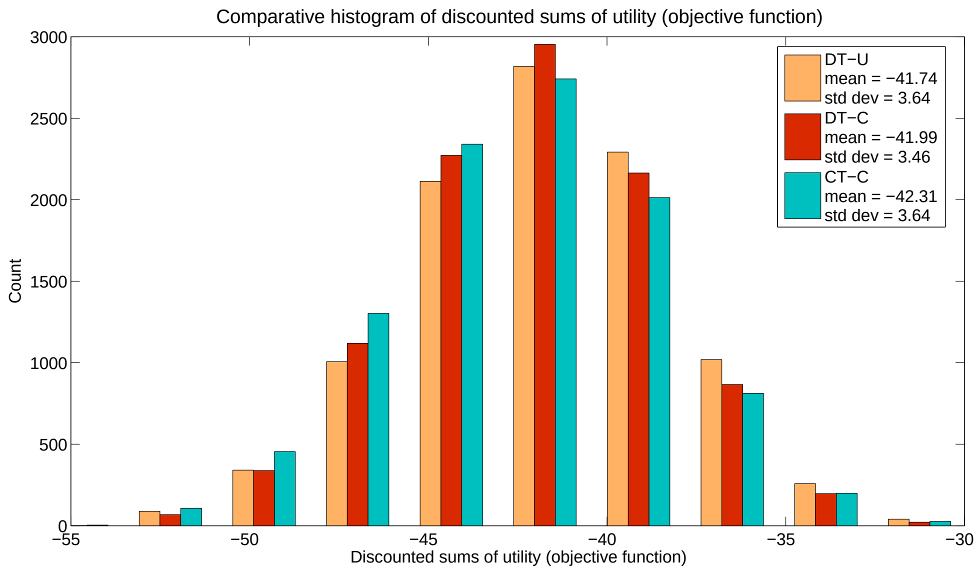

Figure 2.2. Results from 10,000 Sample Runs
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, $\tau = 1$ yr, $N = 100$

The differences in utility among the three strategies can be characterized in terms of the relative efficiency of the poorer strategy with respect to the better one. The relative efficiency of the poorer strategy is the fraction of its initial wealth that the better strategy would require to match the poorer strategy's expected discounted sum of utility. In the context of our current problem, we can also define relative efficiency to be the ratio of the certainty equivalent consumption rate corresponding to the expected discounted sum of utility for the poorer strategy to that of the better one. The two definitions are consistent with each other and yield the same numerical results. This measure is used extensively in Rogers [2001], who considers the effect of time lags on the unconstrained consumption-investment decision.

With parameter values set as in figure (2.2), the CE consumption rates for the DT-U, DT-C and CT-C strategies are 5.72%, 5.65% and 5.57% of initial wealth respectively. These correspond to relative efficiencies of 98.9% (DT-C relative to DT-U), 98.5% (CT-C relative to DT-C) and 97.4% (CT-C relative to DT-U). As expected, the DT-U strategy has the best performance, but it also has the highest volatility. While the DT-U strategy can perform well compared to DT-C and CT-C, it can also perform poorly. The CT-C and DT-C strategies have a smaller range of results. In essence, both are conservative strategies - a direct result of the non-decreasing consumption constraint. The intuition for these results is further developed in the following sub-section (2.3.2).

Similar results are obtained for parameter values as set in a particular example in Rogers ($r = 10\%$, $\mu = 18\%$, $\sigma = 35\%$, $\delta = 10\%$, $C_{0-} = 0$, $1 - \gamma = 4$, $\tau = 0.6$ yrs). The CE consumption rates for the DT-U, DT-C and CT-C strategies are 10.61%, 10.23% and 10.03% of initial wealth respectively. These correspond to relative efficiencies of 96.4% (DT-C relative to DT-U), 98.1% (CT-C relative to DT-C) and 94.5% (CT-C relative to DT-U). In comparison, with the same parameter values Rogers reports a relative efficiency of 96.4% when the unconstrained discrete-time solution is compared to the unconstrained continuous-time solution.

Rogers observed that the utility loss from the time lag in the discrete time setting was smaller than the potential loss from incorrect parameter estimates, but that the time lag effect was much stronger for consumption investors than for wealth-only investors. Our results above indicate that habit formation in the form of a consumption constraint may increase the value of frequent trading opportunities even further than consumption alone.

Similar analysis for parameter values set in Dybvig [1995] was aborted because in the continuous-time case those parameter values result in borrowing from the wealth set aside for the consumption perpetuity which can create infeasibilities in the discrete-time setting. Our discrete-time solution handles this by not allow borrowing due to the chance of ruin over any non-zero period of time.

2.3.2. Results under Two Contrary Scenarios

Figures (2.3(a)) and (2.3(b)) show two different scenarios A and B where, over a period of 100 years, the risky asset increased significantly in value and dropped significantly in value respectively. Figures (2.4(a)), (2.5(a)), (2.6(a)) and (2.4(b)), (2.5(b)), (2.6(b)) show the total wealth, consumption and utility processes over these 100 years for each of the two scenarios.

Under scenario A, where the risky asset value maintains an increasing trend, the non-decreasing consumption constraint will frequently be non-binding in the sense that the ideal consumption rate at any decision epoch will typically be higher than the rate established at the previous decision epoch. Since DT-U is the optimal solution to the unconstrained problem, its performance can be expected to be better than that of either DT-C or CT-C. Under scenario B, the protection from the non-decreasing consumption constraint kicks in - while the consumption rate under the DT-U strategy has significant downside, both the DT-C and the CT-C avoid this pitfall and outperform the DT-U strategy. The DT-C strategy calls for a higher fraction of the wealth to be consumed up front and a lower fraction of the remaining wealth to be invested in the risky asset, as compared to CT-C. In some sense, the CT-C approximation sacrifices some immediate consumption for an increase in the potential for future gains - it does this by investing a higher fraction of the higher amount left over (after guaranteeing the lower consumption rate) in the risky asset. Consequently, it is feasible for CT-C to perform better than DT-C. However, under most scenarios, DT-C will outperform CT-C, which we expect since it is the optimal solution to the constrained problem. One can see that the DT-C and CT-C strategies are conservative in nature when compared with the DT-U strategy, the results of which vary significantly under these two opposing scenarios. In comparison with CT-C, DT-C not only provides more upward potential for consumption (and hence utility) but also provides better protection against downturns.

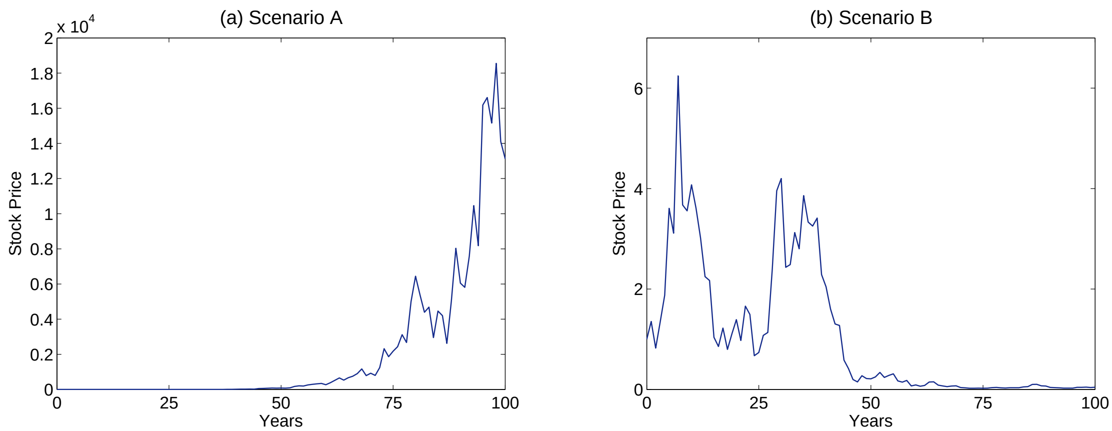

Figure 2.3. Stock prices over 100 years
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1-\gamma = 1.5$, $\tau = 1$ yr, $N = 100$

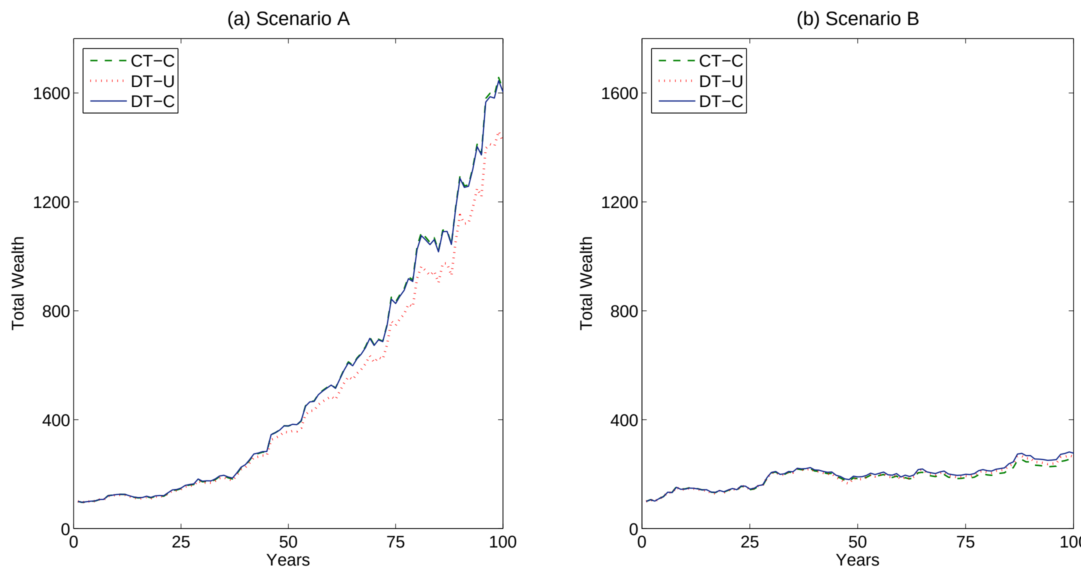

Figure 2.4. Total wealth over 100 years
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, $\tau = 1$ yr, $N = 100$

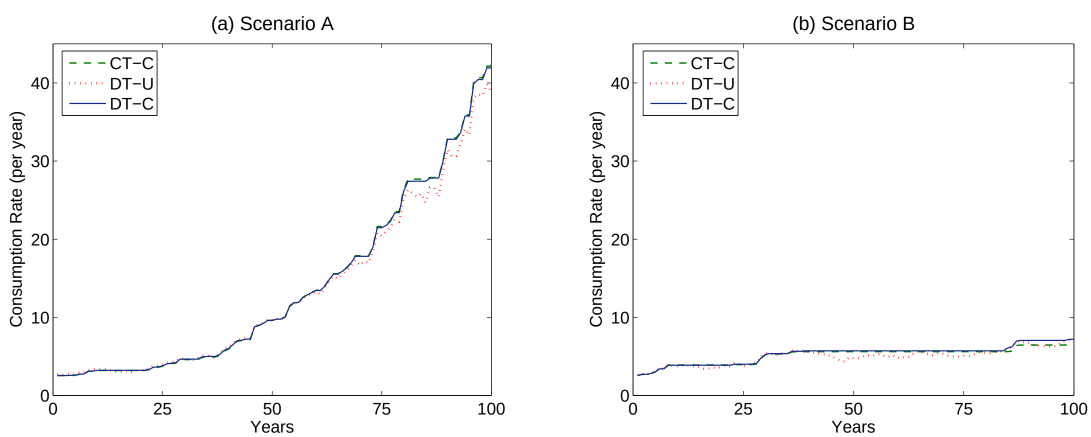

Figure 2.5. Consumption rate over 100 years
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, $\tau = 1$ yr, $N = 100$

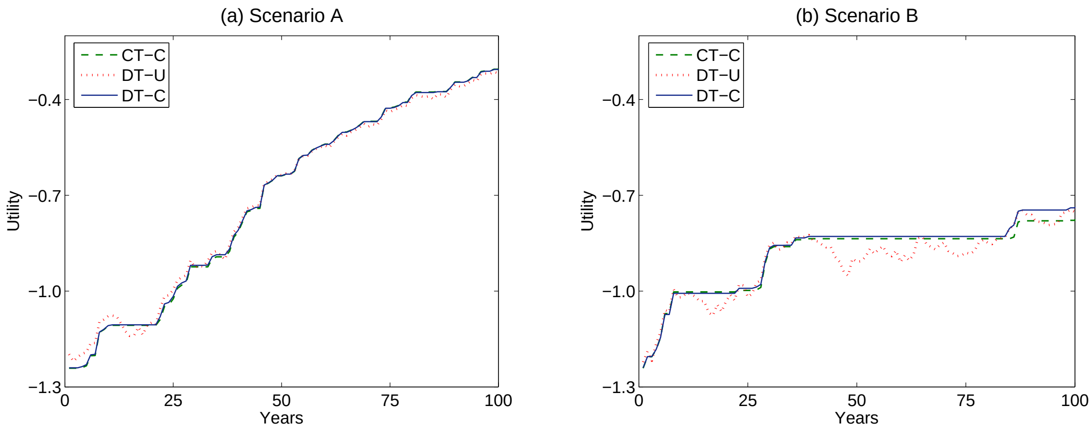

Figure 2.6. Utility over 100 years
$r = 4\%,\ \mu = 6\%,\ \sigma = 30\%,\ \delta = 2\%,\ C_{0-} = 0,\ 1-\gamma = 1.5,\ \tau = 1$ yr, $N = 100$

2.3.3. Impact of the Length of the Time Interval between Decision Epochs

We now turn to examining the impact of the length of the time interval τ on our optimal policy. Using the iteration scheme of Section 2.2.6, we compute the $\gamma^*(\cdot)$, $r^*(\cdot)$ and $\hat{\alpha}(\cdot)$ curves for four different values of τ and compare them to the corresponding constants of the continuous-time case. The parameter values used are $r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$ and $1 - \gamma = 1.5$. 101 ratios were evaluated over 100 iterations. The results for $\gamma^*(\cdot)$, $r^*(\cdot)$ and $\hat{\alpha}(\cdot)$ are shown in figures (2.7(a)), (2.7(b)) and (2.8(a)) respectively.

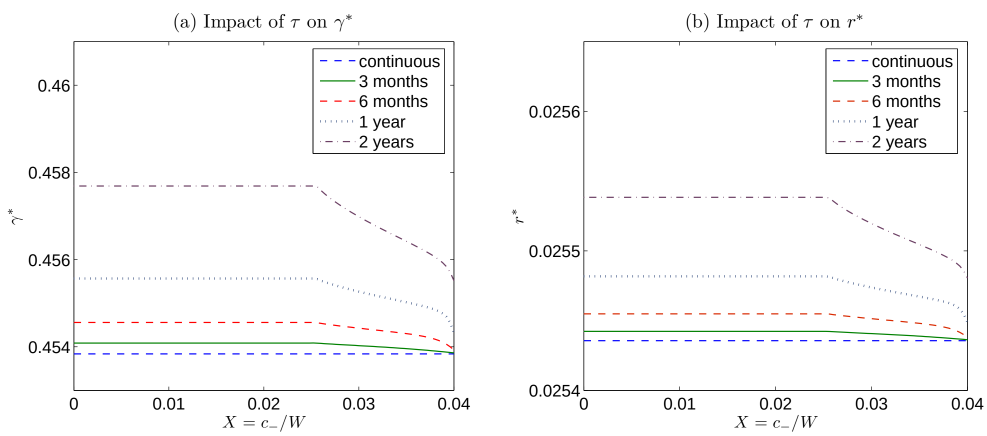

Figure 2.7. Impact of the length of time interval τ on γ^* and r^*
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, $N = 100$

What are the implications of these results? First, as τ increases, the kink in the $\gamma^*(\cdot)$ curve (and consequently in the $r^*(\cdot)$ and $\hat{\alpha}(\cdot)$ curves) shifts to the left - i.e., $\bar{x}$ decreases with an increase in τ. This implies that the *ideal* fraction of wealth to be consumed is decreasing with an increase in τ.

For the same value of the ratio x, the value of γ^* increases with τ. At first this seems odd - why would the risk aversion $(1 - \gamma^*)$ of the agent for the wealth left after guaranteeing consumption drop with a longer time interval between decision epochs? This can be understood by looking at the r^* curve. For a given value of x, $r^*(x)$ increases with τ since it is directly proportional to $\gamma^*(x)$ (see equation (2.20)). Consequently, as the length of the time interval increases, the minimum fraction of wealth being consumed increases as well. The amount required to guarantee this higher rate of consumption also increases and the amount left over for reinvestment is lower for higher values of τ. Now that the agent has locked in a consumption stream at a particular rate (and thereby a utility stream at a corresponding rate), the only way for the agent to increase the utility rate is to generate enough wealth over the subsequent time periods to be able to increase the consumption rate while maintaining the optimal balance between the current consumption rate and the potential for future increases in it. The lower level of risk aversion provides the counter-balance to the higher minimum consumption rate (per unit wealth) that comes with the increase in the length of the time interval.

To complete the picture, we examine what this implies for the investment in the risky asset. As can be seen from figure (2.8(a)), the $\hat{\alpha}(\cdot)$ curve is lowered with an increase in τ. Thus, as expected, with an increase in the length of the time interval a lower fraction of the wealth available for reinvestment is put into the risky asset. Since $\hat{\alpha}$ is a proportion of the total wealth that also includes the perpetuity to maintain previous consumption, we also consider the fraction of wealth after setting aside the perpetuity in the riskfree asset as the *net proportion of wealth invested in the risky asset* $\alpha(\cdot)$ for varying lengths of the time interval. $\alpha(\cdot)$ can be evaluated using a single-state variant of equation (2.21)

$$\alpha(x) = \hat{\alpha}(x)\left(1 - \frac{max(x, r^*(x))}{r}\right) \tag{2.35}$$

The above-mentioned comparison is performed in figures (2.8(a)) and (2.8(b)) - we see that the net proportion invested in the risky asset decreases with an increase in the length of the time interval for all feasible states, which is in line with our intuition. It is to be noted that in the states where the previous step consumption is high relative to what we would ideally prefer give our current wealth (i.e., for $x > \bar{x}$), the investment levels in the risky asset are bunched closer together - however, even for these states the net proportion in risky asset is lower as τ increases.

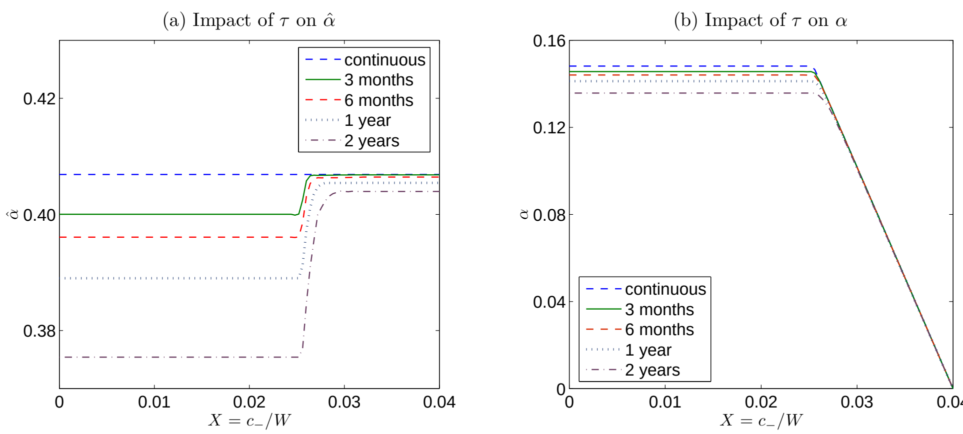

Figure 2.8. Impact of the length of time interval τ on $\hat{\alpha}$ and α
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, $N = 100$

Finally, as τ decreases to 0, both the $\gamma^*(\cdot)$ and $\hat{\alpha}(\cdot)$ curves flatten out and converge to the corresponding constant values of the continuous-time case, as we would expect.

2.3.4. Impact of Negative Skewness and Leptokurtosis

We now examine the possibility of the return distribution of our risky asset possessing negative skewness and leptokurtocity. As we discussed in the introduction, Brooks and Kat [2001] show that the distributions of hedge fund returns typically possess negative skewness and excess kurtosis. If such was the case for the risky asset under our consideration, one could still use the CT-C strategy as an approximation, but the approximation would ignore both the time lag and the lack of normality. We conduct the following analysis to ascertain whether the approximation performs worse than it did when the risky asset returns were normally distributed as would be expected given the normality assumption for the continuous-time result. To test this, we employ the Pearson Type IV distribution as the distribution of the risky asset return. This allows us to compute probability density function values for numerical integration as well as obtain random draws for simulation purposes by simply specifying the mean, standard deviation, skewness and kurtosis of the return distribution. Heinrich [2004] provides an excellent guide to the details of using the Pearson Type IV distribution in the above manner.

Table 2.2 summarizes our findings. To measure the performance of the approximation, instead of looking at the expected discounted sum of utilities (our objective function), we focus on the lower percentiles of the corresponding distribution of the discounted sums of utilities. This provides us with a measure of the downside risk of using the CT-C approximation.

As seen in the tabulated results, the magnitude of the decrease in the objective function value is greater (both in absolute and relative terms) when the risky asset return has negative skewness and excess kurtosis as compared to the case of normally distributed risky asset returns. This ties in with our intuition that the CT-C approximation would afford us a lowered level of downside protection if the risky asset were to have a left-skewed leptokurtic return distribution.

	Return Distribution = Normal				Return Distribution = Pearson Type IV			
	Certainty Equivalent Consumption Perpetuity Rate (% of W_0 per yr)		Magnitude of Decrease caused by CT-C approximation		Certainty Equivalent Consumption Perpetuity Rate (% of W_0 per yr)		Magnitude of Decrease caused by CT-C approximation	
	DT-C	CT-C Approx.	Absolute	Relative	DT-C	CT-C Approx.	Absolute	Relative
1 %ile	3.9372%	3.8258%	0.1114%	2.8307%	3.9907%	3.8685%	0.1222%	3.0626%
2 %ile	4.1051%	3.9928%	0.1123%	2.7361%	4.1448%	4.0214%	0.1234%	2.9771%
5 %ile	4.3570%	4.2452%	0.1118%	2.5654%	4.3926%	4.2671%	0.1255%	2.8568%
10 %ile	4.6080%	4.4943%	0.1136%	2.4658%	4.6531%	4.5333%	0.1197%	2.5727%
20 %ile	4.9321%	4.8301%	0.1020%	2.0678%	4.9672%	4.8582%	0.1090%	2.1950%

Table 2.2. Impact of negative skewness and leptokurtosis in risky asset returns
$r = 4\%$, $\mu = 6\%$, $\sigma = 30\%$, $\delta = 2\%$, $C_{0-} = 0$, $1 - \gamma = 1.5$, $\tau = 1$ yr, $N = 100$
For the Pearson Type IV return distribution, $\sqrt{\beta_1} = -0.25$ and $\beta_2 = 4$
The certainty equivalent consumption perpetuity rate is the fixed rate of perpetual consumption that will give us a von Neumann - Morgenstern utility equal to the objective function of the optimal solution. The above percentiles come from a simulated distribution.

2.4. Conclusion

We have presented a solution method for the discrete-time consumption and investment optimization problem where the consumption rate process is constrained to be nondecreasing. We first prove that the value function for our problem must have a specific form that depends on an unknown function of a bounded state variable, and then use an iterative procedure to complete the solution.

The key element of the solution is to find a function $\gamma^*(\cdot)$ where $1 - \gamma^*(\cdot)$ represents the risk aversion of the agent applied specifically to the wealth remaining after the current consumption rate level has been guaranteed, given a particular state of our system. We have proposed an iterative method to find $\gamma^*(\cdot)$, and, as part of the method, we also find functions $r^*(\cdot)$ and $\hat{\alpha}(\cdot)$, the values of which rely on the value of $\gamma^*(\cdot)$ for the same state.

The structure of the solution is intuitive. Feasibility of the problem is easily checked - does the agent have enough wealth to guarantee herself the current consumption rate by putting all her wealth into the risk-free asset? Once we have feasibility, the agent achieves an optimal outcome as follows:

- At each decision epoch, the agent decides on the current consumption rate based on her current wealth and the consumption rate at the previous step (our state variables). This is done by using the ratcheting formula (A-14) which uses the function $r^*(\cdot)$.
- Next, the agent puts an amount in the risk-free asset that is equivalent to the present value of a perpetuity stream with a payout at the current consumption rate.

- Now, based on the current state of the system, a proportion $\hat{\alpha}(\cdot)$ of the remaining wealth is put in the risky asset, with the rest going into the risk-free asset. This completes the optimal decision process at a particular decision epoch.

As we can see from the simulation runs, the non-decreasing consumption constraint leads to a conservative approach being adopted by the agent which offers significant downside protection without significant losses in certainty-equivalent consumption. The continuous-time solution approximation provides good performance when risky asset values increase but may perform poorly with mediocre or poor risky asset returns. The result is that the optimal discrete-time solution can obtain relatively large gains in certainty-equivalent consumption over the continuous-time solution approximation. This observation implies that investors should beware of using continuous-time approximations for investments that only have limited liquidity.

The impact of an increase in the time interval between decision epochs (τ) is as expected - an increase in the length of the time interval is seen to lower the *ideal* fraction of wealth consumed as well as decrease the net proportion of wealth invested in the risky asset for any feasible state. Note that because of the non-decreasing consumption constraint, it is not always feasible to be at a state where this fraction is at the *ideal* level. Also, with an increase in the length of the time interval, the performance of the continuous-time solution as an approximation to the discrete-time case becomes worse.

Finally, for the case of negatively skewed leptokurtic risky asset return distributions, we have demonstrated the lowered effectiveness of the continuous-time approximation in providing protection against downside risk. In such cases, the ability of our methodology to work with any reasonable asset return distribution proves very useful. This is especially of appeal to institutional investors in hedge funds that may have both limited trading opportunities and non-normal returns.

We could consider some other form of habit formation but we anticipate results that are very similar to those derived in this paper. As an example, the solution method designed in this paper would also work when the consumption rate, at the beginning of a new time interval, is allowed to decrease by a specific relative amount, say 10%. In this case, we can still calculate the amount we need to set aside in a perpetuity to guarantee such a consumption stream and the solution follows. But clearly, this is not as conservative a strategy as non-decreasing consumption, thereby the loss of utility from this method of smoothing consumption can not be more than that from forcing consumption to be non-decreasing. Thus, a broader implication of this paper, when considered in conjunction with Rogers [2001], is that the limitations on trading frequency can have a greater impact in utility terms than smoothing of consumption.

From a technical standpoint, we have devised a novel method for solving dynamic programming problems which have no closed-form solution. While we do not present the method in a formal fashion, we anticipate that a similar approach could apply to a number of dynamic programming problems where some intuition in regards to the structure of the optimal solution is available but not fully exploited.

CHAPTER 3

Hedge Funds: Optimal Portfolio Allocation, the Lockup Premium and the Information Premium

Abstract

We present a systematic approach to the analysis of hedge funds as components of an investor's portfolio. The main differentiator for hedge funds as compared to equities is that they typically require investor funds to be locked up for a pre-specified duration. In this regard, investments in private equity also face similar restrictions. This is illiquidity of a very specific form and has an impact on the attractiveness of the hedge funds as investable assets.

We first provide a framework for the calculation of the hedge fund lockup premium given the ability to calculate the optimal allocation to the hedge fund. Then we study the optimal structure for a portfolio consisting of a bond, a stock and a hedge fund under various settings. The hedge fund lockup premium can then be easily calculated under any of these settings using the framework provided.

Additionally, we introduce the concept of an information premium, which can be thought of as compensation required for a lack of continuously available hedge fund share price information. We present a framework for the calculation of the information premium as well.

3.1. Introduction

In recent years, institutional investors' holdings in hedge funds and private equity have increased significantly. While promising high returns, they also typically impose several restrictions on the investors. The most significant one requires investor funds to be locked up for a pre-specified duration. This requirement represents illiquidity of a very specific form. Longstaff [2001] studies the impact of illiquidity on optimal portfolio choice and the price of the asset. However, the setting is one with limited liquidity whereas we have to deal with not being allowed to trade an asset at all except at specific points in time.

There are a couple of aspects that we would like to study with respect to the lockup restriction. First, it is clear that this imposed illiquidity changes the allocation to the hedge fund as part of an optimal portfolio structure. We will specifically examine a case where we have three assets available to us for investment - a bond, a stock and a hedge fund. The positions in the bond and the stock can be changed continuously, however the position in the hedge fund can only be altered at pre-specified intervals of time.

Within this general setting we will examine the impact of including consumption, as well as describe a method to calculate the value of hedge fund share price information. Consumption is important in the context of institutional investors such as pension funds and university endowments, whereas the no-consumption case is a better description of the setting in which proprietary trading desks or fund of funds operate. The value of hedge fund share price information is important in analyzing the impact of the typical secrecy that surrounds hedge fund performance reporting. We will compare the case where the hedge fund price is available only just before a decision has to be made on adjusting the position in the hedge fund to the default situation where the hedge fund share price is always known.

The second aspect we want to study is the premium associated with the lockup restriction, termed the *lockup premium.* Aragon [2007] demonstrates that the excess returns for funds with lockup restrictions are significantly higher than those for funds without these restrictions. Calculating the lockup premium allows potential investors to check whether the lockup requirement is justified for a particular hedge fund. Derman [2007] and Derman et al. [2007] represent recent attempts at computing this premium. However, there are several issues in their approach. First, these papers rely on persistence among hedge fund returns (the existence of which is quite controversial in nature). Second, they seek to calculate the lockup premium only for *extended* lockup periods relative to the typical one-year lockup requirement. Finally, they do not consider the presence of equities as an alternative investment and by doing so they essentially restrict the applicability of their approach to fund of funds. Our analysis includes a consideration of bonds and equities in addition to hedge funds, which represents a more typical portfolio structure for pension funds and endowments. We will not, however, consider survivorship among hedge funds, something that Derman [2007] and Derman et al. [2007] both consider.

In §3.2, we present a framework for computing the lockup premium for hedge funds, as well as an easier-to-calculate alternative that we call the *lockup penalty.* In §3.3 and §3.5, we present methods to obtain optimal hedge fund allocations under considerations of the utility of terminal wealth and consumption respectively. In §3.4, a method for computing the optimal allocation for the hedge fund to maximize terminal wealth utility is presented, but this time with the hedge fund share price information being available only just before the decision epoch where adjustments to the hedge fund position can be made. Juxtaposing this with §3.3, it is possible to determine the value of hedge fund share price information being continuously available (dubbed the *information premium*), or correspondingly the *information penalty* that should be assessed on hedge funds that do not make this information readily available. This analysis is presented only for the non-consumption case, which as mentioned before is the appropriate setting for fund of funds and proprietary traders. However, it is easy to extend this concept to the consumption case as well, making it applicable for pension funds and university endowments.

3.2. A Framework for Calculating the Hedge Fund Lockup Premium

In this section we present the basic framework for calculating the lockup premium for the hedge fund. First, consider a scenario where there is no lockup restriction on the hedge fund. In this case, the hedge fund is like a typical stock (except that one can not take short positions in it) and the standard rules of portfolio optimization apply and the hedge fund would receive an allocation of α_H. An example of how to obtain α_H would be to utilize the Merton [1969] solution to the portfolio optimization problem.

The next step would be to solve for the optimal allocation to the hedge fund $\hat{\alpha}_H$ when the lockup requirement is in place. Clearly, the hedge fund is no longer as attractive an asset as it was without the restriction, and therefore $\hat{\alpha}_H \leq \alpha_H$, keeping the problem parameters the same across the two scenarios.

Once we can obtain values for α_H and $\hat{\alpha}_H$, to compute the *lockup premium* we pose the following question: at what value μ_H^+ of the expected hedge fund return μ_H would the optimal allocation to the hedge fund be α_H in spite of the lockup being enforced? Or in other words, what is the value μ_H^+ such that when $\mu_H = \mu_H^+$, we have $\hat{\alpha}_H = \alpha_H$, where $\hat{\alpha}_H$ is obtained by whatever method is used to solve the restricted problem?

Upon calculating this value, the lockup premium for the hedge fund is given by

$$p_l^+ = \mu_H^+ - \mu_H \tag{3.1}$$

Since $\hat{\alpha}_H \leq \alpha_H$, we expect that $\mu_H^+ \geq \mu_H$ and thus $p_H^+ \geq 0$.

The process for finding $\hat{\alpha}_H$ might not be in of itself a problem, but it is likely (as we will see in subsequent sections), it is probably not something that is amenable to easily find μ_H^+ given a specific value that is required of $\hat{\alpha}_H$ (α_H). However, we expect that in most problem setups this would require a simple iterative process, along the lines of a line search algorithm.

An alternative approach is to compute a *lockup penalty* instead of a lockup premium. The lockup penalty can be thought of as a deduction to be applied to the expected return for the hedge fund to account for its lockup restrictions. Once we can obtain values for α_H and $\hat{\alpha}_H$, we pose the following question: at what value μ_H^- of the expected hedge fund return μ_H would the optimal allocation to the hedge fund be $\hat{\alpha}_H$ under the non-restricted setting? Or in other words, what is the value μ_H^- such that when $\mu_H = \mu_H^-$, we have $\alpha_H = \hat{\alpha}_H$, where α_H is obtained by the Merton solution or an equivalent (i.e. non-restricted) method?

Upon calculating this value, the lockup penalty for the hedge fund is given by

$$p_l^- = \mu_H^- - \mu_H \tag{3.2}$$

Since $\hat{\alpha}_H \leq \alpha_H$, we expect that $\mu_H^- \leq \mu_H$ and thus $p_H^- \leq 0$.

Finding μ_H^- is relatively straightforward for the Merton framework, or equivalently the mean-variance framework with risk aversion. Because of this, some practitioners might find it easier to deal with the lockup penalty rather than the lockup premium.

3.3. Problem I: Maximizing the Utility of Terminal Wealth with Hedge Fund Price Information Available

In this section, we consider the problem of maximizing the utility of terminal wealth. Initial wealth can be invested in a bond, a stock or a hedge fund, with the hedge fund requiring a lockup period of T. Hedge fund share price information is assumed to be continuously available.

Let $X(t) : t \geq 0$ be the wealth process and $H(t) : t \geq 0$ be the hedge fund investment value process. Let r be the risk-free rate, μ_S and μ_H the expected rates of return on the stock and the hedge fund and σ_S and σ_H the stock and hedge fund volatilities. Define $\nu_S = \mu_S - r$ and $\nu_H = \mu_H - r$. $h(t) \equiv \frac{H(t)}{X(t)}$ (h in brief) is the fraction of total wealth in the hedge fund at time t and $\beta_S(t)$ (β in brief) is the fraction of total wealth invested in the stock at time t. β is a decision variable at all times $t \geq 0$, whereas h is a decision variable only at time $t = 0$. Correspondingly, an alternative notation for $h(0)$ is α_H. Finally, our planning horizon is T, the same as the duration of the lockup period.

The stochastic differential equations for our problem are

$$dH = \mu_H H dt + \sigma_H H dW_H = (\nu_H + r) H dt + \sigma_H H dW_H \tag{3.3}$$

and

$$dX = (rX + \beta\nu_S X + \nu_H H)dt + \sigma_S\beta X dW_S + \sigma_H H dW_H \tag{3.4}$$

where W_H and W_S are Wiener processes with $dW_H dW_S = \rho dt$. We want to maximize $\mathbb{E}\left[\frac{1}{\gamma}X(T)^\gamma\right]$.

Using a result in §8.5 of Arnold [1974], it can be shown that this system of stochastic differential equations have no closed-form solution. Equation (3.3) does however have a closed-form solution by itself.

Now we proceed in a fashion similar to that in Kahl et al. [2003]. Since X and H form a joint Markov process, the value-to-go function $J(X, H, t)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\max\left[\begin{array}{l} \frac{1}{2}(\sigma_S^2\beta^2X^2 + 2\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)J_{XX} \\ +\frac{1}{2}\sigma_H^2H^2J_{HH} + (\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)J_{XH} \\ +(rX + \beta\nu_S X + \nu_H H)J_X + (r+\nu_H)HJ_H + J_t \end{array}\right] = 0 \tag{3.5}$$

The first order optimality condition is $\beta\sigma_S^2X^2J_{XX} + \rho\sigma_S\sigma_H XHJ_{XX} + \rho\sigma_S\sigma_H XHJ_{XH} + \nu_S XJ_X = 0$, thus

$$\beta^* = \frac{-\nu_S}{\sigma_S^2}\frac{J_X}{XJ_{XX}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{H}{X}\frac{J_{XH}}{J_{XX}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{H}{X} \tag{3.6}$$

Note that this expression for β^* is the same as that for ϕ^* in Appendix A of Kahl et al. [2003], even though they consider consumption. Although it may lead one to conclude that including consumption has no impact on the optimal portfolio strategy, this is incorrect - the value functions J will be different in the two cases.

We conjecture that value-to-go function is of the form

$$J(X, H, t) = \frac{X^\gamma}{\gamma} V(h, t) \tag{3.7}$$

We can verify the validity of this conjecture by rewriting the HJB equation in terms of this function $V(h, t)$ and checking that the boundary and terminal conditions are free of X.

We first compute the partial derivatives J_X, J_{XX} and J_{XH} in terms of h and the function $V(h, t)$ and its partial derivatives V_h and V_{hh} as

$$J_X = \frac{1}{\gamma} X^{\gamma-1}(-hV_h + \gamma V) \tag{3.8}$$

$$J_{XX} = \frac{1}{\gamma} X^{\gamma-2}(h^2 V_{hh} + 2(1-\gamma)hV_h + \gamma(\gamma-1)V) \tag{3.9}$$

$$J_{XH} = \frac{1}{\gamma} X^{\gamma-2}(-hV_{hh} + (\gamma-1)V_h) \tag{3.10}$$

We can now write β^* in terms of these quantities

$$\beta^* = \frac{\frac{\sigma_{SH}}{\sigma_S^2} h^2 V_{hh} + \left(\frac{\nu_S}{\sigma_S^2} + (1-\gamma)\frac{\sigma_{SH}}{\sigma_S^2}\right) hV_h - \gamma \frac{\nu_S}{\sigma_S^2} V}{h^2 V_{hh} + 2(1-\gamma)hV_h - \gamma(1-\gamma)V} - \frac{\sigma_{SH}}{\sigma_S^2} h \tag{3.11}$$

Note that the equivalent formula in Kahl et al. [2003] contains a small typographical error.

Our next step is to convert the HJB into a form that uses h and the partial derivatives of the function $V(h, t)$. To do this we also need J_H, J_{HH} and J_t in terms of $V(h, t)$ and its partial derivatives

$$J_H = \frac{X^{\gamma-1}}{\gamma} V_h \tag{3.12}$$

$$J_{HH} = \frac{X^{\gamma-2}}{\gamma} V_{hh} \tag{3.13}$$

$$J_t = \frac{X^{\gamma}}{\gamma} V_t \tag{3.14}$$

Substituting the partial derivatives of $J(X, H, t)$ into the HJB gives us

$$\max \begin{bmatrix} \frac{1}{2}(\sigma_S^2\beta^2X^2 + 2\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)\frac{1}{\gamma}X^{\gamma-2} \\ (h^2V_{hh} + 2(1-\gamma)hV_h - \gamma(1-\gamma)V) \\ +\frac{1}{2}\sigma_H^2H^2\frac{X^{\gamma-2}}{\gamma}V_{hh} + (\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)\frac{1}{\gamma}X^{\gamma-2} \\ (-hV_{hh} - (1-\gamma)V_h) \\ +(rX + \beta\nu_S X + \nu_H H)\frac{1}{\gamma}X^{\gamma-1}(-hV_h + \gamma V) \\ +(r+\nu_H)H\frac{X^{\gamma-1}}{\gamma}V_h + \frac{X^\gamma}{\gamma}V_t \end{bmatrix} = 0$$

Dividing through by $\frac{X^\gamma}{\gamma}$ and rearranging the terms gives us the HJB equation in terms of the function $V(h,t)$

$$\max\left[\begin{array}{l}\left(\begin{array}{l}\frac{1}{2}(\sigma_S^2\beta^2h^2+2\sigma_{SH}\beta h^3+\sigma_H^2h^4)\\+\frac{1}{2}\sigma_H^2h^2-(\sigma_{SH}\beta h^2+\sigma_H^2h^3)\end{array}\right)V_{hh}\\+\left(\begin{array}{l}(1-\gamma)(\sigma_S^2\beta^2h+2\sigma_{SH}\beta h^2+\sigma_H^2h^3)\\-(1-\gamma)(\sigma_{SH}\beta h+\sigma_H^2h^2)\\-(rh+\beta\nu_Sh+\nu_Hh^2)+(rh+\nu_Hh)\end{array}\right)V_h\\+V_t+\left(\begin{array}{l}-\gamma(1-\gamma)\frac{1}{2}(\sigma_S^2\beta^2\\+2\sigma_{SH}\beta h+\sigma_H^2h^2)+\gamma(r+\beta\nu_S+\nu_Hh)\end{array}\right)V\end{array}\right]=0 \qquad (3.15)$$

It is easy to confirm the validity of equation(3.11) by checking it against the value of β^* that would be implied by equation (3.15), thereby confirming the typographical error in the formula in Kahl et al. [2003].

The maximum in equation (3.15) holds when $\beta=\beta^*$ and therefore

$$\begin{array}{l}\left(\begin{array}{l}\frac{1}{2}(\sigma_S^2(\beta^*)^2h^2+2\sigma_{SH}\beta^*h^3+\sigma_H^2h^4)\\+\frac{1}{2}\sigma_H^2h^2-(\sigma_{SH}\beta^*h^2+\sigma_H^2h^3)\end{array}\right)V_{hh}\\+\left(\begin{array}{l}(1-\gamma)(\sigma_S^2(\beta^*)^2h+2\sigma_{SH}\beta^*h^2+\sigma_H^2h^3)\\-(1-\gamma)(\sigma_{SH}\beta^*h+\sigma_H^2h^2)\\-(rh+\beta^*\nu_Sh+\nu_Hh^2)+(rh+\nu_Hh)\end{array}\right)V_h\\+V_t+\left(\begin{array}{l}-\gamma(1-\gamma)\frac{1}{2}(\sigma_S^2(\beta^*)^2+2\sigma_{SH}\beta^*h+\sigma_H^2h^2)\\+\gamma(r+\beta^*\nu_S+\nu_Hh)\end{array}\right)V=0\end{array} \qquad (3.16)$$

We write this in brief as

$$f_1(h,\beta^*)V_{hh} + f_2(h,\beta^*)V_h + V_t + f_3(h,\beta^*)V = 0 \tag{3.17}$$

where

$$\begin{aligned} f_1(h,\beta^*) = \;& \tfrac{1}{2}(\sigma_S^2(\beta^*)^2h^2 + 2\sigma_{SH}\beta^*h^3 + \sigma_H^2h^4) \\ & + \tfrac{1}{2}\sigma_H^2h^2 - (\sigma_{SH}\beta^*h^2 + \sigma_H^2h^3) \end{aligned} \tag{3.18}$$

$$\begin{aligned} f_2(h,\beta^*) = \;& (1-\gamma)(\sigma_S^2(\beta^*)^2h + 2\sigma_{SH}\beta^*h^2 + \sigma_H^2h^3) \\ & -(1-\gamma)(\sigma_{SH}\beta^*h + \sigma_H^2h^2) \\ & -(rh + \beta^*\nu_Sh + \nu_Hh^2) + (rh + \nu_Hh) \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} f_1(h,\beta^*) = \;& -\gamma(1-\gamma)\tfrac{1}{2}(\sigma_S^2(\beta^*)^2 + 2\sigma_{SH}\beta^*h + \sigma_H^2h^2) \\ & +\gamma(r + \beta^*\nu_S + \nu_Hh) \end{aligned} \tag{3.20}$$

We now set up an implicit finite difference scheme to evaluate the function $V(h,t)$. Instead of plugging in the expression for β^* into equation (3.16), we follow Kahl et al. [2003] in linearizing the HJB equation by evaluating the values of β^* using the estimated values of the function $V(h,t)$ and its derivatives at the subsequent time step. As in Kahl et al. [2003], we ensure the accuracy of the scheme by using extremely small time steps.

We use the following central difference estimates for the partial derivatives of V in h

$$V_{hh}(h_i,t_j) = \frac{1}{\Delta h^2}[V(h_{i+1},t_j) - 2V(h_i,t_j) + V(h_{i-1},t_j)] \tag{3.21}$$

$$V_h(h_i, t_j) = \frac{1}{2\Delta h}[V(h_{i+1}, t_j) - V(h_{i-1}, t_j)] \tag{3.22}$$

and a forward difference estimate for V_t

$$V_t(h_i, t_j) = \frac{1}{\Delta t}[V(h_i, t_{j+1}) - V(h_i, t_j)] \tag{3.23}$$

Inserting these estimates into equation (3.17) and rearranging terms gives us the following set of equations for the interior points of our grid

$$\begin{aligned} &\left[\tfrac{f_1(h,\beta^*)}{\Delta h^2} + \tfrac{f_2(h,\beta^*)}{2\Delta h}\right] V(h_{i+1}, t_j) + \left[-2\tfrac{f_1(h,\beta^*)}{\Delta h^2} - \tfrac{1}{\Delta t} + f_3(h, \beta^*)\right] V(h_i, t_j) \\ &+ \left[\tfrac{f_1(h,\beta^*)}{\Delta h^2} - \tfrac{f_2(h,\beta^*)}{2\Delta h}\right] V(h_{i-1}, t_j)] = -\tfrac{1}{\Delta t} V(h_i, t_{j+1}) \end{aligned} \tag{3.24}$$

As seen later, we know the terminal value of $V(h,t)$, i.e. $V(h,T)$ is known for all values of h. Consequently, we will perform a backward recursion to evaluate $V(h,t)$ for all (h,t) combinations. In the above equation, the single term on the right hand side of the equation is indexed at time t_{j+1} and is known. However, the three terms on the left hand side of the equation are indexed at time t_j, are evaluated at different values of h and are unknown, thus making this an implicit scheme.

We now need to determine the terminal and boundary conditions. At $t = T$, we have

$$J(X, H, T) = \frac{X_T^\gamma}{\gamma} \tag{3.25}$$

or equivalently

$$V(h, T) = 1 \tag{3.26}$$

So at $t = T$, we can use

$$V(h_i, t_{N_t}) = 1 \tag{3.27}$$

At $h_1 = 0$, the entire wealth of the investor is in the stock and the bond. Since these two can be traded continuously, the optimal portfolio is obtained using the Merton (1969) solution, i.e. by maintaining a proportion $\beta_M^* = \frac{\nu_S}{\sigma_S^2(1-\gamma)}$ in the stock. Consequently, the value function at time t for $h_1 = 0$ is found as follows

$$J(X, 0, t) = \mathbb{E}\left[\frac{X_t^\gamma}{\gamma}\exp\left(\begin{array}{l}\gamma(r + \beta_M^*\nu_S - \frac{1}{2}(\beta_M^*)^2\sigma_S^2)(T-t)\\ +\gamma\beta_M^*\sigma_S(W_S(T) - W_S(t))\end{array}\right)\right] \tag{3.28}$$

and thus

$$V(0, t) = \exp\left(\gamma(r + \beta_M^*\nu_S)(T-t) - \frac{1}{2}\gamma(1-\gamma)(\beta_M^*)^2\sigma_S^2(T-t)\right) \tag{3.29}$$

So at $h_1 = 0$, we can use

$$V(h_1, t_j) = \exp\left(\gamma(r + \beta_M^*\nu_S)(T-t_j) - \frac{1}{2}\gamma(1-\gamma)(\beta_M^*)^2\sigma_S^2(T-t_j)\right) \tag{3.30}$$

Similarly, at $h_{N_h} = 1$, the entire wealth of the investor is in the hedge fund and stays there until time T. Consequently, the value function at time t for $h_1 = 0$ is found as follows

$$J(X, X, t) = \mathbb{E}\left[\frac{X_t^\gamma}{\gamma}\exp\left(\begin{array}{l}\gamma(r + \nu_H - \frac{1}{2}\sigma_H^2)(T-t)\\ +\gamma\sigma_H(W_H(T) - W_H(t))\end{array}\right)\right] \tag{3.31}$$

and thus

$$V(1,t) = \exp\left(\gamma(r+\nu_H)(T-t) - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2(T-t)\right) \tag{3.32}$$

So at $h_{N_h} = 1$, we can use

$$V(h_{N_h}, t_j) = \exp\left(\gamma\mu_H(T-t_j) - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2(T-t_j)\right) \tag{3.33}$$

3.4. Problem II: Maximizing the Utility of Terminal Wealth with Hedge Fund Price Information Restricted to Decision Epochs

In this section, we consider the problem of maximizing the utility of terminal wealth. Initial wealth can be invested in a bond, a stock or a hedge fund, with the hedge fund requiring a lockup period of T. Hedge fund share price information is assumed to be continuously available.

Let $X(t) : t \geq 0$ be the wealth process, $H(t) : t \geq 0$ be the hedge fund price process, as defined in section [3.3]. Their values are unknown for $0 < t < T$. Let $\hat{H}(t) : t \geq 0$ be the *estimated* hedge fund price, where our estimation process is simply to take the expected value given all the information at time t in terms of the SBM's W_S and W_H. At time t $(0 < t < T)$ we know the exact value of $W_S(t)$ but know nothing about the value of $W_H(s)$ for $0 < s \leq t$.

The stochastic differential equation for $H(t)$ is:

$$dH = \mu_H H dt + \sigma_H H dW_H = (\nu_H + r) H dt + \sigma_H H dW_H \tag{3.34}$$

or alternatively

$$
\begin{aligned}
dH &= \mu_H H dt + \rho\sigma_H H dW_S + \sqrt{1-\rho^2}\sigma_H H dW_I \\
&= (\nu_H + r) H dt + \rho\sigma_H H dW_S + \sqrt{1-\rho^2}\sigma_H H dW_I
\end{aligned}
\tag{3.35}
$$

where W_I is defined to be orthogonal to W_S such that $W_H = \rho W_S + \sqrt{1-\rho^2} W_I$. The solution of this stochastic differential equation is readily available, and $H(t)$ can be written out as

$$
H(t) = H(0)\exp\left[\left(\mu_H - \frac{1}{2}\sigma_H^2\right)t + \rho\sigma_H W_S + \sqrt{1-\rho^2}\sigma_H W_I\right]
\tag{3.36}
$$

The estimated value of the hedge fund price at time t can be calculated as follows

$$
\begin{aligned}
\hat{H}(t) \quad &= \mathbb{E}\left[H(0)\exp\left[\left(\mu_H - \frac{1}{2}\sigma_H^2\right)t + \rho\sigma_H W_S(t) + \sqrt{1-\rho^2}\sigma_H W_I(t) \mid W_S(t)\right]\right] \\
&= H(0)\exp\left[\left(\mu_H - \frac{1}{2}\sigma_H^2\right)t + \rho\sigma_H W_S(t) + \frac{1}{2}(1-\rho^2)\sigma_H^2 t\right]
\end{aligned}
$$

Thus

$$
\hat{H}(t) = H(0)\exp\left[\left(\mu_H - \frac{1}{2}\rho^2\sigma_H^2\right)t + \rho\sigma_H W_S(t)\right]
\tag{3.37}
$$

Let $\hat{X}(t) : t \geq 0$ be the corresponding *estimated* total wealth process. Then

$$
\hat{h} = \frac{\hat{H}}{\hat{X}}
\tag{3.38}
$$

is the *estimated* fraction of total wealth in the hedge fund at time t with $\hat{h}(0) = h(0) = \alpha_H$ and $\beta_S(t)$ (β in brief) is the fraction of *estimated* total wealth invested in the stock at time t. β is a decision variable at all times $t \geq 0$, whereas h is a decision variable only at time $t = 0$. Correspondingly, an alternative notation for $h(0)$ is α_H. Finally, our planning horizon is T, i.e. at time $t = T$, the investor is allowed to adjust the investment in the hedge fund, and essentially faces the same problem again as at time $t = 0$.

The stochastic differential equations for our problem are

$$d\hat{H} = \mu_H \hat{H} dt + \rho \sigma_H \hat{H} dW_S = (\nu_H + r)\hat{H} dt + \rho \sigma_H \hat{H} dW_S \tag{3.39}$$

and

$$d\hat{X} = (r\hat{X} + \beta \nu_S \hat{X} + \nu_H \hat{H})dt + (\sigma_S \beta \hat{X} + \rho \sigma_H \hat{H})dW_S \tag{3.40}$$

where W_S is a Wiener process. We want to maximize $\mathbb{E}\left[\frac{1}{\gamma} X(T)^\gamma\right]$.

Now we proceed in a fashion similar to that in Kahl et al. [2003]. Since $\hat{X}$ and $\hat{H}$ form a joint Markov process, the value-to-go function $J(\hat{X}, \hat{H}, t)$ satisfies the following HJB equation

$$\max \begin{bmatrix} \frac{1}{2}(\sigma_S^2 \beta^2 \hat{X}^2 + 2\rho\sigma_S\sigma_H \beta \hat{X}\hat{H} + \rho^2 \sigma_H^2 \hat{H}^2) J_{\hat{X}\hat{X}} \\ +\frac{1}{2}\rho^2\sigma_H^2 \hat{H}^2 J_{\hat{H}\hat{H}} + (\rho\sigma_S\sigma_H \beta \hat{X}\hat{H} + \rho^2\sigma_H^2 \hat{H}^2) J_{\hat{X}\hat{H}} \\ +(r\hat{X} + \beta\nu_S\hat{X} + \nu_H\hat{H}) J_{\hat{X}} + (r + \nu_H)\hat{H} J_{\hat{H}} + J_t \end{bmatrix} = 0 \tag{3.41}$$

The first order optimality condition is $\beta\sigma_S^2\hat{X}^2 J_{\hat{X}\hat{X}} + \rho\sigma_S\sigma_H\hat{X}\hat{H}J_{\hat{X}\hat{X}} + \rho\sigma_S\sigma_H\hat{X}\hat{H}J_{\hat{X}\hat{H}} + \nu_S\hat{X}J_{\hat{X}} = 0$, thus

$$\beta^* = \frac{-\nu_S}{\sigma_S^2}\frac{J_{\hat{X}}}{\hat{X}J_{\hat{X}\hat{X}}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{\hat{H}}{\hat{X}}\frac{J_{\hat{X}\hat{H}}}{J_{\hat{X}\hat{X}}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{\hat{H}}{\hat{X}} \tag{3.42}$$

Note that this expression for β^* is similar to that for β^* in §3.3, except that X and H have been replaced by $\hat{X}$ and $\hat{H}$ respectively. It is also similar to the expression for ϕ^* in Appendix A of Kahl et al. [2003], even though they consider consumption. Again, it is incorrect to conclude that including consumption has no impact on the optimal portfolio strategy - in addition to the substitution of $\hat{X}$ and $\hat{H}$ for X and H respectively, the value functions J will be different in the two cases.

Unfortunately, for this problem there is no clear way of establishing whether the value function J is in either of the following forms:

$$J(\hat{X}, \hat{H}, t) = \frac{X^\gamma}{\gamma}V(\hat{h}, t)$$

or

$$J(\hat{X}, \hat{H}, t) = \frac{\hat{X}^\gamma}{\gamma}V(\hat{h}, t)$$

We are therefore unable to reduce the dimensionality of the HJB equation (3.41).

Now, the maximum in equation (3.41) holds when $\beta = \beta^*$. Thus

$$\begin{aligned}
&\tfrac{1}{2}(\sigma_S^2\beta^{*2}\hat{X}^2 + 2\rho\sigma_S\sigma_H\beta^*\hat{X}\hat{H} + \rho^2\sigma_H^2\hat{H}^2)J_{\hat{X}\hat{X}} \\
&+\tfrac{1}{2}\rho^2\sigma_H^2\hat{H}^2 J_{\hat{H}\hat{H}} + (\rho\sigma_S\sigma_H\beta^*\hat{X}\hat{H} + \rho^2\sigma_H^2\hat{H}^2)J_{\hat{X}\hat{H}} \\
&+(r\hat{X} + \beta^*\nu_S\hat{X} + \nu_H\hat{H})J_{\hat{X}} + (r + \nu_H)\hat{H}J_{\hat{H}} + J_t = 0
\end{aligned} \tag{3.43}$$

In this case, we have to solve for $J(\hat{X}, \hat{H}, t)$ by building a grid on a 2-dimensional state space. Ideally, we would like to use the Alternating Direction Implicit (ADI) scheme (due to Peaceman and Rachford) in a fashion similar to that described in Brandimarte [2006]. The basic idea is to introduce an intermediate time layer $t_{k-\frac{1}{2}}$ when stepping from t_k to t_{k-1} as part of the backward recursion. The approximation scheme for the partial derivatives is implicit with respect to one space dimension and explicit with respect to the other when going from t_k to $t_{k-\frac{1}{2}}$ and the roles get reversed when going from $t_{k-\frac{1}{2}}$ to t_{k-1}. The net effect is to solve the 2-D problem as a series of 1-D problems. This scheme allows us to bypass the difficulties inherent in implementing implicit schemes in higher dimensions.

However, the major difference in our problem with respect to the two-dimensional heat equation example demonstrated in Brandimarte is that here we have a mixed second-order derivative $J_{\hat{X}\hat{H}}$. Unfortunately, this can not be eliminated. A second, smaller issue is the following: the amount estimated to be in the hedge fund $\hat{H}$ can not be larger than the total estimated wealth $\hat{X}$. Thus, one of the boundaries for our problem is $\hat{H} = \hat{X}$, leading us to implement a rectangular grid on a two-dimensional feasible state space that is triangular in shape. To resolve this second issue, we replace $\hat{H}$ by $\hat{h}$ as a state variable. Clearly, we lose no information by making this change, but the feasible state space now becomes rectangular, with the bounds on both state variables becoming independent of each other.

Since

$$J_{\hat{H}} = \frac{J_{\hat{h}}}{\hat{X}} \tag{3.44}$$

$$J_{\hat{H}\hat{H}} = \frac{J_{\hat{h}\hat{h}}}{\hat{X}^2} \tag{3.45}$$

$$J_{\hat{X}\hat{H}} = \frac{J_{\hat{X}\hat{h}}}{\hat{X}} \tag{3.46}$$

equation (3.42) becomes

$$\beta^* = \frac{-\nu_S}{\sigma_S^2}\frac{J_{\hat{X}}}{\hat{X}J_{\hat{X}\hat{X}}} - \frac{\sigma_{SH}}{\sigma_S^2}\hat{h}\frac{J_{\hat{X}\hat{h}}}{\hat{X}J_{\hat{X}\hat{X}}} - \frac{\sigma_{SH}}{\sigma_S^2}\hat{h} \tag{3.47}$$

and equation (3.43) changes to

$$\begin{aligned}&\tfrac{1}{2}(\sigma_S^2\beta^{*2} + 2\rho\sigma_S\sigma_H\beta^*\hat{h} + \rho^2\sigma_H^2\hat{h}^2)\hat{X}^2 J_{\hat{X}\hat{X}}\\&+\tfrac{1}{2}\rho^2\sigma_H^2\hat{h}^2 J_{\hat{h}\hat{h}} + (\rho\sigma_S\sigma_H\beta^* + \rho^2\sigma_H^2\hat{h})\hat{X}\hat{h}J_{\hat{X}\hat{h}}\\&+(r + \beta^*\nu_S + \nu_H\hat{h})\hat{X}J_{\hat{X}} + (r + \nu_H)\hat{h}J_{\hat{h}} + J_t = 0\end{aligned} \tag{3.48}$$

which we write in brief as

$$\begin{aligned}&g_1(\hat{X},\hat{h},\beta^*)J_{\hat{X}\hat{X}} + g_2(\hat{X},\hat{h},\beta^*)J_{\hat{h}\hat{h}} + g_3(\hat{X},\hat{h},\beta^*)J_{\hat{X}\hat{h}}\\&+g_4(\hat{X},\hat{h},\beta^*)J_{\hat{X}} + g_5(\hat{X},\hat{h},\beta^*)J_{\hat{h}} + J_t = 0\end{aligned} \tag{3.49}$$

where

$$g_1(\hat{X},\hat{h},\beta^*) = \frac{1}{2}(\sigma_S^2\beta^{*2} + 2\rho\sigma_S\sigma_H\beta^*\hat{h} + \rho^2\sigma_H^2\hat{h}^2)\hat{X}^2 \tag{3.50}$$

$$g_2(\hat{X},\hat{h},\beta^*) = \frac{1}{2}\rho^2\sigma_H^2\hat{h}^2 \tag{3.51}$$

$$g_3(\hat{X},\hat{h},\beta^*) = (\rho\sigma_S\sigma_H\beta^* + \rho^2\sigma_H^2\hat{h})\hat{X}\hat{h} \tag{3.52}$$

$$g_4(\hat{X},\hat{h},\beta^*) = (r + \beta^*\nu_S + \nu_H\hat{h})\hat{X} \tag{3.53}$$

and

$$g_5(\hat{X},\hat{h},\beta^*) = (r + \nu_H)\hat{h} \tag{3.54}$$

We continue with setting up the ADI finite difference scheme to evaluate the function $J(\hat{X},\hat{h},t)$. Instead of plugging in the expression for β^* into equation (3.48), we follow Kahl et al. [2003] in linearizing the HJB equation by evaluating the values of β^* using the estimated values of the function $V(h,t)$ and its derivatives at the subsequent time step. As in Kahl et al. [2003], we ensure the accuracy of the scheme by using extremely small time steps.

In going from t_k to $t_{k-\frac{1}{2}}$, we use the following approximations for the derivatives

$$J_{\hat{X}\hat{X}}(\hat{X}_i,\hat{h}_j,t_k) = \frac{1}{\Delta\hat{X}^2}\left[\begin{array}{l} J(\hat{X}_{i+1},\hat{h}_j,t_{k-\frac{1}{2}}) - 2J(\hat{X}_i,\hat{h}_j,t_{k-\frac{1}{2}}) \\ +J(\hat{X}_{i-1},\hat{h}_j,t_{k-\frac{1}{2}}) \end{array}\right] \tag{3.55}$$

$$J_{\hat{h}\hat{h}}(\hat{X}_i,\hat{h}_j,t_k) = \frac{1}{\Delta\hat{h}^2}[J(\hat{X}_i,\hat{h}_{j+1},t_k) - 2J(\hat{X}_i,\hat{h}_j,t_k) + J(\hat{X}_i,\hat{h}_{j-1},t_k)] \tag{3.56}$$

$$J_{\hat{X}\hat{h}}(\hat{X}_i,\hat{h}_j,t_k) = \frac{1}{4\Delta\hat{X}\Delta\hat{h}}\left[\begin{array}{l} J(\hat{X}_{i+1},\hat{h}_{j+1},t_k) - J(\hat{X}_{i+1},\hat{h}_{j-1},t_k) \\ -J(\hat{X}_{i-1},\hat{h}_{j+1},t_k) + J(\hat{X}_{i-1},\hat{h}_{j-1},t_k) \end{array}\right] \tag{3.57}$$

$$J_{\hat{X}}(\hat{X}_i, \hat{h}_j, t_k) = \frac{1}{2\Delta\hat{X}}[J(\hat{X}_{i+1}, \hat{h}_j, t_{k-\frac{1}{2}}) - J(\hat{X}_{i-1}, \hat{h}_j, t_{k-\frac{1}{2}})] \tag{3.58}$$

$$J_{\hat{h}}(\hat{X}_i, \hat{h}_j, t_k) = \frac{1}{2\Delta\hat{h}}[J(\hat{X}_i, \hat{h}_{j+1}, t_k) - J(\hat{X}_i, \hat{h}_{j-1}, t_k)] \tag{3.59}$$

$$J_t(\hat{X}_i, \hat{h}_j, t_k) = \frac{1}{\Delta t}[J(\hat{X}_i, \hat{h}_j, t_k) - J(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}})] \tag{3.60}$$

Accordingly, for the half-step from t_k to $t_{k-\frac{1}{2}}$, equation (3.49) becomes

$$\begin{aligned}
&g_1(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{\Delta\hat{X}^2}\left[\ J(\hat{X}_{i+1}, \hat{h}_j, t_{k-\frac{1}{2}}) - 2J(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) + J(\hat{X}_{i-1}, \hat{h}_j, t_{k-\frac{1}{2}})\ \right] \\
&+g_2(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{\Delta\hat{h}^2}[J(\hat{X}_i, \hat{h}_{j+1}, t_k) - 2J(\hat{X}_i, \hat{h}_j, t_k) + J(\hat{X}_i, \hat{h}_{j-1}, t_k)] \\
&+g_3(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{4\Delta\hat{X}\Delta\hat{h}}\left[\begin{array}{l} J(\hat{X}_{i+1}, \hat{h}_{j+1}, t_k) - J(\hat{X}_{i+1}, \hat{h}_{j-1}, t_k) \\ -J(\hat{X}_{i-1}, \hat{h}_{j+1}, t_k) + J(\hat{X}_{i-1}, \hat{h}_{j-1}, t_k) \end{array}\right] \\
&+g_4(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{2\Delta\hat{X}}[J(\hat{X}_{i+1}, \hat{h}_j, t_{k-\frac{1}{2}}) - J(\hat{X}_{i-1}, \hat{h}_j, t_{k-\frac{1}{2}})] \\
&+g_5(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{2\Delta\hat{h}}[J(\hat{X}_i, \hat{h}_{j+1}, t_k) - J(\hat{X}_i, \hat{h}_{j-1}, t_k)] \\
&+\tfrac{1}{\Delta t}[J(\hat{X}_i, \hat{h}_j, t_k) - J(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}})] = 0
\end{aligned}$$

Rearranging the terms gives us

$$
\begin{aligned}
&\left[\tfrac{g_1(\hat{X},\hat{h},\beta^*)}{\Delta\hat{X}^2}+\tfrac{g_4(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{X}}\right]J(\hat{X}_{i+1},\hat{h}_j,t_{k-\frac{1}{2}})\\
&-\left[2\tfrac{g_1(\hat{X},\hat{h},\beta^*)}{\Delta\hat{X}^2}+\tfrac{1}{\Delta t}\right]J(\hat{X}_i,\hat{h}_j,t_{k-\frac{1}{2}})\\
&+\left[\tfrac{g_1(\hat{X},\hat{h},\beta^*)}{\Delta\hat{X}^2}-\tfrac{g_4(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{X}}\right]J(\hat{X}_{i-1},\hat{h}_j,t_{k-\frac{1}{2}})\\
&=-\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i+1},\hat{h}_{j+1},t_k)+\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i+1},\hat{h}_{j-1},t_k)\\
&-\left[\tfrac{g_2(\hat{X},\hat{h},\beta^*)}{\Delta\hat{h}^2}+\tfrac{g_5(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{h}}\right]J(\hat{X}_i,\hat{h}_{j+1},t_k)\\
&+\left[2\tfrac{g_2(\hat{X},\hat{h},\beta^*)}{\Delta\hat{h}^2}-\tfrac{1}{\Delta t}\right]J(\hat{X}_i,\hat{h}_j,t_k)\\
&+\left[-\tfrac{g_2(\hat{X},\hat{h},\beta^*)}{\Delta\hat{h}^2}+\tfrac{g_5(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{h}}\right]J(\hat{X}_i,\hat{h}_{j-1},t_k)\\
&+\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i-1},\hat{h}_{j+1},t_k)-\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i-1},\hat{h}_{j-1},t_k)
\end{aligned}
\tag{3.61}
$$

Note that all unknowns $J(\cdot,\cdot,t_{k-\frac{1}{2}})$ are to be evaluated at $\hat{h}_j$. This allows us to solve a system of equations for each $\hat{h}_j$ individually, effectively making this a series of one-dimensional problems.

In going from $t_{k-\frac{1}{2}}$ to t_{k-1}, we use the following approximations for the derivatives

$$
J_{\hat{X}\hat{X}}(\hat{X}_i,\hat{h}_j,t_{k-\frac{1}{2}})=\frac{1}{\Delta\hat{X}^2}\left[\begin{array}{l}J(\hat{X}_{i+1},\hat{h}_j,t_{k-\frac{1}{2}})-2J(\hat{X}_i,\hat{h}_j,t_{k-\frac{1}{2}})\\+J(\hat{X}_{i-1},\hat{h}_j,t_{k-\frac{1}{2}})\end{array}\right]\tag{3.62}
$$

$$
J_{\hat{h}\hat{h}}(\hat{X}_i,\hat{h}_j,t_{k-\frac{1}{2}})=\frac{1}{\Delta\hat{h}^2}\left[\begin{array}{l}J(\hat{X}_i,\hat{h}_{j+1},t_{k-1})-2J(\hat{X}_i,\hat{h}_j,t_{k-1})\\+J(\hat{X}_i,\hat{h}_{j-1},t_{k-1})\end{array}\right]\tag{3.63}
$$

$$J_{\hat{X}\hat{h}}(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) = \frac{1}{4\Delta\hat{X}\Delta\hat{h}} \begin{bmatrix} J(\hat{X}_{i+1}, \hat{h}_{j+1}, t_{k-\frac{1}{2}}) \\ -J(\hat{X}_{i+1}, \hat{h}_{j-1}, t_{k-\frac{1}{2}}) \\ -J(\hat{X}_{i-1}, \hat{h}_{j+1}, t_{k-\frac{1}{2}}) \\ +J(\hat{X}_{i-1}, \hat{h}_{j-1}, t_{k-\frac{1}{2}}) \end{bmatrix} \tag{3.64}$$

$$J_{\hat{X}}(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) = \frac{1}{2\Delta\hat{X}}[J(\hat{X}_{i+1}, \hat{h}_j, t_{k-\frac{1}{2}}) - J(\hat{X}_{i-1}, \hat{h}_j, t_{k-\frac{1}{2}})] \tag{3.65}$$

$$J_{\hat{h}}(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) = \frac{1}{2\Delta\hat{h}}[J(\hat{X}_i, \hat{h}_{j+1}, t_{k-1}) - J(\hat{X}_i, \hat{h}_{j-1}, t_{k-1})] \tag{3.66}$$

$$J_t(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) = \frac{1}{\Delta t}[J(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) - J(\hat{X}_i, \hat{h}_j, t_{k-1})] \tag{3.67}$$

Accordingly, for the half-step from $t_{k-\frac{1}{2}}$ to t_{k-1}, equation (3.49) becomes

$$\begin{aligned}
& g_1(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{\Delta\hat{X}^2} \begin{bmatrix} J(\hat{X}_{i+1}, \hat{h}_j, t_{k-\frac{1}{2}}) - 2J(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) \\ +J(\hat{X}_{i-1}, \hat{h}_j, t_{k-\frac{1}{2}}) \end{bmatrix} \\
& +g_2(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{\Delta\hat{h}^2} \begin{bmatrix} J(\hat{X}_i, \hat{h}_{j+1}, t_{k-1}) - 2J(\hat{X}_i, \hat{h}_j, t_{k-1}) \\ +J(\hat{X}_i, \hat{h}_{j-1}, t_{k-1}) \end{bmatrix} \\
& +g_3(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{4\Delta\hat{X}\Delta\hat{h}} \begin{bmatrix} J(\hat{X}_{i+1}, \hat{h}_{j+1}, t_{k-\frac{1}{2}}) - J(\hat{X}_{i+1}, \hat{h}_{j-1}, t_{k-\frac{1}{2}}) \\ -J(\hat{X}_{i-1}, \hat{h}_{j+1}, t_{k-\frac{1}{2}}) + J(\hat{X}_{i-1}, \hat{h}_{j-1}, t_{k-\frac{1}{2}}) \end{bmatrix} \\
& +g_4(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{2\Delta\hat{X}}[J(\hat{X}_{i+1}, \hat{h}_j, t_{k-\frac{1}{2}}) - J(\hat{X}_{i-1}, \hat{h}_j, t_{k-\frac{1}{2}})] \\
& +g_5(\hat{X}, \hat{h}, \beta^*)\tfrac{1}{2\Delta\hat{h}}[J(\hat{X}_i, \hat{h}_{j+1}, t_{k-1}) - J(\hat{X}_i, \hat{h}_{j-1}, t_{k-1})] \\
& +\tfrac{1}{\Delta t}[J(\hat{X}_i, \hat{h}_j, t_{k-\frac{1}{2}}) - J(\hat{X}_i, \hat{h}_j, t_{k-1})] = 0
\end{aligned}$$

Rearranging the terms gives us

$$\begin{aligned}
&\left[\tfrac{g_2(\hat{X},\hat{h},\beta^*)}{\Delta\hat{h}^2}+\tfrac{g_5(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{h}}\right]J(\hat{X}_i,\hat{h}_{j+1},t_{k-1})\\
&-\left[2\tfrac{g_2(\hat{X},\hat{h},\beta^*)}{\Delta\hat{h}^2}+\tfrac{1}{\Delta t}\right]J(\hat{X}_i,\hat{h}_j,t_{k-1})\\
&+\left[\tfrac{g_2(\hat{X},\hat{h},\beta^*)}{\Delta\hat{h}^2}-\tfrac{g_5(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{h}}\right]J(\hat{X}_i,\hat{h}_{j-1},t_{k-1})\\
&=-\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i+1},\hat{h}_{j+1},t_{k-\frac{1}{2}})\\
&-\left[\tfrac{g_1(\hat{X},\hat{h},\beta^*)}{\Delta\hat{X}^2}+\tfrac{g_4(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{X}}\right]J(\hat{X}_{i+1},\hat{h}_j,t_{k-\frac{1}{2}})\\
&+\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i+1},\hat{h}_{j-1},t_{k-\frac{1}{2}})\\
&+\left[2\tfrac{g_1(\hat{X},\hat{h},\beta^*)}{\Delta\hat{X}^2}-\tfrac{1}{\Delta t}\right]J(\hat{X}_i,\hat{h}_j,t_{k-\frac{1}{2}})\\
&+\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i-1},\hat{h}_{j+1},t_{k-\frac{1}{2}})\\
&+\left[-\tfrac{g_1(\hat{X},\hat{h},\beta^*)}{\Delta\hat{X}^2}+\tfrac{g_4(\hat{X},\hat{h},\beta^*)}{2\Delta\hat{X}}\right]J(\hat{X}_{i-1},\hat{h}_j,t_{k-\frac{1}{2}})\\
&-\tfrac{g_3(\hat{X},\hat{h},\beta^*)}{4\Delta\hat{X}\Delta\hat{h}}J(\hat{X}_{i-1},\hat{h}_{j-1},t_{k-\frac{1}{2}})
\end{aligned}\tag{3.68}$$

Again, note that all unknowns $J(\cdot,\cdot,t_{k-1})$ are to be evaluated at $\hat{X}_i$. This allows us to solve a system of equations for each $\hat{X}_i$ individually, effectively making this a series of one-dimensional problems.

We now need to determine the terminal and boundary conditions for our problem. At $t=T$, $\hat{X}_T-\hat{H}_T$ is the amount in the stock and the bond and we have

$$\begin{aligned}
J(\hat{X},\hat{h},T) &= \mathbb{E}\left[\frac{X_T^\gamma}{\gamma}\mid \hat{X}_T,\hat{h}_T\right]\\
&= \mathbb{E}_{W_H(T)}\left[\frac{1}{\gamma}\begin{pmatrix}\hat{X}_T-\hat{H}_T\\ +H_0\exp\left((\mu_H-\frac{1}{2}\sigma_H^2)T+\sigma_H W_H(T)\right)\end{pmatrix}^\gamma \mid \hat{X}_T,\hat{h}_T\right]
\end{aligned}$$

Using

$$H_0 = \hat{H}_T \exp\left(-\left(\mu_H - \frac{1}{2}\rho^2\sigma_H^2\right)T - \rho\sigma_H W_S(T)\right)$$

and

$$W_H(T) = \rho W_S(T) + \sqrt{1-\rho^2}W_I(T)$$

we get

$$J(\hat{X}, \hat{h}, T) = \mathbb{E}_{W_I(T)}\left[\frac{\hat{X}_T^\gamma}{\gamma}\begin{pmatrix} 1-\hat{h}_T \\ +\hat{h}_T \exp\begin{pmatrix} -\frac{1}{2}(1-\rho^2)\sigma_H^2 T \\ +\sqrt{1-\rho^2}\sigma_H W_I(T)\end{pmatrix}\end{pmatrix}^\gamma \mid \hat{X}_T, \hat{h}_T\right] \tag{3.69}$$

which must be evaluated numerically.

For $\hat{X} = 0$, the HJB equation (3.48) reduces to

$$J_t = 0 \tag{3.70}$$

Consequently, at $\hat{X}_1 = 0$, we can use

$$J(\hat{X}_1, \hat{h}_j, t_k) = J(\hat{X}_1, \hat{h}_j, t_{k+1}) \tag{3.71}$$

At the grid's upper bound for $\hat{X}$ (large but finite), we simply assign the utility of this wealth (at time T) to the value function. We are basically assuming that the wealth is already large enough that any growth will have minimal incremental impact on the utility derived at T.

$$J(\hat{X}_{gridmax}, \hat{h}, t) = \frac{\hat{X}^{\gamma}_{gridmax}}{\gamma}$$

So at $\hat{X}_{N_X} = \hat{X}_{gridmax}$, we have

$$J(\hat{X}_{N_X}, \hat{h}_j, t_k) = \frac{\hat{X}^{\gamma}_{N_X}}{\gamma}$$

For $\hat{h} = 0$, the entire portfolio is in the stock and the bond, and we are essentially back in the Merton world with two assets. Note that in this case, $\hat{X}(t) = X(t)$, i.e. the total wealth is known at all times. Thus, the value function for $\hat{h} = 0$ is

$$J(\hat{X}, 0, t) = \frac{\hat{X}^{\gamma}_t}{\gamma} \exp\left(\gamma(r + \beta^*_M \nu_S)(T - t) - \frac{1}{2}\gamma(1-\gamma)(\beta^*_M)^2 \sigma^2_S (T - t)\right)$$

where $\beta^*_M = \frac{\nu_S}{\sigma^2_S(1-\gamma)}$ Consequently, at $\hat{h}_1 = 0$, we can use

$$J(\hat{X}_i, \hat{h}_1, t_k) = \frac{\hat{X}^{\gamma}_i}{\gamma} \exp\left(\gamma(r + \beta^*_M \nu_S)(T - t_k) - \frac{1}{2}\gamma(1-\gamma)(\beta^*_M)^2 \sigma^2_S (T - t_k)\right)$$

Similarly, for $\hat{h} = 1$, the entire wealth of the investor is in the hedge fund and stays there until time T. The corresponding value function is

$$\begin{aligned} J(\hat{X}, 1, t) \quad &= \mathbb{E}\left[\frac{X^{\gamma}_T}{\gamma} \mid \hat{X}_t = \hat{H}_t\right] = \mathbb{E}\left[\frac{H^{\gamma}_T}{\gamma} \mid \hat{X}_t = \hat{H}_t\right] \\ &= \mathbb{E}_{W_H(T)}\left[\frac{1}{\gamma}\left(H_0 \exp\left((\mu_H - \frac{1}{2}\sigma^2_H)T + \sigma_H W_H(T)\right)\right)^{\gamma} \mid \hat{X}_t = \hat{H}_t\right] \end{aligned}$$

Now, if we have $\hat{X}_t = \hat{H}_t$ for some time t, then equation (3.37) gives us

$$H_0 = \hat{X}_t \exp\left(-\left(\mu_H - \frac{1}{2}\rho^2\sigma_H^2\right)t - \rho\sigma_H W_S(t)\right)$$

Then, using $\rho W_S = W_H - \sqrt{1-\rho^2}W_I$ and the independence of W_I and W_H over non-overlapping time intervals, we get

$$\begin{aligned}
J(\hat{X},1,t) \quad &= \mathbb{E}_{W_H(T)}\left[\frac{1}{\gamma}\begin{pmatrix} \hat{X}_t \exp\left(-\left(\mu_H - \frac{1}{2}\rho^2\sigma_H^2\right)t - \rho\sigma_H W_S(t)\right) \\ \exp\left((\mu_H - \frac{1}{2}\sigma_H^2)T + \sigma_H W_H(T)\right) \end{pmatrix}^{\gamma} \mid \hat{X}_t = \hat{H}_t\right] \\
&= \frac{\hat{X}_t^\gamma}{\gamma}\mathbb{E}_{W_H(T)}\left[\exp\begin{pmatrix} \gamma\mu_H(T-t) - \frac{1}{2}\gamma\sigma_H^2(T-\rho^2 t) \\ -\gamma\rho\sigma_H W_S(t) + \gamma\sigma_H W_H(T) \end{pmatrix}\right] \\
&= \frac{\hat{X}_t^\gamma}{\gamma}\mathbb{E}_{W_I(t),W_H(T)-W_H(t)}\left[\exp\begin{pmatrix} \gamma\mu_H(T-t) - \frac{1}{2}\gamma\sigma_H^2(T-\rho^2 t) \\ +\gamma\sqrt{1-\rho^2}\sigma_H W_I(t) \\ +\gamma\sigma_H(W_H(T) - W_H(t)) \end{pmatrix}\right] \\
&= \frac{\hat{X}_t^\gamma}{\gamma}\exp\left(\gamma\mu_H(T-t) - \frac{1}{2}\gamma\sigma_H^2(T-\rho^2 t)\right) \\
&\quad \mathbb{E}_{W_I(t)}\left[\exp\left(\gamma\sqrt{1-\rho^2}\sigma_H W_I(t)\right)\right] \\
&\quad \mathbb{E}_{W_H(T)-W_H(t)}\left[\exp\left(\gamma\sigma_H(W_H(T)-W_H(t))\right)\right]
\end{aligned}$$

Thus

$$J(\hat{X},1,t) = \frac{\hat{X}_t^\gamma}{\gamma}\exp\left(\gamma\mu_H(T-t) - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2(T-\rho^2 t)\right)$$

So at $h_{N_h} = 1$, we can use

$$J(\hat{X}_i, \hat{h}_{N_h}, t_k) = \frac{\hat{X}_i^\gamma}{\gamma}\exp\left(\gamma\mu_H(T-t_k) - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2(T-\rho^2 t_k)\right) \tag{3.72}$$

3.4.1. A Framework for Calculating the Hedge Fund Information Premium

In §3.3 and the initial part of §3.4 above, we have the methodology to compute the allocations to the hedge fund that maximize the utility of terminal wealth under two scenarios - one in which the hedge fund price information is readily available, and the other in which the hedge fund price information is only available just before the decision epochs. Armed with this, we can use the framework outlined in §**??** to obtain the lockup premiums $p_l^+(A)$ and $p_l^+(R)$ under the above two scenarios. A and R denote *available* and *restricted* respectively, in reference to the hedge fund share price information.

Then, the *information premium* for the hedge fund is given by

$$p_i^+ = p_l^+(R) - p_l^+(A) \tag{3.73}$$

We expect that $p_l^+(R) \geq p_l^+(A)$ and thus $p_i^+ \geq 0$.

An alternative approach is to compute the *information penalty* which we define as

$$p_i^- = p_l^-(A) - p_l^-(R) \tag{3.74}$$

where $p_l^-(A)$ and $p_l^-(R)$ are the lockup penalties associated with the *information available* and *restricted information* scenarios respectively. We expect that $p_l^-(A) \leq p_l^-(R)$ and thus $p_i^- \leq 0$.

3.5. Problem III: Maximizing the Utility of Consumption with Hedge Fund Price Information Available

In this section, we consider the problem of maximizing the utility of consumption. Initial wealth can be invested in a bond, a stock or a hedge fund, with the hedge fund requiring a lockup period of T. Hedge fund share price information is assumed to be continuously available.

Assume the notation as laid out in §3.3. The investor now consumes at a rate C, which is a decision made at time 0. $c(t) \equiv \frac{C}{X(t)}$ (c in brief) is correspondingly the consumption rate as a fraction of the total wealth. Note that while C is fixed over the time period under analysis, $c(t)$ is a function of time due to the changes in wealth.

As before, our planning horizon is T, i.e. at T, the investor is allowed to rebalance the investment in the hedge fund, and essentially faces the same problem again as at time $t = 0$.

The stochastic differential equations for our problem are

$$dH = \mu_H H dt + \sigma_H H dW_H = (\nu_H + r) H dt + \sigma_H H dW_H \tag{3.75}$$

and

$$dX = (rX + \beta \nu_S X + \nu_H H - C)dt + \sigma_S \beta X dW_S + \sigma_H H dW_H \tag{3.76}$$

where W_H and W_S are Wiener processes with $dW_H dW_S = \rho dt$. We want to maximize

$$\begin{aligned} & J(X(0), H^*(0), C^*(0), 0) \\ & \quad = \mathbb{E}\left[\int_0^\infty e^{-\delta t} \tfrac{1}{\gamma} C(t)^\gamma dt\right] \\ & \quad = \int_0^T e^{-\delta t} \tfrac{1}{\gamma} (C^*(0))^\gamma dt + e^{-\delta T} \mathbb{E}\left[J(X(T), H^*(T), C^*(T), T)\right] \end{aligned} \tag{3.77}$$

where δ is the discount rate for utility and $H^*(0), C^*(0)$ and $H^*(T), C^*(T)$ are determined according to our resulting optimal policy.

Now we proceed in a fashion similar to that in Kahl et al. [2003]. Since X and H form a joint Markov process, the value-to-go function $J(X,H,t)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\max \left[\begin{array}{l} \frac{1}{2}(\sigma_S^2\beta^2X^2 + 2\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)J_{XX} \\ +\frac{1}{2}\sigma_H^2H^2J_{HH} + (\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)J_{XH} \\ +(rX + \beta\nu_S X + \nu_H H - C)J_X \\ +(r+\nu_H)HJ_H + J_t + e^{-\delta t}\frac{1}{\gamma}C^\gamma \end{array} \right] = 0 \tag{3.78}$$

The first order optimality conditions for β is

$$\beta\sigma_S^2X^2J_{XX} + \rho\sigma_S\sigma_H XHJ_{XX} + \rho\sigma_S\sigma_H XHJ_{XH} + \nu_S XJ_X = 0$$

Thus

$$\beta^* = \frac{-\nu_S}{\sigma_S^2}\frac{J_X}{XJ_{XX}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{H}{X}\frac{J_{XH}}{J_{XX}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{H}{X} \tag{3.79}$$

Note that this expression for β^* is the same as that for β^* in §3.3 as well as that for ϕ^* in Appendix A of Kahl et al. [2003].

We conjecture that value-to-go function is of the form

$$J(X,H,C,t) = \frac{X^\gamma}{\gamma}V(h,c,t) \tag{3.80}$$

We can verify the validity of this conjecture by rewriting the HJB equation in terms of this function $V(h,c,t)$ and checking that the boundary and terminal conditions are free of X.

We first compute the partial derivatives J_X, J_{XX} and J_{XH} in terms of h, c and the function $V(h,c,t)$ and its partial derivatives V_h, V_{hh}, V_c and V_{cc} as

$$J_X = \frac{X^{\gamma-1}}{\gamma}\left(-hV_h - cV_c + \gamma V\right) \tag{3.81}$$

$$J_{XX} = \frac{X^{\gamma-2}}{\gamma}\left(h^2V_{hh} + c^2V_{cc} + 2(1-\gamma)hV_h + 2(1-\gamma)cV_c - \gamma(1-\gamma)V\right) \tag{3.82}$$

$$J_{XH} = \frac{X^{\gamma-2}}{\gamma}\left(-hV_{hh} + V_{hc} - (1-\gamma)V_h\right) \tag{3.83}$$

We can now write β^* in terms of these quantities

$$\beta^* = \frac{\frac{\sigma_{SH}}{\sigma_S^2}h^2V_{hh} - \frac{\sigma_{SH}}{\sigma_S^2}hV_{hc} + \left((1-\gamma)\frac{\sigma_{SH}}{\sigma_S^2} + \frac{\nu_S}{\sigma_S^2}\right)hV_h + \frac{\nu_S}{\sigma_S^2}cV_c - \frac{\nu_S}{\sigma_S^2}\gamma V}{h^2V_{hh} + c^2V_{cc} + 2(1-\gamma)hV_h + 2(1-\gamma)cV_c - \gamma(1-\gamma)V} - \frac{\sigma_{SH}}{\sigma_S^2}h \tag{3.84}$$

In this case, comparing this formula with that in Kahl et al. [2003], we see that the formulae are significantly different. This is because in Kahl et al. [2003], consumption is a decision at all points in time. In our case however, consumption is a decision only at time 0 and later shows up through the state variable c.

Also, assuming that we do not allow borrowing, it is easy to show that given values for h and c, the maximum allowed value for β at any time t is $\beta_{max}(t) = 1 - h - \frac{c}{r}(1 - e^{-r(T-t)}$. Depending on the values for the problem parameters, it might be necessary to enforce this restriction. However, we will concentrate on cases where the stock is not an overly attractive investment and its allocation is well below this bound.

Our next step is to convert the HJB into a form that uses h and the partial derivatives of the function $V(h,c,t)$. To do this we also need the J_H, J_{HH} and J_t in terms of $V(h,c,t)$ and its partial derivatives

$$J_H = \frac{X^{\gamma-1}}{\gamma} V_h \tag{3.85}$$

$$J_{HH} = \frac{X^{\gamma-2}}{\gamma} V_{hh} \tag{3.86}$$

$$J_t = \frac{X^{\gamma}}{\gamma} V_t \tag{3.87}$$

Substituting the partial derivatives of $J(X,H,C,t)$ into the HJB gives us

$$\max \left[\begin{array}{l} \frac{1}{2}(\sigma_S^2\beta^2X^2 + 2\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)\frac{X^{\gamma-2}}{\gamma} \\ \left(\begin{array}{l} h^2V_{hh} + c^2V_{cc} + 2(1-\gamma)hV_h \\ +2(1-\gamma)cV_c - \gamma(1-\gamma)V \end{array} \right) + \frac{1}{2}\sigma_H^2H^2\frac{X^{\gamma-2}}{\gamma}V_{hh} \\ +(\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)\frac{X^{\gamma-2}}{\gamma}\left(-hV_{hh} + V_{hc} - (1-\gamma)V_h\right) \\ +(rX + \beta\nu_S X + \nu_H H - C)\frac{X^{\gamma-1}}{\gamma}\left(-hV_h - cV_c + \gamma V\right) \\ +(r+\nu_H)H\frac{X^{\gamma-1}}{\gamma}V_h + \frac{X^\gamma}{\gamma}V_t + e^{-\delta t}\frac{1}{\gamma}C^\gamma \end{array} \right] = 0$$

Dividing through by $\frac{X^\gamma}{\gamma}$ and rearranging the terms gives us the HJB equation in terms of the function $V(h,c,t)$

$$\max\left[\begin{array}{l}
\left(\begin{array}{l}\frac{1}{2}(\beta^2\sigma_S^2h^2+2\beta\sigma_{SH}h^3+\sigma_H^2h^4)\\ +\frac{1}{2}\sigma_H^2h^2-(\beta\sigma_{SH}h^2+\sigma_H^2h^3)\end{array}\right)V_{hh}\\
+(\beta\sigma_{SH}h+\sigma_H^2h^2)V_{hc}\\
+\frac{1}{2}(\beta^2\sigma_S^2c^2+2\beta\sigma_{SH}hc^2+\sigma_H^2h^2c^2)V_{cc}\\
+\left(\begin{array}{l}(1-\gamma)(\beta^2\sigma_S^2h+2\beta\sigma_{SH}h^2+\sigma_H^2h^3)\\ -(rh+\beta\nu_Sh+\nu_Hh^2-hc)\\ -(1-\gamma)(\beta\sigma_{SH}h+\sigma_H^2h^2)+(rh+\nu_Hh)\end{array}\right)V_h\\
+\left(\begin{array}{l}(1-\gamma)(\beta^2\sigma_S^2c+2\sigma_{SH}\beta hc+\sigma_H^2h^2c)\\ -(rc+\beta\nu_Sc+\nu_Hhc-c^2)\end{array}\right)V_c\\
+\left(\begin{array}{l}-\gamma(1-\gamma)\frac{1}{2}(\beta^2\sigma_S^2+2\beta\sigma_{SH}h+\sigma_H^2h^2)\\ +\gamma(r+\beta\nu_S+\nu_Hh-c)\end{array}\right)V\\
+V_t+e^{-\delta t}c^\gamma
\end{array}\right]=0 \tag{3.88}$$

The maximum in equation (3.88) holds when $\beta = \beta^*$ and therefore

$$\begin{aligned}
&\left(\begin{array}{l} \frac{1}{2}((\beta^*)^2\sigma_S^2 h^2 + 2\beta^*\sigma_{SH}h^3 + \sigma_H^2 h^4) \\ +\frac{1}{2}\sigma_H^2 h^2 - (\beta^*\sigma_{SH}h^2 + \sigma_H^2 h^3) \end{array}\right) V_{hh} \\
&+(\beta^*\sigma_{SH}h + \sigma_H^2 h^2)V_{hc} \\
&+\tfrac{1}{2}((\beta^*)^2\sigma_S^2 c^2 + 2\beta^*\sigma_{SH}hc^2 + \sigma_H^2 h^2 c^2)V_{cc} \\
&+\left(\begin{array}{l} (1-\gamma)((\beta^*)^2\sigma_S^2 h + 2\beta^*\sigma_{SH}h^2 + \sigma_H^2 h^3) \\ -(rh + \beta^*\nu_S h + \nu_H h^2 - hc) \\ -(1-\gamma)(\beta^*\sigma_{SH}h + \sigma_H^2 h^2) + (rh + \nu_H h) \end{array}\right) V_h \\
&+\left(\begin{array}{l} (1-\gamma)((\beta^*)^2\sigma_S^2 c + 2\sigma_{SH}\beta^* hc + \sigma_H^2 h^2 c) \\ -(rc + \beta^*\nu_S c + \nu_H hc - c^2) \end{array}\right) V_c \\
&+\left(\begin{array}{l} -\gamma(1-\gamma)\frac{1}{2}((\beta^*)^2\sigma_S^2 + 2\beta^*\sigma_{SH}h + \sigma_H^2 h^2) \\ +\gamma(r + \beta^*\nu_S + \nu_H h - c) \end{array}\right) V \\
&+V_t + e^{-\delta t}c^\gamma = 0
\end{aligned} \tag{3.89}$$

As in the case of Problem II (§3.4), we have to solve for the value function by building a grid on a 2-dimensional state space. Again, we would like to use the Alternating Direction Implicit (ADI) scheme but we are faced with the same issues as before. First, we have a mixed second-order derivative V_{hc} which can not be eliminated. The second issue is the following: the maximum allowed consumption rate depends on the amount invested in the hedge fund. We enforce this condition at time 0, and then assume that an appropriate amount is invested in the bond to meet the consumption requirements. This ties in with the earlier condition on the maximum allowed value of β. Once the consumption rate is fixed at C, to guarantee this consumption stream we require to put $\frac{C}{r}(1 - e^{-rT})$ in the bond. Clearly this amount increases with an increase in C. Then, given an initial investment $H(0)$ in the hedge fund, the maximum allowed consumption rate C_{max} would satisfy the following equation

$$X(0) = \frac{C_{max}}{r}(1 - e^{-rT}) + H(0) \tag{3.90}$$

Dividing by $X(0)$, we get an expression for the maximum initial value for c_{max}, which is the consumption rate expressed as a fraction of the total wealth

$$c_{max} = \frac{r}{1 - e^{-rT}}(1 - h) \tag{3.91}$$

Thus, one of the boundaries for our problem is $c = \frac{r}{1-e^{-rT}}(1-h)$, leading us to implement a rectangular grid on a two-dimensional feasible state space that is triangular in shape. To resolve this issue, we replace c by d as a state variable, where

$$d = \frac{c}{1 - h} \tag{3.92}$$

d can be interpreted as the consumption rate as a fraction of the *liquid* wealth. Then

$$d_{max} = \frac{r}{1 - e^{-rT}} \tag{3.93}$$

Clearly, we lose no information by making this change, but the feasible state space now becomes rectangular, with the bounds on both state variables becoming independent of each other. Since

$$V_c = \frac{1}{1 - h}V_d \tag{3.94}$$

$$V_{cc} = \frac{1}{(1-h)^2} V_{dd} \tag{3.95}$$

$$V_{hc} = \frac{1}{1-h} V_{hd} \tag{3.96}$$

equation (3.84) becomes

$$\begin{aligned}\beta^* = \; & \frac{\frac{\sigma_{SH}}{\sigma_S^2} h^2 V_{hh} - \frac{\sigma_{SH}}{\sigma_S^2} \frac{h}{1-h} V_{hd} + \left((1-\gamma) \frac{\sigma_{SH}}{\sigma_S^2} + \frac{\nu_S}{\sigma_S^2} \right) h V_h + \frac{\nu_S}{\sigma_S^2} d V_d - \frac{\nu_S}{\sigma_S^2} \gamma V}{h^2 V_{hh} + d^2 V_{dd} + 2(1-\gamma) h V_h + 2(1-\gamma) d V_d - \gamma(1-\gamma) V} \\ & - \frac{\sigma_{SH}}{\sigma_S^2} h\end{aligned} \tag{3.97}$$

and equation (3.89) changes to

$$\begin{aligned}& \begin{pmatrix} \frac{1}{2}((\beta^*)^2 \sigma_S^2 h^2 + 2\beta^* \sigma_{SH} h^3 + \sigma_H^2 h^4) \\ + \frac{1}{2} \sigma_H^2 h^2 - (\beta^* \sigma_{SH} h^2 + \sigma_H^2 h^3) \end{pmatrix} V_{hh} + (\beta^* \sigma_{SH} + \sigma_H^2 h) \tfrac{h}{1-h} V_{hd} \\ & + \tfrac{1}{2}((\beta^*)^2 \sigma_S^2 d^2 + 2\beta^* \sigma_{SH} h d^2 + \sigma_H^2 h^2 d^2) V_{dd} \\ & + \begin{pmatrix} (1-\gamma)((\beta^*)^2 \sigma_S^2 h + 2\beta^* \sigma_{SH} h^2 + \sigma_H^2 h^3) \\ -(rh + \beta^* \nu_S h + \nu_H h^2 - h(1-h)d) \\ -(1-\gamma)(\beta^* \sigma_{SH} h + \sigma_H^2 h^2) + (rh + \nu_H h) \end{pmatrix} V_h \\ & + \begin{pmatrix} (1-\gamma)((\beta^*)^2 \sigma_S^2 d + 2\sigma_{SH} \beta^* h d + \sigma_H^2 h^2 d) \\ -(rd + \beta^* \nu_S d + \nu_H h d - (1-h) d^2) \end{pmatrix} V_d \\ & + \begin{pmatrix} -\gamma(1-\gamma) \frac{1}{2}((\beta^*)^2 \sigma_S^2 + 2\beta^* \sigma_{SH} h + \sigma_H^2 h^2) \\ + \gamma(r + \beta^* \nu_S + \nu_H h - (1-h)d) \end{pmatrix} V \\ & + V_t + e^{-\delta t}(1-h)^\gamma d^\gamma = 0\end{aligned} \tag{3.98}$$

or in brief as

$$f_1(h,d,\beta^*)V_{hh} + f_2(h,d,\beta^*)V_{hd} + f_3(h,d,\beta^*)V_{dd} + f_4(h,d,\beta^*)V_h \\ +f_5(h,d,\beta^*)V_d + V_t + f_6(h,d,\beta^*)V + f_7(h,d) = 0 \tag{3.99}$$

where

$$\begin{aligned} f_1(h,d,\beta^*) = \; & \tfrac{1}{2}((\beta^*)^2\sigma_S^2 h^2 + 2\beta^*\sigma_{SH}h^3 + \sigma_H^2 h^4) \\ & +\tfrac{1}{2}\sigma_H^2 h^2 - (\beta^*\sigma_{SH}h^2 + \sigma_H^2 h^3) \end{aligned} \tag{3.100}$$

$$f_2(h,d,\beta^*) = (\beta^*\sigma_{SH} + \sigma_H^2 h)\frac{h}{1-h} \tag{3.101}$$

$$f_3(h,d,\beta^*) = \frac{1}{2}((\beta^*)^2\sigma_S^2 d^2 + 2\beta^*\sigma_{SH}hd^2 + \sigma_H^2 h^2 d^2) \tag{3.102}$$

$$\begin{aligned} f_4(h,d,\beta^*) = \; & (1-\gamma)((\beta^*)^2\sigma_S^2 h + 2\beta^*\sigma_{SH}h^2 + \sigma_H^2 h^3) \\ & -(rh + \beta^*\nu_S h + \nu_H h^2 - h(1-h)d) \\ & -(1-\gamma)(\beta^*\sigma_{SH}h + \sigma_H^2 h^2) + (rh + \nu_H h) \end{aligned} \tag{3.103}$$

$$\begin{aligned} f_5(h,d,\beta^*) = \; & (1-\gamma)((\beta^*)^2\sigma_S^2 d + 2\sigma_{SH}\beta^* hd + \sigma_H^2 h^2 d) \\ & -(rd + \beta^*\nu_S d + \nu_H hd - (1-h)d^2) \end{aligned} \tag{3.104}$$

$$\begin{aligned} f_6(h,d,\beta^*) = \; & -\gamma(1-\gamma)\tfrac{1}{2}((\beta^*)^2\sigma_S^2 + 2\beta^*\sigma_{SH}h + \sigma_H^2 h^2) \\ & +\gamma(r + \beta^*\nu_S + \nu_H h - (1-h)d) \end{aligned} \tag{3.105}$$

and

$$f_7(h,d) = e^{-\delta t}(1-h)^\gamma d^\gamma \tag{3.106}$$

We continue with setting up the ADI finite difference scheme to evaluate the function $V(h,d,t)$. Instead of plugging in the expression for β^* into equation (3.89), we follow Kahl et al. [2003] in linearizing the HJB equation by evaluating the values of β^* using the estimated values of the function $V(h,d,t)$ and its derivatives at the subsequent time step. As in Kahl et al. [2003], we ensure the accuracy of the scheme by using extremely small time steps.

In going from t_k to $t_{k-\frac{1}{2}}$, we use the following approximations for the derivatives

$$V_{hh}(h_i,d_j,t_k) = \frac{1}{\Delta h^2}\begin{bmatrix} V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - 2V(h_i,d_j,t_{k-\frac{1}{2}}) \\ +V(h_{i-1},d_j,t_{k-\frac{1}{2}}) \end{bmatrix} \tag{3.107}$$

$$V_{hd}(h_i,d_j,t_k) = \frac{1}{4\Delta h\Delta d}\begin{bmatrix} V(h_{i+1},d_{j+1},t_k) - V(h_{i+1},d_{j-1},t_k) \\ -V(h_{i-1},d_{j+1},t_k) + V(h_{i-1},d_{j-1},t_k) \end{bmatrix} \tag{3.108}$$

$$V_{dd}(h_i,d_j,t_k) = \frac{1}{\Delta d^2}[V(h_i,d_{j+1},t_k) - 2V(h_i,d_j,t_k) + V(h_i,d_{j-1},t_k)] \tag{3.109}$$

$$V_h(h_i,d_j,t_k) = \frac{1}{2\Delta h}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \tag{3.110}$$

$$V_d(h_i,d_j,t_k) = \frac{1}{2\Delta d}[V(h_i,d_{j+1},t_k) - V(h_i,d_{j-1},t_k)] \tag{3.111}$$

$$V_t(h_i,d_j,t_k) = \frac{1}{\Delta t}[V(h_i,d_j,t_k) - V(h_i,d_j,t_{k-\frac{1}{2}})] \tag{3.112}$$

Accordingly, for the half-step from t_k to $t_{k-\frac{1}{2}}$, equation (3.99) becomes

$$
\begin{aligned}
& f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - 2V(h_i,d_j,t_{k-\frac{1}{2}}) + V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
& +f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\left[\begin{array}{l} V(h_{i+1},d_{j+1},t_k) - V(h_{i+1},d_{j-1},t_k) \\ -V(h_{i-1},d_{j+1},t_k) + V(h_{i-1},d_{j-1},t_k) \end{array}\right] \\
& +f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2}[V(h_i,d_{j+1},t_k) - 2V(h_i,d_j,t_k) + V(h_i,d_{j-1},t_k)] \\
& +f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
& +f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}[V(h_i,d_{j+1},t_k) - V(h_i,d_{j-1},t_k)] \\
& +\tfrac{1}{\Delta t}[V(h_i,d_j,t_k) - V(h_i,d_j,t_{k-\frac{1}{2}})] \\
& +f_6(h,d,\beta^*)V(h_i,d_j,t_k) + f_7(h,d) = 0
\end{aligned}
$$

Rearranging the terms gives us

$$
\begin{aligned}
& \left[f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2} + f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}\right] V(h_{i+1},d_j,t_{k-\frac{1}{2}}) \\
& - \left[2f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2} + \tfrac{1}{\Delta t}\right] V(h_i,d_j,t_{k-\frac{1}{2}}) \\
& + \left[f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2} - f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}\right] V(h_{i-1},d_j,t_{k-\frac{1}{2}}) \\
& = - \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i+1},d_{j+1},t_k) \\
& + \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i+1},d_{j-1},t_k) \\
& - \left[f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2} + f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}\right] V(h_i,d_{j+1},t_k) \\
& + \left[2f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2} - \tfrac{1}{\Delta t} - f_6(h,d,\beta^*)\right] V(h_i,d_j,t_k) \\
& - \left[f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2} - f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}\right] V(h_i,d_{j-1},t_k) \\
& + \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i-1},d_{j+1},t_k) \\
& - \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i-1},d_{j-1},t_k) - f_7(h,d)
\end{aligned}
\tag{3.113}
$$

Note that all unknowns $(V(\cdot,\cdot,t_{k-\frac{1}{2}})$ are to be evaluated at d_j. This allows us to solve a system of equations for each d_j individually, effectively making this a series of one-dimensional problems.

In going from $t_{k-\frac{1}{2}}$ to t_{k-1}, we use the following approximations for the derivatives

$$V_{hh}(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{\Delta h^2}\begin{bmatrix} V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - 2V(h_i,d_j,t_{k-\frac{1}{2}}) \\ +V(h_{i-1},d_j,t_{k-\frac{1}{2}}) \end{bmatrix} \tag{3.114}$$

$$V_{hd}(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{4\Delta h\Delta d}\begin{bmatrix} V(h_{i+1},d_{j+1},t_{k-\frac{1}{2}}) \\ -V(h_{i+1},d_{j-1},t_{k-\frac{1}{2}}) \\ -V(h_{i-1},d_{j+1},t_{k-\frac{1}{2}}) \\ +V(h_{i-1},d_{j-1},t_{k-\frac{1}{2}}) \end{bmatrix} \tag{3.115}$$

$$V_{dd}(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{\Delta d^2}\begin{bmatrix} V(h_i,d_{j+1},t_{k-1}) - 2V(h_i,d_j,t_{k-1}) \\ +V(h_i,d_{j-1},t_{k-1}) \end{bmatrix} \tag{3.116}$$

$$V_h(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{2\Delta h}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \tag{3.117}$$

$$V_d(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{2\Delta d}[V(h_i,d_{j+1},t_{k-1}) - V(h_i,d_{j-1},t_{k-1})] \tag{3.118}$$

$$V_t(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{\Delta t}[V(h_i,d_j,t_{k-\frac{1}{2}}) - V(h_i,d_j,t_{k-1})] \tag{3.119}$$

Accordingly, for the half-step from $t_{k-\frac{1}{2}}$ to t_{k-1}, equation (3.99) becomes

$$
\begin{aligned}
& f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - 2V(h_i,d_j,t_{k-\frac{1}{2}}) + V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
& +f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\left[\begin{array}{c} V(h_{i+1},d_{j+1},t_{k-\frac{1}{2}}) - V(h_{i+1},d_{j-1},t_{k-\frac{1}{2}}) \\ -V(h_{i-1},d_{j+1},t_{k-\frac{1}{2}}) + V(h_{i-1},d_{j-1},t_{k-\frac{1}{2}}) \end{array}\right] \\
& +f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2}[V(h_i,d_{j+1},t_{k-1}) - 2V(h_i,d_j,t_{k-1}) + V(h_i,d_{j-1},t_{k-1})] \\
& +f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
& +f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}[V(h_i,d_{j+1},t_{k-1}) - V(h_i,d_{j-1},t_{k-1})] \\
& +\tfrac{1}{\Delta t}[V(h_i,d_j,t_{k-\frac{1}{2}}) - V(h_i,d_j,t_{k-1})] \\
& +f_6(h,d,\beta^*)V(h_i,d_j,t_{k-\frac{1}{2}}) + f_7(h,d) = 0
\end{aligned}
$$

Rearranging the terms gives us

$$
\begin{aligned}
& \left[\tfrac{f_3(h,d,\beta^*)}{\Delta d^2} + \tfrac{f_5(h,d,\beta^*)}{2\Delta d}\right] V(h_i,d_{j+1},t_{k-1}) \\
& -\left[2\tfrac{f_3(h,d,\beta^*)}{\Delta d^2} + \tfrac{1}{\Delta t}\right] V(h_i,d_j,t_{k-1}) \\
& +\left[\tfrac{f_3(h,d,\beta^*)}{\Delta d^2} - \tfrac{f_5(h,d,\beta^*)}{2\Delta d}\right] V(h_i,d_{j-1},t_{k-1}) \\
& = -\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d} V(h_{i+1},d_{j+1},t_{k-\frac{1}{2}}) \\
& -\left[\tfrac{f_1(h,d,\beta^*)}{\Delta h^2} + \tfrac{f_4(h,d,\beta^*)}{2\Delta h}\right] V(h_{i+1},d_j,t_{k-\frac{1}{2}}) \\
& +\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d} V(h_{i+1},d_{j-1},t_{k-\frac{1}{2}}) \\
& +\left[2\tfrac{f_1(h,d,\beta^*)}{\Delta h^2} - \tfrac{1}{\Delta t} - f_6(h,d,\beta^*)\right] V(h_i,d_j,t_{k-\frac{1}{2}}) \\
& +\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d} V(h_{i-1},d_{j+1},t_{k-\frac{1}{2}}) \\
& -\left[\tfrac{f_1(h,d,\beta^*)}{\Delta h^2} - \tfrac{f_4(h,d,\beta^*)}{2\Delta h}\right] V(h_{i-1},d_j,t_{k-\frac{1}{2}}) \\
& -\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d} V(h_{i-1},d_{j-1},t_{k-\frac{1}{2}}) - f_7(h,d)
\end{aligned}
\tag{3.120}
$$

Again, note that all unknowns ($V(\cdot,\cdot,t_{k-1})$ are to be evaluated at h_i. This allows us to solve a system of equations for each h_i individually, effectively making this a series of one-dimensional problems.

We now need to determine the terminal and boundary conditions for our problem. At $t = T$, we are able to rebalance our entire portfolio. Essentially, we face the same problem as at time 0, except that our wealth level is now $X(T)$ instead of $X(0)$. So we must have

$$J(X, H^*(X,T), C^*(X,T), T) = J^*(X, H^*(X,0), C^*(X,0), 0) \tag{3.121}$$

where $H^*(X,0)$ denotes the optimal value of H if the wealth at time 0 were to be X. Note that this boundary condition is different from that in Kahl et al. [2003] because of the problem context. So at $t = T$, we could use

$$V(h,d,T) = V^*(h^*(0), d^*(0), 0) = V^* \tag{3.122}$$

This however, is problematic. We obviously do not know the value V^* until we solve for the function $V(h,d,t)$ by completing the backward recursion, and we can not perform the backward recursion until we have the terminal and boundary conditions. As it were, even the boundary conditions are reliant on V^*.

One option is an iterative procedure based on an initial guess of V^* and then updated estimates based on the outcome of the backward recursion. But we prefer an alternative route to the solution. Let us consider N time periods of this problem, and assume that we are dealing with the last one. Assuming that after the N^{th} period is over, the investor immediately consumes the left-over wealth. If so, for this time period, the terminal condition is

$$J_N(X, H, C, T) = \frac{X^\gamma}{\gamma} \tag{3.123}$$

and therefore

$$V_N(h, d, T) = 1 \tag{3.124}$$

Assuming we are able to then solve the n^{th} period problem, then we can use V_n^*, the optimal value function for the n^{th} period problem, as the terminal condition for the $(n-1)^{th}$ problem. Then, as $n \to \infty$, the solution to the first period problem converges to the solution to our problem.

Now, we set up the boundary conditions for the generic n^{th} problem with the terminal condition

$$V(h, d, T) = V_n^* \tag{3.125}$$

At $t_{N_t} = T$, we can use

$$V(h_i, d_j, t_{N_t}) = V_n^* \tag{3.126}$$

At $d = 0$, the consumption rate is 0 and therefore the value function at time t is the expected value of the value function at time T

$$V(h,0,t) = e^{-\delta(T-t)}V_n^* \ \mathbb{E}\left[\left(\frac{X(T)}{X(t)}\right)^\gamma\right]$$
$$= e^{-\delta(T-t)}V_n^*$$
$$\mathbb{E}\left[\frac{1}{X(t)^\gamma}\left(\begin{array}{l} H(t)e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t)+\sigma_H(W_H(T)-W_H(t))} \\ +(X(t)-H(t))e^{(\hat{\beta}(\mu_S-\frac{1}{2}\sigma_S^2)+(1-\hat{\beta})r)(T-t)+\hat{\beta}\sigma_S(W_S(T)-W_S(t))} \end{array}\right)^\gamma\right]$$

where $\hat{\beta} = \frac{\beta}{1-h}$. Note that this is an approximation that assumes that we will keep $\hat{\beta}$ constant, but in reality we have β as a function of h, and h itself varies over time. However, we expect this approximation to be quite close. Thus

$$V(h,0,t) = e^{-\delta(T-t)}V_n^* \mathbb{E}\left[\left(\begin{array}{l} h(t)e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t)+\sigma_H(W_H(T)-W_H(t))} \\ +(1-h(t))e^{(\hat{\beta}(\mu_S-\frac{1}{2}\sigma_S^2)+(1-\hat{\beta})r)(T-t)+\hat{\beta}\sigma_S(W_S(T)-W_S(t))} \end{array}\right)^\gamma\right] \quad (3.127)$$

which has to be evaluated numerically. So at $d_1 = 0$, we can use

$$V(h_i,d_1,t_k) = e^{-\delta(T-t_k)}V_n^* \mathbb{E}\left[\left(\begin{array}{l} h_i(t_k)e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t_k)+\sigma_H(W_H(T)-W_H(t_k))} \\ +(1-h_i(t_k))e^{(\hat{\beta}(\mu_S-\frac{1}{2}\sigma_S^2)+(1-\hat{\beta})r)(T-t_k)+\hat{\beta}\sigma_S(W_S(T)-W_S(t_k))} \end{array}\right)^\gamma\right] \quad (3.128)$$

At $d = d_{max}$, we use the facts that the consumption rate (as a fraction of wealth) is at its maximum $c_{max} = \frac{r}{1-e^{-rT}}(1-h)$ and β is 0 since all the liquid wealth is being used to guarantee the consumption stream. Therefore

$$\begin{aligned} V(h,d_{max},t) &= \int_0^{(T-t)} e^{-\delta s} c_{max}^{\gamma} ds + e^{-\delta(T-t)} V_n^* \ \mathbb{E}\left[\left(\frac{X(T)}{X(t)}\right)^{\gamma}\right] \\ &= \left(\frac{r}{1-e^{-rT}}\right)^{\gamma} \left(\frac{1-e^{-\delta(T-t)}}{\delta}\right)(1-h)^{\gamma} \\ &\quad + e^{-\delta(T-t)} V_n^* \ \mathbb{E}\left[h^{\gamma}\left(e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t)+\sigma_H(W_H(T)-W_H(t))}\right)^{\gamma}\right] \end{aligned}$$

Thus

$$\begin{aligned} V(h,d_{max},t) = & \left(\tfrac{r}{1-e^{-rT}}\right)^{\gamma}\left(\tfrac{1-e^{-\delta(T-t)}}{\delta}\right)(1-h)^{\gamma} \\ & + e^{-\delta(T-t)} V_n^* \ \ h^{\gamma} e^{(\gamma\mu_H-\frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t)} \end{aligned} \tag{3.129}$$

So at $d_{N_d} = d_{max}$, we can use

$$\begin{aligned} V(h_i,d_{N_d},t_k) = & \left(\tfrac{r}{1-e^{-rT}}\right)^{\gamma}\left(\tfrac{1-e^{-\delta(T-t_k)}}{\delta}\right)(1-h_i)^{\gamma} \\ & + e^{-\delta(T-t_k)} V_n^* \ \ h_i^{\gamma} e^{(\gamma\mu_H-\frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t_k)} \end{aligned} \tag{3.130}$$

At $h = 0$, the entire wealth of the investor is in the stock or the bond, and therefore

$$\begin{aligned} V(0,d,t) &= \int_0^{(T-t)} e^{-\delta s} d^{\gamma} ds + e^{-\delta(T-t)} V_n^* \ \mathbb{E}\left[\left(\frac{X(T)}{X(t)}\right)^{\gamma}\right] \\ &= \left(\frac{1-e^{-\delta(T-t)}}{\delta}\right) d^{\gamma} \\ &\quad + e^{-\delta(T-t)} V_n^* \ \mathbb{E}\left[\left(e^{(r+\beta\nu_S-d-\frac{1}{2}\beta^2\sigma_S^2)(T-t)+\beta\sigma_S(W_S(T)-W_S(t))}\right)^{\gamma}\right] \end{aligned}$$

Thus

$$\begin{aligned} V(0,d,t) = & \left(\tfrac{1-e^{-\delta(T-t)}}{\delta}\right) d^{\gamma} \\ & + e^{-\delta(T-t)} V_n^* \ \ e^{(\gamma(r+\beta\nu_S-d)-\frac{1}{2}\gamma(1-\gamma)\beta^2\sigma_S^2)(T-t)} \end{aligned} \tag{3.131}$$

So at $h_1 = 0$, we can use

$$V(h_1, d_j, t_k) = \left(\frac{1-e^{-\delta(T-t_k)}}{\delta}\right) d_j^{\gamma} + e^{-\delta(T-t_k)} V_n^* \;\; e^{(\gamma(r+\beta\nu_S - d_j) - \frac{1}{2}\gamma(1-\gamma)\beta^2\sigma_S^2)(T-t_k)} \tag{3.132}$$

At $h = 1$, the entire wealth is in the hedge fund, hence there is no liquid wealth that can be consumed. Therefore

$$V(1, 0, t) = e^{-\delta(T-t)} V_n^* \;\; e^{(\gamma\mu_H - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t)} \tag{3.133}$$

So at $h_{N_h} = 1$, we can use

$$V(h_{N_h}, d_j, t_k) = e^{-\delta(T-t_k)} V_n^* \;\; e^{(\gamma\mu_H - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t_k)} \tag{3.134}$$

3.6. Summary

We have presented a systematic framework for the analysis of hedge funds as investable assets. We have looked at the impact of illiquidity of a very specific form - where investor funds are required to be locked up in the hedge fund for a pre-specified duration - on two fronts.

First, we have provided a framework for the calculation of the hedge fund lockup premium as well as an alternative measure of the impact of the lockup requirement that we call the lockup penalty. Second, we have studied the optimal structure for a portfolio consisting of a bond, a stock and a hedge fund under both terminal wealth and consumption utility considerations. These two scenarios apply to different types of institutional investors, with fund of funds and proprietary traders falling under the no-consumption setting and pension funds and university endowments falling under the utility of consumption setting. Under both scenarios, we account for the availability of equities as an alternative risky investment. While fund of funds do not usually maintain equity positions, that is not a restriction imposed on them due to the hedge fund lockup requirement. Consequently, in computing the lockup premium it is appropriate to account for the presence of equities even if they are not part of the investor's portfolio.

An interesting aside is the following - the positions in the bond and the stock can be changed continuously, however the position in the hedge fund can only be altered at prespecified intervals of time. Thus, this setting falls neither under the Merton continuous-time framework, nor under the Samuelson discrete-time framework. We take blatant advantage of our first-mover status and christen this the mixed continuous-discrete time framework problem.

We have also introduced the concept of an information premium, as well as a framework to calculate this premium and an alternative measure of the value of hedge fund price information that we call the information penalty. The value of hedge fund share price information is important in analyzing the impact of the typical *secrecy* (although in most cases this is simply a result of a lack of market price information for highly illiquid investments made by the private equity or hedge funds) that surrounds hedge fund performance reporting - often hedge fund share prices are available only just before a decision has to be made on adjusting the position in the hedge fund.

CHAPTER 4

Consumption and Equity-Hedge Fund Portfolio Optimization given an Intolerance for a Decline in Standard of Living

Abstract

Institutional portfolios of today are more complex than ever. Pension funds and university endowments require steady payout streams and must try and avoid major declines in these payouts. Moreover, their portfolios typically also include significant exposure to alternative assets such as hedge funds. The major complication that arises from the inclusion of hedge funds in the investor's portfolio is that they often have a lockup requirement, where the investor is restricted from withdrawing funds from the hedge fund for a pre-specified duration of time.

Considering a situation where the investor has an intolerance for a decline in standard of living, or in other words, where the investor seeks guaranteed non-decreasing consumption, we provide a framework for obtaining the optimal consumption and investment policies. The investor is allowed to invest in a bond, a stock or a hedge fund, with the bond and the stock being continuously tradeable and the hedge fund requiring a lockup period of given duration.

4.1. Introduction

In their classic papers on lifetime portfolio selection given continuous- and discrete-time settings respectively, Merton [1969] and Samuelson [1969] both show that under a constant relative risk aversion (CRRA) utility assumption, optimal consumption (*rate* for continuous-time, *amount* for discrete-time) at any point in time is in direct proportion to the then current wealth. The consumption pattern can therefore vary significantly, and given losses on investments in risky assets, a significant loss of utility may result.

Long-term institutional investors such as pension funds and university endowments that might require quarterly or annual payouts from their portfolios would want to avoid major downturns in these payouts. Since the classical optimal consumption and investment policies would not help achieve these goals, in Chapter 2 we looked at a scenario where these institutional investors impose a constraint that their payouts be non-decreasing. Note that since we impose a constraint on the classical problem, we expect a decrease in the overall value to be gained.

Clearly, this is about as conservative as one can get in regard to downturns in consumption and not a policy one would expect to be implemented. However, by studying this scenario we analyze a *worst-case* scenario of sorts. Weaker restrictions on the consumption pattern would yield results that are somewhere in between those from the classical problem and those from the non-decreasing consumption problem, and the decrease in value relative to the classical case would consequently be lower than that in the non-decreasing consumption problem. If desired, this solution framework can easily be extended to other forms of habit formation, as long as they can be specified as a constraint on the consumption process.

In Chapter 3, we analyze the inclusion of hedge funds in institutional investors' portfolios. Specifically, one of the things we study is the impact that the lockup restrictions imposed by the hedge fund have on the allocation to the hedge fund as part of the optimal portfolio where we have three assets available to us for investment - a bond, a stock and a hedge fund. The positions in the bond and the stock can be changed continuously, however the position in the hedge fund can only be altered at pre-specified intervals of time.

Here we combine these two restrictions and lay out a framework to determine the optimal consumption and investment policies when we have the three assets to invest in, the hedge fund imposes a lockup period and the investor desires non-decreasing consumption.

4.2. Problem IV: Maximizing the Utility of Non-Decreasing Consumption with Hedge Fund Price Information Available

In this section, we consider the problem of maximizing the utility of consumption, which is restricted to be non-decreasing. Initial wealth can be invested in a bond, a stock or a hedge fund, with the hedge fund requiring a lockup period of T. Hedge fund share price information is assumed to be continuously available.

Assume the notation as laid out in §3.3. The investor decides to consume at a rate C, which is a decision made at time 0. $c(t) \equiv \frac{C}{X(t)}$ (c in brief) is correspondingly the consumption rate as a fraction of the total wealth. Note that while C is fixed over the time period under analysis, $c(t)$ is a function of time due to the changes in wealth.

Assume that the inherited consumption rate is C_- and therefore we must have $C \geq C_-$. Equivalently, define $c_-(t) \equiv \frac{C_-}{X(t)}$ (c_- in brief). Once we set $C(0) \geq C_-(0)$, we automatically have that $c(t) \geq c_-(t)$ for all $0 \leq t \leq T$. This is an important fact and we will utilize this in our solution framework.

As before, our planning horizon is T, i.e. at T, the investor is allowed to rebalance the investment in the hedge fund, and essentially faces the same problem again as at time $t = 0$, except that the wealth and the inherited consumption rate have now changed.

The stochastic differential equations for our problem are

$$dH = \mu_H H dt + \sigma_H H dW_H = (\nu_H + r) H dt + \sigma_H H dW_H \tag{4.1}$$

and

$$dX = (rX + \beta \nu_S X + \nu_H H - C)dt + \sigma_S \beta X dW_S + \sigma_H H dW_H \tag{4.2}$$

where W_H and W_S are Wiener processes with $dW_H dW_S = \rho dt$. We want to maximize

$$\begin{aligned} &J(X(0), H^*(0), C^*(0), C_-, 0) \\ &\quad = \mathbb{E}\left[\int_0^\infty e^{-\delta t} \tfrac{1}{\gamma} C(t)^\gamma dt\right] \\ &\quad = \int_0^T e^{-\delta t} \tfrac{1}{\gamma} (C^*(0))^\gamma dt + e^{-\delta T} \mathbb{E}\left[J(X(T), H^*(T), C^*(T), C^*(0), T)\right] \end{aligned} \tag{4.3}$$

subject to the constraint that $C^*(0) \geq C_-$. δ is the discount rate for utility and $H^*(0), C^*(0)$ and $H^*(T), C^*(T)$ are determined according to our resulting optimal policy. Note that although C_- is a state variable, it influences the value function only through the restriction it imposes on C.

Now we proceed in a fashion similar to that in Kahl et al. [2003]. Since X and H form a joint Markov process, the value-to-go function $J(X, H, C, C_-, t)$ satisfies the following

Hamilton-Jacobi-Bellman (HJB) equation

$$\max \left[\begin{array}{l} \frac{1}{2}(\sigma_S^2\beta^2X^2 + 2\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)J_{XX} \\ +\frac{1}{2}\sigma_H^2H^2J_{HH} + (\rho\sigma_S\sigma_H\beta XH + \sigma_H^2H^2)J_{XH} \\ +(rX + \beta\nu_SX + \nu_HH - C)J_X \\ +(r+\nu_H)HJ_H + J_t + e^{-\delta t}\frac{1}{\gamma}C^\gamma \end{array} \right] = 0 \tag{4.4}$$

The first order optimality conditions for β is

$$\beta\sigma_S^2X^2J_{XX} + \rho\sigma_S\sigma_HXHJ_{XX} + \rho\sigma_S\sigma_HXHJ_{XH} + \nu_SXJ_X = 0$$

Thus

$$\beta^* = \frac{-\nu_S}{\sigma_S^2}\frac{J_X}{XJ_{XX}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{H}{X}\frac{J_{XH}}{J_{XX}} - \frac{\sigma_{SH}}{\sigma_S^2}\frac{H}{X} \tag{4.5}$$

Note that this expression for β^* is the same as that for β^* in §3.3 as well as that for ϕ^* in Appendix A of Kahl et al. [2003].

We conjecture that value-to-go function is of the form

$$J(X,H,C,C_-,t) = \frac{X^\gamma}{\gamma}V(h,c,c_-,t) \tag{4.6}$$

We can verify the validity of this conjecture by rewriting the HJB equation in terms of this function $V(h,c,c_-,t)$ and checking that the boundary and terminal conditions are free of X. Also, similar to the relationship between $J(X,H,C,C_-,t)$ and C_-, c_- influences $V(h,c,c_-,t)$ only through the restriction it imposes on c.

We first compute the partial derivatives J_X, J_{XX} and J_{XH} in terms of h, c and the function $V(h,c,c_-,t)$ and its partial derivatives V_h, V_{hh}, V_c and V_{cc} as

$$J_X = \frac{X^{\gamma-1}}{\gamma}\left(-hV_h - cV_c + \gamma V\right) \tag{4.7}$$

$$J_{XX} = \frac{X^{\gamma-2}}{\gamma}\big(h^2V_{hh} + c^2V_{cc} + 2(1-\gamma)hV_h + 2(1-\gamma)cV_c - \gamma(1-\gamma)V\big) \tag{4.8}$$

$$J_{XH} = \frac{X^{\gamma-2}}{\gamma}\left(-hV_{hh} + V_{hc} - (1-\gamma)V_h\right) \tag{4.9}$$

We can now write β^* in terms of these quantities

$$\beta^* = \frac{\frac{\sigma_{SH}}{\sigma_S^2}h^2V_{hh} - \frac{\sigma_{SH}}{\sigma_S^2}hV_{hc} + \left((1-\gamma)\frac{\sigma_{SH}}{\sigma_S^2} + \frac{\nu_S}{\sigma_S^2}\right)hV_h + \frac{\nu_S}{\sigma_S^2}cV_c - \frac{\nu_S}{\sigma_S^2}\gamma V}{h^2V_{hh} + c^2V_{cc} + 2(1-\gamma)hV_h + 2(1-\gamma)cV_c - \gamma(1-\gamma)V} - \frac{\sigma_{SH}}{\sigma_S^2}h \tag{4.10}$$

Assuming that we do not allow borrowing, it is easy to show that given values for h and c, the maximum allowed value for β at any time t is $\beta_{max}(t) = 1 - h - \frac{c}{r}(1 - e^{-r(T-t)}$. Depending on the values for the problem parameters, it might be necessary to enforce this restriction. However, we will concentrate on cases where the stock is not an overly attractive investment and its allocation is well below this bound.

Our next step is to convert the HJB into a form that uses h and the partial derivatives of the function $V(h,c,c_-,t)$. To do this we also need the J_H, J_{HH} and J_t in terms of $V(h,c,c_-,t)$ and its partial derivatives

$$J_H = \frac{X^{\gamma-1}}{\gamma}V_h \tag{4.11}$$

$$J_{HH} = \frac{X^{\gamma-2}}{\gamma} V_{hh} \tag{4.12}$$

$$J_t = \frac{X^{\gamma}}{\gamma} V_t \tag{4.13}$$

Substituting the partial derivatives of $J(X, H, C, C_-, t)$ into the HJB gives us

$$\max \begin{bmatrix} \frac{1}{2}(\sigma_S^2 \beta^2 X^2 + 2\rho\sigma_S\sigma_H\beta XH + \sigma_H^2 H^2)\frac{X^{\gamma-2}}{\gamma} \\ \begin{pmatrix} h^2 V_{hh} + c^2 V_{cc} + 2(1-\gamma)hV_h \\ +2(1-\gamma)cV_c - \gamma(1-\gamma)V \end{pmatrix} + \frac{1}{2}\sigma_H^2 H^2 \frac{X^{\gamma-2}}{\gamma} V_{hh} \\ +(\rho\sigma_S\sigma_H\beta XH + \sigma_H^2 H^2)\frac{X^{\gamma-2}}{\gamma}\left(-hV_{hh} + V_{hc} - (1-\gamma)V_h\right) \\ +(rX + \beta\nu_S X + \nu_H H - C)\frac{X^{\gamma-1}}{\gamma}\left(-hV_h - cV_c + \gamma V\right) \\ +(r+\nu_H)H\frac{X^{\gamma-1}}{\gamma}V_h + \frac{X^{\gamma}}{\gamma}V_t + e^{-\delta t}\frac{1}{\gamma}C^{\gamma} \end{bmatrix} = 0$$

Dividing through by $\frac{X^\gamma}{\gamma}$ and rearranging the terms gives us the HJB equation in terms of the function $V(h, c, c_-, t)$

$$\max \left[\begin{array}{l} \left(\begin{array}{l} \frac{1}{2}(\beta^2\sigma_S^2 h^2 + 2\beta\sigma_{SH}h^3 + \sigma_H^2 h^4) \\ +\frac{1}{2}\sigma_H^2 h^2 - (\beta\sigma_{SH}h^2 + \sigma_H^2 h^3) \end{array} \right) V_{hh} \\ +(\beta\sigma_{SH}h + \sigma_H^2 h^2)V_{hc} \\ +\frac{1}{2}(\beta^2\sigma_S^2 c^2 + 2\beta\sigma_{SH}hc^2 + \sigma_H^2 h^2 c^2)V_{cc} \\ + \left(\begin{array}{l} (1-\gamma)(\beta^2\sigma_S^2 h + 2\beta\sigma_{SH}h^2 + \sigma_H^2 h^3) \\ -(rh + \beta\nu_S h + \nu_H h^2 - hc) \\ -(1-\gamma)(\beta\sigma_{SH}h + \sigma_H^2 h^2) + (rh + \nu_H h) \end{array} \right) V_h \\ + \left(\begin{array}{l} (1-\gamma)(\beta^2\sigma_S^2 c + 2\sigma_{SH}\beta hc + \sigma_H^2 h^2 c) \\ -(rc + \beta\nu_S c + \nu_H hc - c^2) \end{array} \right) V_c \\ + \left(\begin{array}{l} -\gamma(1-\gamma)\frac{1}{2}(\beta^2\sigma_S^2 + 2\beta\sigma_{SH}h + \sigma_H^2 h^2) \\ +\gamma(r + \beta\nu_S + \nu_H h - c) \end{array} \right) V \\ +V_t + e^{-\delta t}c^\gamma \end{array} \right] = 0 \tag{4.14}$$

The maximum in equation (4.14) holds when $\beta = \beta^*$ and therefore

$$\begin{aligned}
&\begin{pmatrix} \frac{1}{2}((\beta^*)^2\sigma_S^2 h^2 + 2\beta^*\sigma_{SH}h^3 + \sigma_H^2 h^4) \\ +\frac{1}{2}\sigma_H^2 h^2 - (\beta^*\sigma_{SH}h^2 + \sigma_H^2 h^3) \end{pmatrix} V_{hh} \\
&+(\beta^*\sigma_{SH}h + \sigma_H^2 h^2)V_{hc} \\
&+\tfrac{1}{2}((\beta^*)^2\sigma_S^2 c^2 + 2\beta^*\sigma_{SH}hc^2 + \sigma_H^2 h^2 c^2)V_{cc} \\
&+\begin{pmatrix} (1-\gamma)((\beta^*)^2\sigma_S^2 h + 2\beta^*\sigma_{SH}h^2 + \sigma_H^2 h^3) \\ -(rh + \beta^*\nu_S h + \nu_H h^2 - hc) \\ -(1-\gamma)(\beta^*\sigma_{SH}h + \sigma_H^2 h^2) + (rh + \nu_H h) \end{pmatrix} V_h \\
&+\begin{pmatrix} (1-\gamma)((\beta^*)^2\sigma_S^2 c + 2\sigma_{SH}\beta^* hc + \sigma_H^2 h^2 c) \\ -(rc + \beta^*\nu_S c + \nu_H hc - c^2) \end{pmatrix} V_c \\
&+\begin{pmatrix} -\gamma(1-\gamma)\frac{1}{2}((\beta^*)^2\sigma_S^2 + 2\beta^*\sigma_{SH}h + \sigma_H^2 h^2) \\ +\gamma(r + \beta^*\nu_S + \nu_H h - c) \end{pmatrix} V \\
&+V_t + e^{-\delta t}c^\gamma = 0
\end{aligned} \tag{4.15}$$

As in the case of Problem II (§3.4), we have to solve for the value function by building a grid on a 2-dimensional state space (h,c). Again, we would like to use the Alternating Direction Implicit (ADI) scheme but we are faced with the same issues as before. First, we have a mixed second-order derivative V_{hc} which can not be eliminated. The second issue is the following: in addition to the maximum allowed consumption rate depending on the amount invested in the hedge fund, the minimum allowed consumption rate depends on the inherited consumption rate. We enforce these condition at time 0, and then assume that an appropriate amount is invested in the bond to meet the consumption requirements. This ties in with the earlier condition on the maximum allowed value of β. Once the consumption rate is fixed at $C \geq C_-$, to guarantee this consumption stream we require to put $\frac{C}{r}(1 - e^{-rT})$ in the bond. Clearly this amount increases with an increase in C. Then, given an initial investment $H(0)$ in the hedge fund, the maximum allowed consumption rate C_{max} would satisfy the following equation

$$X(0) = \frac{C_{max}}{r}(1 - e^{-rT}) + H(0) \tag{4.16}$$

Dividing by $X(0)$, we get an expression for the maximum initial value for c_{max}, which is the consumption rate expressed as a fraction of the total wealth

$$c_{max} = \frac{r}{1 - e^{-rT}}(1 - h) \tag{4.17}$$

Thus, one of the boundaries for our problem is $c = \frac{r}{1-e^{-rT}}(1 - h)$, leading us to implement a rectangular grid on a two-dimensional feasible state space that is triangular in shape. To resolve this issue, we replace c by d as a state variable, where

$$d = \frac{c}{1 - h} \tag{4.18}$$

d can be interpreted as the consumption rate as a fraction of the *liquid* wealth. Then

$$d_{max} = \frac{r}{1 - e^{-rT}} \tag{4.19}$$

Similarly

$$d_{min} = \frac{c_-}{1 - h} \equiv d_- \tag{4.20}$$

Clearly, we lose no information by making this change, but the feasible state space now becomes rectangular, with the bounds on both state variables becoming independent of each other. Since

$$V_c = \frac{1}{1-h} V_d \tag{4.21}$$

$$V_{cc} = \frac{1}{(1-h)^2} V_{dd} \tag{4.22}$$

$$V_{hc} = \frac{1}{1-h} V_{hd} \tag{4.23}$$

equation (4.10) becomes

$$\beta^* = \frac{\frac{\sigma_{SH}}{\sigma_S^2} h^2 V_{hh} - \frac{\sigma_{SH}}{\sigma_S^2} \frac{h}{1-h} V_{hd} + \left((1-\gamma) \frac{\sigma_{SH}}{\sigma_S^2} + \frac{\nu_S}{\sigma_S^2} \right) h V_h + \frac{\nu_S}{\sigma_S^2} d V_d - \frac{\nu_S}{\sigma_S^2} \gamma V}{h^2 V_{hh} + d^2 V_{dd} + 2(1-\gamma) h V_h + 2(1-\gamma) d V_d - \gamma(1-\gamma) V} - \frac{\sigma_{SH}}{\sigma_S^2} h \tag{4.24}$$

and equation (4.15) changes to

$$\left(\begin{array}{l}\frac{1}{2}((\beta^*)^2\sigma_S^2h^2+2\beta^*\sigma_{SH}h^3+\sigma_H^2h^4)\\+\frac{1}{2}\sigma_H^2h^2-(\beta^*\sigma_{SH}h^2+\sigma_H^2h^3)\end{array}\right)V_{hh}+(\beta^*\sigma_{SH}+\sigma_H^2h)\tfrac{h}{1-h}V_{hd}$$
$$+\tfrac{1}{2}((\beta^*)^2\sigma_S^2d^2+2\beta^*\sigma_{SH}hd^2+\sigma_H^2h^2d^2)V_{dd}$$
$$+\left(\begin{array}{l}(1-\gamma)((\beta^*)^2\sigma_S^2h+2\beta^*\sigma_{SH}h^2+\sigma_H^2h^3)\\-(rh+\beta^*\nu_Sh+\nu_Hh^2-h(1-h)d)\\-(1-\gamma)(\beta^*\sigma_{SH}h+\sigma_H^2h^2)+(rh+\nu_Hh)\end{array}\right)V_h \tag{4.25}$$
$$+\left(\begin{array}{l}(1-\gamma)((\beta^*)^2\sigma_S^2d+2\sigma_{SH}\beta^*hd+\sigma_H^2h^2d)\\-(rd+\beta^*\nu_Sd+\nu_Hhd-(1-h)d^2)\end{array}\right)V_d$$
$$+\left(\begin{array}{l}-\gamma(1-\gamma)\frac{1}{2}((\beta^*)^2\sigma_S^2+2\beta^*\sigma_{SH}h+\sigma_H^2h^2)\\+\gamma(r+\beta^*\nu_S+\nu_Hh-(1-h)d)\end{array}\right)V$$
$$+V_t+e^{-\delta t}(1-h)^\gamma d^\gamma=0$$

or in brief as

$$\begin{aligned}&f_1(h,d,\beta^*)V_{hh}+f_2(h,d,\beta^*)V_{hd}+f_3(h,d,\beta^*)V_{dd}+f_4(h,d,\beta^*)V_h\\&+f_5(h,d,\beta^*)V_d+V_t+f_6(h,d,\beta^*)V+f_7(h,d)=0\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}f_1(h,d,\beta^*)=\;&\tfrac{1}{2}((\beta^*)^2\sigma_S^2h^2+2\beta^*\sigma_{SH}h^3+\sigma_H^2h^4)\\&+\tfrac{1}{2}\sigma_H^2h^2-(\beta^*\sigma_{SH}h^2+\sigma_H^2h^3)\end{aligned} \tag{4.27}$$

$$f_2(h,d,\beta^*)=(\beta^*\sigma_{SH}+\sigma_H^2h)\frac{h}{1-h} \tag{4.28}$$

$$f_3(h,d,\beta^*) = \frac{1}{2}((\beta^*)^2\sigma_S^2 d^2 + 2\beta^*\sigma_{SH} h d^2 + \sigma_H^2 h^2 d^2) \tag{4.29}$$

$$\begin{aligned} f_4(h,d,\beta^*) = \quad & (1-\gamma)((\beta^*)^2\sigma_S^2 h + 2\beta^*\sigma_{SH} h^2 + \sigma_H^2 h^3) \\ & -(rh + \beta^*\nu_S h + \nu_H h^2 - h(1-h)d) \\ & -(1-\gamma)(\beta^*\sigma_{SH} h + \sigma_H^2 h^2) + (rh + \nu_H h) \end{aligned} \tag{4.30}$$

$$\begin{aligned} f_5(h,d,\beta^*) = \quad & (1-\gamma)((\beta^*)^2\sigma_S^2 d + 2\sigma_{SH}\beta^* h d + \sigma_H^2 h^2 d) \\ & -(rd + \beta^*\nu_S d + \nu_H h d - (1-h)d^2) \end{aligned} \tag{4.31}$$

$$\begin{aligned} f_6(h,d,\beta^*) = \quad & -\gamma(1-\gamma)\tfrac{1}{2}((\beta^*)^2\sigma_S^2 + 2\beta^*\sigma_{SH} h + \sigma_H^2 h^2) \\ & +\gamma(r + \beta^*\nu_S + \nu_H h - (1-h)d) \end{aligned} \tag{4.32}$$

and

$$f_7(h,d) = e^{-\delta t}(1-h)^\gamma d^\gamma \tag{4.33}$$

We continue with setting up the ADI finite difference scheme to evaluate the function $V(h,d,t)$. Instead of plugging in the expression for β^* into equation (4.15), we follow Kahl et al. [2003] in linearizing the HJB equation by evaluating the values of β^* using the estimated values of the function $V(h,d,t)$ and its derivatives at the subsequent time step. As in Kahl et al. [2003], we ensure the accuracy of the scheme by using extremely small time steps.

In going from t_k to $t_{k-\frac{1}{2}}$, we use the following approximations for the derivatives

$$V_{hh}(h_i, d_j, t_k) = \frac{1}{\Delta h^2}\begin{bmatrix} V(h_{i+1}, d_j, t_{k-\frac{1}{2}}) - 2V(h_i, d_j, t_{k-\frac{1}{2}}) \\ +V(h_{i-1}, d_j, t_{k-\frac{1}{2}}) \end{bmatrix} \tag{4.34}$$

$$V_{hd}(h_i, d_j, t_k) = \frac{1}{4\Delta h \Delta d}\begin{bmatrix} V(h_{i+1}, d_{j+1}, t_k) - V(h_{i+1}, d_{j-1}, t_k) \\ -V(h_{i-1}, d_{j+1}, t_k) + V(h_{i-1}, d_{j-1}, t_k) \end{bmatrix} \tag{4.35}$$

$$V_{dd}(h_i, d_j, t_k) = \frac{1}{\Delta d^2}[V(h_i, d_{j+1}, t_k) - 2V(h_i, d_j, t_k) + V(h_i, d_{j-1}, t_k)] \tag{4.36}$$

$$V_h(h_i, d_j, t_k) = \frac{1}{2\Delta h}[V(h_{i+1}, d_j, t_{k-\frac{1}{2}}) - V(h_{i-1}, d_j, t_{k-\frac{1}{2}})] \tag{4.37}$$

$$V_d(h_i, d_j, t_k) = \frac{1}{2\Delta d}[V(h_i, d_{j+1}, t_k) - V(h_i, d_{j-1}, t_k)] \tag{4.38}$$

$$V_t(h_i, d_j, t_k) = \frac{1}{\Delta t}[V(h_i, d_j, t_k) - V(h_i, d_j, t_{k-\frac{1}{2}})] \tag{4.39}$$

Accordingly, for the half-step from t_k to $t_{k-\frac{1}{2}}$, equation (4.26) becomes

$$
\begin{aligned}
& f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - 2V(h_i,d_j,t_{k-\frac{1}{2}}) + V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
& + f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\begin{bmatrix} V(h_{i+1},d_{j+1},t_k) - V(h_{i+1},d_{j-1},t_k) \\ -V(h_{i-1},d_{j+1},t_k) + V(h_{i-1},d_{j-1},t_k) \end{bmatrix} \\
& + f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2}[V(h_i,d_{j+1},t_k) - 2V(h_i,d_j,t_k) + V(h_i,d_{j-1},t_k)] \\
& + f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
& + f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}[V(h_i,d_{j+1},t_k) - V(h_i,d_{j-1},t_k)] \\
& + \tfrac{1}{\Delta t}[V(h_i,d_j,t_k) - V(h_i,d_j,t_{k-\frac{1}{2}})] \\
& + f_6(h,d,\beta^*)V(h_i,d_j,t_k) + f_7(h,d) = 0
\end{aligned}
$$

Rearranging the terms gives us

$$
\begin{aligned}
& \left[f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2} + f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}\right] V(h_{i+1},d_j,t_{k-\frac{1}{2}}) \\
& - \left[2f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2} + \tfrac{1}{\Delta t}\right] V(h_i,d_j,t_{k-\frac{1}{2}}) \\
& + \left[f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2} - f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}\right] V(h_{i-1},d_j,t_{k-\frac{1}{2}}) \\
& = - \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i+1},d_{j+1},t_k) \\
& + \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i+1},d_{j-1},t_k) \\
& - \left[f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2} + f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}\right] V(h_i,d_{j+1},t_k) \\
& + \left[2f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2} - \tfrac{1}{\Delta t} - f_6(h,d,\beta^*)\right] V(h_i,d_j,t_k) \\
& - \left[f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2} - f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}\right] V(h_i,d_{j-1},t_k) \\
& + \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i-1},d_{j+1},t_k) \\
& - \left[f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\right] V(h_{i-1},d_{j-1},t_k) - f_7(h,d)
\end{aligned}
\tag{4.40}
$$

Note that all unknowns $(V(\cdot,\cdot,t_{k-\frac{1}{2}})$ are to be evaluated at d_j. This allows us to solve a system of equations for each d_j individually, effectively making this a series of one-dimensional problems.

In going from $t_{k-\frac{1}{2}}$ to t_{k-1}, we use the following approximations for the derivatives

$$V_{hh}(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{\Delta h^2}\begin{bmatrix} V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - 2V(h_i,d_j,t_{k-\frac{1}{2}}) \\ +V(h_{i-1},d_j,t_{k-\frac{1}{2}}) \end{bmatrix} \tag{4.41}$$

$$V_{hd}(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{4\Delta h\Delta d}\begin{bmatrix} V(h_{i+1},d_{j+1},t_{k-\frac{1}{2}}) \\ -V(h_{i+1},d_{j-1},t_{k-\frac{1}{2}}) \\ -V(h_{i-1},d_{j+1},t_{k-\frac{1}{2}}) \\ +V(h_{i-1},d_{j-1},t_{k-\frac{1}{2}}) \end{bmatrix} \tag{4.42}$$

$$V_{dd}(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{\Delta d^2}\begin{bmatrix} V(h_i,d_{j+1},t_{k-1}) - 2V(h_i,d_j,t_{k-1}) \\ +V(h_i,d_{j-1},t_{k-1}) \end{bmatrix} \tag{4.43}$$

$$V_h(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{2\Delta h}[V(h_{i+1},d_j,t_{k-\frac{1}{2}}) - V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \tag{4.44}$$

$$V_d(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{2\Delta d}[V(h_i,d_{j+1},t_{k-1}) - V(h_i,d_{j-1},t_{k-1})] \tag{4.45}$$

$$V_t(h_i,d_j,t_{k-\frac{1}{2}}) = \frac{1}{\Delta t}[V(h_i,d_j,t_{k-\frac{1}{2}}) - V(h_i,d_j,t_{k-1})] \tag{4.46}$$

Accordingly, for the half-step from $t_{k-\frac{1}{2}}$ to t_{k-1}, equation (4.26) becomes

$$
\begin{aligned}
&f_1(h,d,\beta^*)\tfrac{1}{\Delta h^2}[V(h_{i+1},d_j,t_{k-\frac{1}{2}})-2V(h_i,d_j,t_{k-\frac{1}{2}})+V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
&+f_2(h,d,\beta^*)\tfrac{1}{4\Delta h\Delta d}\left[\begin{array}{l} V(h_{i+1},d_{j+1},t_{k-\frac{1}{2}})-V(h_{i+1},d_{j-1},t_{k-\frac{1}{2}}) \\ -V(h_{i-1},d_{j+1},t_{k-\frac{1}{2}})+V(h_{i-1},d_{j-1},t_{k-\frac{1}{2}}) \end{array}\right] \\
&+f_3(h,d,\beta^*)\tfrac{1}{\Delta d^2}[V(h_i,d_{j+1},t_{k-1})-2V(h_i,d_j,t_{k-1})+V(h_i,d_{j-1},t_{k-1})] \\
&+f_4(h,d,\beta^*)\tfrac{1}{2\Delta h}[V(h_{i+1},d_j,t_{k-\frac{1}{2}})-V(h_{i-1},d_j,t_{k-\frac{1}{2}})] \\
&+f_5(h,d,\beta^*)\tfrac{1}{2\Delta d}[V(h_i,d_{j+1},t_{k-1})-V(h_i,d_{j-1},t_{k-1})] \\
&+\tfrac{1}{\Delta t}[V(h_i,d_j,t_{k-\frac{1}{2}})-V(h_i,d_j,t_{k-1})] \\
&+f_6(h,d,\beta^*)V(h_i,d_j,t_{k-\frac{1}{2}})+f_7(h,d)=0
\end{aligned}
$$

Rearranging the terms gives us

$$
\begin{aligned}
&\left[\tfrac{f_3(h,d,\beta^*)}{\Delta d^2}+\tfrac{f_5(h,d,\beta^*)}{2\Delta d}\right]V(h_i,d_{j+1},t_{k-1}) \\
&-\left[2\tfrac{f_3(h,d,\beta^*)}{\Delta d^2}+\tfrac{1}{\Delta t}\right]V(h_i,d_j,t_{k-1}) \\
&+\left[\tfrac{f_3(h,d,\beta^*)}{\Delta d^2}-\tfrac{f_5(h,d,\beta^*)}{2\Delta d}\right]V(h_i,d_{j-1},t_{k-1}) \\
&=-\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d}V(h_{i+1},d_{j+1},t_{k-\frac{1}{2}}) \\
&-\left[\tfrac{f_1(h,d,\beta^*)}{\Delta h^2}+\tfrac{f_4(h,d,\beta^*)}{2\Delta h}\right]V(h_{i+1},d_j,t_{k-\frac{1}{2}}) \\
&+\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d}V(h_{i+1},d_{j-1},t_{k-\frac{1}{2}}) \\
&+\left[2\tfrac{f_1(h,d,\beta^*)}{\Delta h^2}-\tfrac{1}{\Delta t}-f_6(h,d,\beta^*)\right]V(h_i,d_j,t_{k-\frac{1}{2}}) \\
&+\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d}V(h_{i-1},d_{j+1},t_{k-\frac{1}{2}}) \\
&-\left[\tfrac{f_1(h,d,\beta^*)}{\Delta h^2}-\tfrac{f_4(h,d,\beta^*)}{2\Delta h}\right]V(h_{i-1},d_j,t_{k-\frac{1}{2}}) \\
&-\tfrac{f_2(h,d,\beta^*)}{4\Delta h\Delta d}V(h_{i-1},d_{j-1},t_{k-\frac{1}{2}})-f_7(h,d)
\end{aligned}
\tag{4.47}
$$

Again, note that all unknowns ($V(\cdot,\cdot,t_{k-1})$ are to be evaluated at h_i. This allows us to solve a system of equations for each h_i individually, effectively making this a series of one-dimensional problems.

We now need to determine the terminal and boundary conditions for our problem. At $t = T$, we are able to rebalance our entire portfolio. Essentially, we face the same problem as at time 0, except that our wealth level is now $X(T)$ instead of $X(0)$. So we must have

$$\begin{aligned} &J(X, H^*(X, C^*(X, C_-, 0), T), C^*(X, C^*(X, C_-, 0), T), T) \\ &\quad = J(X, H^*(X, C_-, 0), C^*(X, C_-, 0), 0) \end{aligned} \tag{4.48}$$

where $H^*(X, C_-, 0)$ and $C^*(X, C_-, 0)$ denote the optimal value of H and C if the wealth at time 0 were to be X and the inherited consumption rate was C_-. Note that this boundary condition is different from that in Kahl et al. [2003] because of the problem context. So at $t = T$, we could use

$$V(h, d, T) = V^*(h^*(d, 0), d^*(d, 0), 0) = V^* \tag{4.49}$$

where $h^*(d, 0)$ and $d^*(d, 0)$ denote optimal values determined using d as the inherited consumption rate as a fraction of liquid wealth. This however, is problematic. We obviously do not know the value V^* until we solve for the function $V(h, d, t)$ by completing the backward recursion, and we can not perform the backward recursion until we have the terminal and boundary conditions. As it were, even the boundary conditions are reliant on V^*.

One option is an iterative procedure based on an initial guess of V^* and then updated estimates based on the outcome of the backward recursion. But we prefer an alternative route to the solution. Let us consider N time periods of this problem, and assume that we are dealing with the last one. Assuming that after the N^{th} period is over, the investor immediately consumes the left-over wealth. If so, for this time period, the terminal condition is

$$J(X, H, C, C_{-}, T) = \frac{X^{\gamma}}{\gamma} \tag{4.50}$$

and therefore

$$V(h, d, T) = 1 \tag{4.51}$$

Assuming we are able to then solve the n^{th} period problem, then we can use V_n^*, the optimal value function for the n^{th} period problem, as the terminal condition for the $(n-1)^{th}$ problem. Then, as $n \to \infty$, the solution to the first period problem converges to the solution to our problem.

The other complication in this process is the handling of the inherited consumption rate in its various forms (C_{-}, c_{-}, d_{-}). For the n^{th} problem, when we are performing a backward recursion there is obviously no way of knowing what the inherited consumption rate is going to be. Luckily, d_{-} is bounded below at 0 and above at the same bound that d has, i.e. d_{max}. So we will solve the two-dimensional finite difference problem for different values of d_{-} and store the optimal solutions for each of these values at every n, essentially introducing a third dimension into our problem, albeit without bringing it into the finite difference grid. Then, when we solve the $(n-1)^{th}$ problem, the consumption rate d_{n-1} we decide on will determine the terminal value of a certain amount of wealth, this value being generated by an interpolation of the stored values of $V_n^*(d_{-})$.

Now, we set up the boundary conditions for the generic n^{th} problem with the terminal condition

$$V(h,d,T) = V_n^* \tag{4.52}$$

At $t_{N_t} = T$, we can use

$$V(h_i,d_j,t_{N_t}) = V_n^* \tag{4.53}$$

At $d = d_-$, the consumption rate is c_- and therefore

$$\begin{aligned} V(h,d_-,t) &= \int_0^{(T-t)} e^{-\delta s} c_-^\gamma ds + e^{-\delta(T-t)} V_n^* \ \mathbb{E}\left[\left(\frac{X(T)}{X(t)}\right)^\gamma\right] \\ &= \left(\frac{1-e^{-\delta(T-t)}}{\delta}\right)(d_-(1-h))^\gamma + e^{-\delta(T-t)} V_n^* \\ &\mathbb{E}\left[\frac{1}{X(t)^\gamma}\left(\begin{array}{l} H(t)e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t)+\sigma_H(W_H(T)-W_H(t))} \\ +(X(t)-H(t))e^{(\hat{\beta}(\mu_S-\frac{1}{2}\sigma_S^2)+(1-\hat{\beta})r)(T-t)+\hat{\beta}\sigma_S(W_S(T)-W_S(t))} \end{array}\right)^\gamma\right] \end{aligned}$$

where $\hat{\beta} = \frac{\beta}{1-h}$. Note that this is an approximation that assumes that we will keep $\hat{\beta}$ constant, but in reality we have β as a function of h, and h itself varies over time. However, we expect this approximation to be quite close. Thus

$$\begin{aligned} V(h,d_-,t) = &\left(\frac{1-e^{-\delta(T-t)}}{\delta}\right)(d_-(1-h))^\gamma + e^{-\delta(T-t)} V_n^* \\ &\mathbb{E}\left[\left(\begin{array}{l} h(t)e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t)+\sigma_H(W_H(T)-W_H(t))} \\ +(1-h(t))e^{(\hat{\beta}(\mu_S-\frac{1}{2}\sigma_S^2)+(1-\hat{\beta})r)(T-t)+\hat{\beta}\sigma_S(W_S(T)-W_S(t))} \end{array}\right)^\gamma\right] \end{aligned} \tag{4.54}$$

which has to be evaluated numerically. So at $d_1 = d_-$, we can use

$$V(h_i, d_1, t_k) = \left(\frac{1-e^{-\delta(T-t_k)}}{\delta}\right)(d_1(1-h_i))^\gamma + e^{-\delta(T-t_k)}V_n^* \\ \mathbb{E}\left[\left(\begin{array}{l} h_i(t_k)e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t_k)+\sigma_H(W_H(T)-W_H(t_k))} \\ +(1-h_i(t_k))e^{(\hat{\beta}(\mu_S-\frac{1}{2}\sigma_S^2)+(1-\hat{\beta})r)(T-t_k)+\hat{\beta}\sigma_S(W_S(T)-W_S(t_k))} \end{array}\right)^\gamma\right] \tag{4.55}$$

At $d = d_{max}$, we use the facts that the consumption rate (as a fraction of wealth) is at its maximum $c_{max} = \frac{r}{1-e^{-rT}}(1-h)$ and β is 0 since all the liquid wealth is being used to guarantee the consumption stream. Therefore

$$\begin{aligned} V(h, d_{max}, t) \quad &= \int_0^{(T-t)} e^{-\delta s} c_{max}^\gamma ds + e^{-\delta(T-t)}V_n^* \ \mathbb{E}\left[\left(\frac{X(T)}{X(t)}\right)^\gamma\right] \\ &= \left(\frac{r}{1-e^{-rT}}\right)^\gamma \left(\frac{1-e^{-\delta(T-t)}}{\delta}\right)(1-h)^\gamma \\ &\quad +e^{-\delta(T-t)}V_n^* \ \mathbb{E}\left[h^\gamma\left(e^{(\mu_H-\frac{1}{2}\sigma_H^2)(T-t)+\sigma_H(W_H(T)-W_H(t))}\right)^\gamma\right] \end{aligned}$$

Thus

$$\begin{aligned} V(h, d_{max}, t) = \quad &\left(\tfrac{r}{1-e^{-rT}}\right)^\gamma \left(\tfrac{1-e^{-\delta(T-t)}}{\delta}\right)(1-h)^\gamma \\ &+e^{-\delta(T-t)}V_n^* \quad h^\gamma e^{(\gamma\mu_H-\frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t)} \end{aligned} \tag{4.56}$$

So at $d_{N_d} = d_{max}$, we can use

$$\begin{aligned} V(h_i, d_{N_d}, t_k) = \quad &\left(\tfrac{r}{1-e^{-rT}}\right)^\gamma \left(\tfrac{1-e^{-\delta(T-t_k)}}{\delta}\right)(1-h_i)^\gamma \\ &+e^{-\delta(T-t_k)}V_n^* \quad h_i^\gamma e^{(\gamma\mu_H-\frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t_k)} \end{aligned} \tag{4.57}$$

At $h = 0$, the entire wealth of the investor is in the stock or the bond, and therefore

$$\begin{aligned} V(0,d,t) &= \int_0^{(T-t)} e^{-\delta s} d^\gamma ds + e^{-\delta(T-t)} V_n^* \ \mathbb{E}\left[\left(\frac{X(T)}{X(t)} \right)^\gamma \right] \\ &= \left(\frac{1 - e^{-\delta(T-t)}}{\delta} \right) d^\gamma \\ &\quad + e^{-\delta(T-t)} V_n^* \ \mathbb{E}\left[\left(e^{(r+\beta^*\nu_S - d - \frac{1}{2}(\beta^*)^2\sigma_S^2)(T-t) + \beta^*\sigma_S(W_S(T)-W_S(t))} \right)^\gamma \right] \end{aligned}$$

Thus

$$\begin{aligned} V(0,d,t) = \ & \left(\tfrac{1-e^{-\delta(T-t)}}{\delta} \right) d^\gamma \\ & + e^{-\delta(T-t)} V_n^* \ \ e^{(\gamma(r+\beta^*\nu_S - d) - \frac{1}{2}\gamma(1-\gamma)(\beta^*)^2\sigma_S^2)(T-t)} \end{aligned} \tag{4.58}$$

So at $h_1 = 0$, we can use

$$\begin{aligned} V(h_1,d_j,t_k) = \ & \left(\tfrac{1-e^{-\delta(T-t_k)}}{\delta} \right) d_j^\gamma \\ & + e^{-\delta(T-t_k)} V_n^* \ \ e^{(\gamma(r+\beta^*\nu_S - d_j) - \frac{1}{2}\gamma(1-\gamma)(\beta^*)^2\sigma_S^2)(T-t_k)} \end{aligned} \tag{4.59}$$

At $h = 1$, the entire wealth is in the hedge fund, hence there is no liquid wealth that can be consumed. Therefore

$$V(1,0,t) = e^{-\delta(T-t)} V_n^* \ \ e^{(\gamma\mu_H - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t)} \tag{4.60}$$

So at $h_{N_h} = 1$, we can use

$$V(h_{N_h},d_j,t_k) = e^{-\delta(T-t_k)} V_n^* \ \ e^{(\gamma\mu_H - \frac{1}{2}\gamma(1-\gamma)\sigma_H^2)(T-t_k)} \tag{4.61}$$

4.3. Summary

We have analyzed the situation of an investor who is allowed to invest in a bond, a stock or a hedge fund, with the bond and the stock being continuously tradeable and the hedge fund requiring a lockup period of given duration. The investor wants to guarantee that consumption is non-decreasing - an extreme form of habit formation. We have provided a framework for obtaining the optimal consumption and investment policies under such a scenario.

It is not expected that such a conservative restriction would ever be implemented. However, by studying this scenario, we essentially have performed a *worst-case* analysis in that weaker restrictions on consumption patterns would result in a decrease in value relative to the classical case that is lower than that in the non-decreasing consumption problem.

Pension funds and university endowments require steady payout streams and must try and avoid major declines in these payouts. Moreover, their portfolios typically also include significant exposure to alternative assets such as hedge funds. The major complication that arises from the inclusion of hedge funds in the investor's portfolio is that they often have a lockup requirement, where the investor is restricted from withdrawing funds from the hedge fund for a pre-specified duration of time. Consequently, the approach laid out here is of immense importance to these institutional investors, whose portfolio structures must account for more complications than ever before.

CHAPTER 5

Summary

In Chapter 2, we have presented a solution method for the discrete-time consumption and investment optimization problem where the consumption rate process is constrained to be non-decreasing.

We first prove that the value function for our problem must have a specific form that depends on an unknown function of a bounded state variable, and then use an iterative procedure to complete the solution.

The structure of the solution is intuitive. Feasibility of the problem is easily checked - does the agent have enough wealth to guarantee herself the current consumption rate by putting all her wealth into the risk-free asset? Once we have feasibility, the agent achieves an optimal outcome as follows:

- At each decision epoch, the agent decides on the current consumption rate based on her current wealth and the consumption rate at the previous step.
- Next, the agent puts an amount in the risk-free asset that is equivalent to the present value of a perpetuity stream with a payout at the current consumption rate.
- Now, based on the current state of the system, a proportion of the remaining wealth is put in the risky asset, with the rest going into the risk-free asset.

The non-decreasing consumption constraint leads to a conservative approach being adopted by the agent which offers significant downside protection without significant losses in certainty-equivalent consumption. The impact of an increase in the time interval between decision epochs (τ) is as expected - an increase in the length of the time interval is seen to lower the *ideal* fraction of wealth consumed as well as decrease the net proportion of wealth invested in the risky asset for any feasible state.

Continuous-time approximations do not work well for investments that only have limited liquidity or for increased lengths of the time interval. This is also true of assets that have negatively skewed leptokurtic risky return distributions. In such cases, the ability of our methodology to work with any reasonable asset return distribution proves very useful. This is especially of appeal to institutional investors in hedge funds that may have both limited trading opportunities and non-normal returns.

We could consider some other form of habit formation but we anticipate results that are very similar to those derived in this paper. Other restrictions that are implemented to smooth consumption are not as conservative a strategy as non-decreasing consumption, thereby the loss of utility from other methods of smoothing consumption can not be more than that from forcing consumption to be non-decreasing. Thus, a broader implication of this paper, when considered in conjunction with Rogers [2001], is that the limitations on trading frequency can have a greater impact in utility terms than smoothing of consumption.

In Chapter 3 we first provide a framework for the calculation of the hedge fund lockup premium and an alternative measure that we refer to as the lockup penalty. Second, we have studied the optimal structure for a portfolio consisting of a bond, a stock and a hedge fund under both terminal wealth and consumption utility considerations. These two scenarios apply to different types of institutional investors, with fund of funds and proprietary traders falling under the no-consumption setting and pension funds and university endowments falling under the utility of consumption setting. Under both scenarios, we account for the availability of equities as an alternative risky investment. While fund of funds do not usually maintain equity positions, that is not a restriction imposed on them due to the hedge fund lockup requirement. Consequently, in computing the lockup premium it is appropriate to account for the presence of equities even if they are not part of the investor's portfolio.

Chapter 3 also introduces the concept of an information premium, as well as a framework to calculate this premium and an alternative measure of the value of hedge fund price information that we refer to as the information penalty. Often hedge fund share prices are available only just before a decision has to be made on adjusting the position in the hedge fund - our framework essentially allows the investor to determine the fair value for obtaining access to the share price information on a continuous basis.

In Chapter 4 we combine the restrictions on consumption and liquidity from Chapters 2 and 3 and provide a framework for obtaining the optimal consumption and investment policies under such a scenario.

Pension funds and university endowments require steady payout streams and must try and avoid major declines in these payouts. Moreover, their portfolios typically also include significant exposure to alternative assets such as hedge funds. The major complication that arises from the inclusion of hedge funds in the investor's portfolio is that they often have a lockup requirement, where the investor is restricted from withdrawing funds from the hedge fund for a pre-specified duration of time. Consequently, the approach described in Chapter 4 is of significant use to these institutional investors, whose portfolio structures must account for more complications than ever before.

References

G. O. Aragon. Share restrictions and asset pricing: evidence from the hedge fund industry. *Journal of Financial Economics*, 83(1):33–58, January 2007.

L. Arnold. *Stochastic differential equations: theory and applications.* John Wiley & Sons, Inc., New York, NY, 1974. ISBN 0471033596.

D. P. Bertsekas. *Dynamic programming and optimal control*, volume I & II. Athena Scientific, Belmont, MA, 2 edition, 1995. ISBN 1886529086.

D. P. Bertsekas and S. E. Shreve. *Stochastic optimal control: the discrete-time case.* Athena Scientific, Nashua, NH, 1978, reprinted 1996. ISBN 1886529035. URL `http://hdl.handle.net/1721.1/4852`.

F. Black and A. F. Perold. Theory of constant proportions portfolio insurance. *Economic Dynamics and Control*, 16:403–426, 1992.

P. Brandimarte. *Numerical methods in finance and economics: A MATLAB-based introduction.* John Wiley & Sons, Inc., New York, NY, 2006. ISBN 0471745030.

C. Brooks and H. M. Kat. The statistical properties of hedge fund index returns and their implications for investors. *ISMA Centre Discussion Papers in Finance*, 2001-09, November 2001.

G. M. Constantinides. Habit formation: a resolution of the equity premium puzzle. *Journal of Political Economy*, 98(3):519–543, June 1990.

E. Derman. A simple model for the expected premium for hedge fund lockups. *Journal of Investment Management*, 5(3):5–15, Third Quarter 2007.

E. Derman, K. Park, and W. Whitt. Markov chain models to estimate the premium for extended hedge fund lockups. Working Paper, 2007. URL `http://www.cfe.columbia.edu/pdf-files/Derman_Park_Whitt_2008.pdf`.

J. S. Duesenberry. *Income, saving, and the theory of consumer behavior*. Harvard University Press, 1949.

P. H. Dybvig. Duesenberry's ratcheting of consumption: Optimal dynamic consumption and investment given intolerance for any decline in standard of living. *The Review of Economic Studies*, 62(2):287–313, April 1995.

J. Heinrich. A guide to the Pearson Type IV distribution. Technical Report Memo 6820, The Collider Detector at Fermilab, Fermilab, Batavia, Illinois, Dec., 2004. URL `http://www-cdf.fnal.gov/publications/cdf6820_pearson4.pdf`.

M. Kahl, J. Liu, and F. A. Longstaff. Paper millionaires: howvaluable is stock to a stockholder who is restricted from selling it? *Journal of Financial Economics*, 67(3): 385–410, March 2003.

D. Kahneman and A. Tversky. Prospect theory: An analysis of of decisions under risk. *Econometrica*, 47(2):276–287, March 1979.

F. A. Longstaff. Optimal portfolio choice and the valuation of illiquid securities. *Review of Financial Studies*, 14(2):407–431, 2001.

R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, August 1969.

J. A. Ohlson and W. T. Ziemba. Portfolio selection in a lognormal market when the investor has a power utility function. *The Journal of Financial and Quantitative Analysis*, 11(1):57–71, March 1976.

L. Rogers. The relaxed investor and parameter uncertainty. *Finance and Stochastics*, 5 (2):131–154, April 2001.

P. A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51(3):239–246, August 1969.

P. A. Samuelson and R. C. Merton. A complete model of warrant pricing that maximizes utility. *Industrial Management Review*, 10(2):17–46, 1969.

APPENDIX A

Proofs of Results in Chapter 2

To prove Theorem 1, we first need to establish some required results. We will return to this proof once we have done so. We continue to use γ^*, r^*, α and $\hat{\alpha}$ as a briefer notation for $\gamma^*(W_n, C_{n-1})$, $r^*(W_n, C_{n-1})$, $\alpha(W_n, C_{n-1})$ and $\hat{\alpha}(W_n, C_{n-1})$ respectively. These still represent functions of the state variables and are not constants.

In some of the following lemmas, we will utilize results from Chapter 5 of Bertsekas and Shreve [1978, reprinted 1996] as well as from Chapters 1 and 3 (Vol. II) of Bertsekas [1995]. The results in these chapters are for a minimization problem - but the problem can be interpreted as a minimization of the negative of its objective function, which is the expected discounted present value of the utility stream. We use the prefix [BS5] to refer to results obtained in Bertsekas and Shreve [1978, reprinted 1996] and the prefix [B1] and [B3] for those obtained in Chapters 1 and 3 of Bertsekas [1995] respectively.

Lemma A-1. *The value function $V^*(\cdot)$ corresponding to the optimal solution to our problem must satisfy the following optimality equation,*

$$\begin{aligned} V(W_n, c_{n-1}) = \max_{\substack{0 \le \alpha_n \le 1 \\ c_n \ge c_{n-1}}} \quad & \tfrac{1-e^{-\delta\tau}}{\delta} u(c_n) \\ + \Delta^{-1} \mathbb{E}_{\tilde{R}} & \left[V(W_n[(1-\alpha_n)R + \alpha_n \tilde{R}_n] - \tfrac{c_n}{r}(R-1), c_n) \middle| W_n, c_{n-1} \right] \end{aligned} \tag{A-1}$$

where $u(\cdot)$, Δ, R and $\tilde{R}$ are as defined before.

Proof: The reward function in this case (written in the style of [BS5]) is

$$g([W_n, c_{n-1}], [c_n, \alpha_n], \tilde{R}) = -\frac{1 - e^{-\delta\tau}}{\delta} u(c_n) = -\frac{1 - e^{-\delta\tau}}{\delta} \frac{1}{\gamma} (c_n)^\gamma \tag{A-2}$$

Thus, since the consumption c_n will always be positive, $g([W_n, c_{n-1}], [c_n, \alpha_n], \tilde{R})$ is positive (negative) if γ is negative (positive). Remember that $\gamma < 1$.

Assume for now that $c_{0-} > 0$. Then, for all feasible states and actions, we have either

$$g([W_n, c_{n-1}], [c_n, \alpha_n], \tilde{R}) \leq 0 \quad \forall \quad n \geq 0 \qquad if \quad \gamma \geq 0 \tag{A-3}$$

or

$$g([W_n, c_{n-1}], [c_n, \alpha_n], \tilde{R}) \geq 0 \quad \forall \quad n \geq 0 \qquad if \quad \gamma < 0 \tag{A-4}$$

Note that the above relationships hold for $c_{0-} = 0$ in the limit.

By proposition ([BS5].5.12), if equation (A-3) holds, then assumptions ([BS5].D), ([BS5].D1) and ([BS5].D2) hold, with the scalar in ([BS5].D2) equal to Δ^{-1}. Similarly, if equation (A-4) holds, then assumptions ([BS5].I), ([BS5].I1) and ([BS5].I2) hold, with the scalar in ([BS5].I2) equal to Δ^{-1}. In either case, all results in ([BS5]) apply to our problem.

Using either proposition ([BS5].5.2) (if $\gamma < 0$) or proposition ([BS5].5.3) (if $\gamma \geq 0$), there exists a function V^* that satisfies the optimality equation

$$V^* = T(V^*) \tag{A-5}$$

where

$$T(V)([W_n, c_{n-1}]) = \max_{\substack{0 \le \alpha_n \le 1 \\ c_n \ge c_{n-1}}} \quad H([W_n, c_{n-1}], [c_n, \alpha_n], V) \tag{A-6}$$

and in turn

$$\begin{aligned} H([W_n, c_{n-1}], [c_n, \alpha_n], V) =& \mathbb{E}_{\tilde{R}} \left[g([W_n, c_{n-1}], [c_n, \alpha_n], \tilde{R}) \right. \\ & \left. + \ \Delta^{-1} V(W_{n+1}(c_n, \alpha_n, \tilde{R}), c_n) \middle| W_n, c_{n-1} \right] \end{aligned} \tag{A-7}$$

Note that the above equations incorporate the reversal from the minimization form of [BS5] to our maximization form. Putting equations (A-2), (A-5), (A-6) and (A-7) together, we obtain

$$V^*(W_n, c_{n-1}) = \max_{\substack{0 \le \alpha_n \le 1 \\ c_n \ge c_{n-1}}} \mathbb{E}_{\tilde{R}} \left[\begin{array}{l} \frac{1-e^{-\delta\tau}}{\delta} u(c_n) \\ +\Delta^{-1} V^*(W_{n+1}(c_n, \alpha_n, \tilde{R}), c_n) \big| W_n, c_{n-1} \end{array} \right] \tag{A-8}$$

Using the independence of $u(c_n)$ from the behavior of the risky asset during the $(n+1)^{th}$ time interval (as characterized by $\tilde{R}$), and the wealth dynamics in equation (2.11), we conclude that the optimality equation (A-1) holds. □

Examining equation (A-1), we can expect that the optimal policies c_n and α_n will be stationary - i.e., the history $\mathcal{H}_n \equiv \{W_0, c_0, \alpha_0, W_1, c_1, \alpha_1, \ldots W_n\}$ impacts the optimal policies only through the current states W_n and c_{n-1}. However, since we are dealing with uncountable state and action spaces and generalizations of dynamic programming results are tricky in these settings, it is prudent to formalize this expected result.

Once we have established the existence of a stationary solution in the following lemma, we switch our notation to the stationary form as well - from here on, w, c, x, α will represent the current wealth, current consumption rate, current ratio and current investment in risky asset respectively, and the suffixes $+$ and $-$ will be used to denote the next-step and previous-step values of these variables. Note that our state variables are w, c_- and x, whereas c and α are our optimal stationary policies, should they exist.

Lemma A-2. *There exists an optimal stationary policy* $[c(w, c_-), \alpha(w, c_-)]$ *defined for all feasible states* (w, c_-)*. Moreover, the optimality equation (A-1) can be rewritten in stationary form as*

$$V(w, c_-) = \max_{\substack{0 \leq \alpha \leq 1 \\ c \geq c_-}} \left[\begin{array}{l} \frac{1-e^{-\delta\tau}}{\delta} u(c) \\ +\Delta^{-1} \mathbb{E}_{\tilde{R}} \left[V\left(w[(1-\alpha)R + \alpha\tilde{R}] - \frac{c}{r}(R-1), c \right) \middle| w, c_- \right] \end{array} \right] \tag{A-9}$$

where (w, c_-) *is any feasible state. The value function* $V_s(\cdot)$ *generated by the optimal stationary policy satisfies the optimality equation (A-9) and consequently, also equation (A-1). We therefore have* $V_s(\cdot) = V^*(\cdot)$

Proof: For all feasible states (w, c_-), there is an optimal policy in the continuous-time framework; with finite time intervals between decision epochs, we can only do worse. Also, the action space is bounded. So for all feasible states, there exists an optimal policy. Hence proposition [BS5].5.4 applies. When $\gamma < 0$, we know that assumptions ([BS5].I), ([BS5].I1) and ([BS5].I2) hold and we can use proposition ([BS5].5.4) to state that some stationary policy $[c(w, c_-), \alpha(w, c_-)]$ defined for all feasible states (w, c_-) achieves optimality.

An equivalent general result is not available under assumptions ([BS5].D), ([BS5].D1) and ([BS5].D2). Fortunately, we need not get bogged down in the details, and a brief explanation will suffice - in the equivalent general problem we would be minimizing unbounded negative rewards (or maximizing unbounded positive rewards, as is the case in our problem for $\gamma > 0$) and counterexamples exist to show that an equivalent general result proving the existence of optimal stationary solutions cannot be obtained. Consequently, we need to obtain the desired result for our problem in a somewhat convoluted manner.

The basic idea is to convert the problem from one that maximizes the discounted sum of positive rewards to one that maximizes the discounted sum of negative rewards. We know that the continuous-time solution would obtain a higher value for any state. Therefore, if we define the discrete-time rewards as the excess over those in continuous-time for any given state, we are now trying to maximize negative rewards.

To make this a little more formal, given state $y_n \equiv [W_n, c_{n-1}]$, the continuous-time value is $V_c(y_n)$. We look at each sample path ω that goes from time n to $n+1$. If we follow the continuous-time policy, suppose we earn $r_c(y_n, \omega)$ and land in state $y_c(y_n, \omega)$. Let

$$\bar{r_c}(y_n) = \mathbb{E}_\omega \left[r_c(y_n, \omega) + \Delta^{-1}[V_c(y_c(y_n, \omega)) - V_c(y_{n+1} \middle| y_n, \mu_n^*)] \right] \tag{A-10}$$

where μ_n^* is the optimal policy for the discrete-time problem at time n. We know that we cannot do better than earning this reward in the discrete-time case before the next rebalancing because, if we did, we would then exceed the continuous-time optimal value in the limit. The discrete-time problem is equivalent to a problem to maximize $V(y) - V_c(y)$.

Now, the rewards are all negative, and we are trying to maximize their discounted sum. Assumptions ([BS5].I), ([BS5].I1) and ([BS5].I2) now hold, and again we use proposition ([BS5].5.4) to state that some stationary policy $[c(w, c_{-}), \alpha(w, c_{-})]$ defined for all feasible states (w, c_{-}) achieves optimality.

Thus, for both cases we have an optimal stationary policy - consequently, the value function $V_s(\cdot)$ generated by this stationary policy must satisfy equation (A-1). Consequently, $V_s(\cdot) = V^*(\cdot)$.

Finally, equation (A-9) is clearly the stationary form of the optimality equation (A-1) and holds because of the existence of an optimal stationary policy. □

In general, it is not necessary that there will be a unique fixed point of the operator T (see Example [B1].1.2), so we need to formally establish its uniqueness. This would allow us to declare any stationary policy that generates a fixed point of the operator T to be an optimal policy.

Lemma A-3. *If the value function $V_s(\cdot)$ corresponding to an optimal stationary policy is of the form $V_s(c_{-}, w) = U_s(\frac{c_{-}}{w})w^{\gamma}$ where $U_s(\cdot)$ is a bounded function for all feasible values of $\frac{c_{-}}{w}$, then $V_s(\cdot)$ is a unique solution of the optimality equation (A-9) - i.e., $V_s(\cdot)$ is a unique fixed point of the operator T. Consequently, any stationary policy that generates a value function that is of the above form and is a fixed point of T must be an optimal stationary policy.*

Proof: In lemma A-2 we have shown that the problem under assumptions ([BS5].D), ([BS5].D1) and ([BS5].D2) can be converted to a problem under assumptions ([BS5].I), ([BS5].I1) and ([BS5].I2). Consequently, we only show the uniqueness of the solution $V^*(\cdot)$ when assumptions ([BS5].I), ([BS5].I1) and ([BS5].I2) hold.

In this case, proposition [B1].1.2.3 tells us that a stationary policy $[c_s^*, \alpha_s^*]$ is optimal if and only if it solves $TV_s = T_{[c_s^*,\alpha_s^*]} V_s$ where $T_{[c_s^*,\alpha_s^*]}$ is defined as

$$\begin{aligned} & T_{[c_s^*,\alpha_s^*]}(V)([w, c_-]) \\ = \quad & \mathbb{E}_{\tilde{R}} \left[\begin{array}{l} g([w, c_-], [c_s^*(w, c_-), \alpha_s^*(w, c_-)], \tilde{R}) \\ +\Delta^{-1} V(w_+(c_s^*(w, c_-), \alpha_s^*(w, c_-), \tilde{R}), c(w, c_-)) \big| w, c_- \end{array} \right] \end{aligned} \tag{A-11}$$

¿From lemma A-1, we know that a solution V^* to $TV = V$ exists. We now need to show that V^* is also the V_s that solves $TV = T_{[c_s^*,\alpha_s^*]} V$. Suppose $V_s(c_-, w) = U_s(\frac{c_-}{w}) w^\gamma$, and we have a solution $V^*(c_-, w) = U^*(\frac{c_-}{w}) w^\gamma = TV^*$ where $U^*(\frac{c_-}{w})$ is bounded. Now, we can use Exercise [B3].3.5.

U^* being bounded implies that there exists $0 < \lambda < \infty$ such that $U_s(\frac{c_-}{w}) + \lambda \geq U^*(\frac{c_-}{w})$. Now, we also have

$$T\left(\left(U_s\left(\frac{c_-}{w}\right) + \lambda\right) w^\gamma\right) \geq T\left(U^*\left(\frac{c_-}{w}\right) w^\gamma\right) = V^*$$

where

$$T\left(\left(U_s\left(\frac{c_-}{w}\right) + \lambda\right) w^\gamma\right) = V_s + \Delta^{-1} \lambda w^\gamma$$

Doing this n times yields

$$V_s + \Delta^{-n} \lambda w^\gamma \geq V^*$$

or, in the limit, $V_s \geq V^*$. Therefore, $V^* = V_s$. Consequently, T has a unique fixed point, and if the value function generated by a stationary policy solves $TV = V$, then that stationary policy must be optimal. □

Lemma A-4. *$V^*(\cdot)$ is continuous in the current state variables (w, c_-) if the consumption policy c_n is continuous in (w_n, c_{n-1}).*

Proof: The continuity of the function V^* follows from its representation as the expected discounted present value at decision epoch 0 of the utility stream under the optimal policy

$$V^*(w, c_-) = \mathbb{E}_0^{\mu^*}\left[\sum_{n=0}^{\infty}(\Delta^{-1})^n u\left(c_n(w_n, c_{n-1})\right)\middle| w_0 = w, c_{-0} = c_-\right] \tag{A-12}$$

Clearly, if $u(\cdot)$ is continuous in c_n and the consumption policy c_n is itself continuous in (w_n, c_{n-1}), we have that $V^*(\cdot)$ is continuous in the current state variables (w, c_-). □

Lemma A-5. *The function V^* that solves (A-9) takes values in $\left(\frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma, 0\right)$ if $\gamma < 0$ and in $\left(\frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma, \infty\right)$ if $0 \leq \gamma < 1$.*

Proof: V^* is real valued. Given any feasible state, any policy that:

- produces a feasible consumption stream, and,
- for which one can explicitly calculate the value of the right hand side of the optimality equation, (A-9)

immediately gives us a lower bound on the actual value of V^* for that state.

Given any feasible state $(w, x) \in \mathcal{S}$, if the agent puts all her wealth into a consumption perpetuity, then the resulting consumption stream of rw at every decision epoch is feasible;

we can explicitly calculate the value of the right hand side of equation (A-9) as the present value of the utility perpetuity, since that is the only source of value. This present value is $\frac{1}{\delta}u(rw) = \frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma$. Therefore, $\frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma$ is a lower bound on the value function V^* for any feasible state $(w, x) \in \mathcal{S}$.

Moreover, this lower bound is not achievable if the risky asset is attractive enough - the precise condition being $\mu > \delta$. To see this, consider the scenario where $\lambda > 0$ (small) is put into the risky asset at the current decision epoch, and the rest is put into the consumption perpetuity. At the next decision epoch, the entire wealth is moved into the consumption perpetuity. Then, the value obtained from this strategy will be

$$V_\lambda(w, c_-) = \left(\frac{1-\Delta^{-1}}{\delta}\right)\frac{(r(w-\lambda))^\gamma}{\gamma} + \frac{\Delta^{-1}}{\delta}\mathbb{E}_{\tilde{R}}\left[\frac{1}{\delta}\frac{\left(r(\lambda\tilde{R}+w-\lambda)\right)^\gamma}{\gamma}\right]$$

Then we have

$$\begin{aligned}\tfrac{\partial V_\lambda(w,c_-)}{\partial \lambda} &= -\left(\tfrac{1-\Delta^{-1}}{\delta}\right) r^\gamma (w-\lambda)^{\gamma-1} \\ &\quad + \tfrac{\Delta^{-1}}{\delta} r^\gamma \mathbb{E}_{\tilde{R}}\left[(\tilde{R}-1)\left(w+\lambda(\tilde{R}-1)\right)^{\gamma-1}\right]\end{aligned}$$

and, for λ close to 0

$$\frac{\partial V_\lambda(w,c_-)}{\partial \lambda} \approx -\left(\frac{1-\Delta^{-1}}{\delta}\right) r^\gamma w^{\gamma-1} + \frac{\Delta^{-1}}{\delta} r^\gamma \mathbb{E}_{\tilde{R}}\left[(\tilde{R}-1)w^{\gamma-1}\right]$$

For $\frac{\partial V_\lambda(w,c_-)}{\partial \lambda}$ to be positive (thereby implying increasing value with an increase in λ from 0), we end up with the simple condition that $\mu > \delta$. Intuitively, the risky asset needs to have an expected return rate greater than the pure rate of time preference for it to be attractive enough to warrant investment. We will assume this to be the case, since otherwise the problem is trivial and the optimal strategy is to put everything into the consumption perpetuity.

Now we move to the upper bound - V^* represents the objective function of our optimization problem, which is a discounted sum of utility obtained at each decision epoch. If $\gamma < 0$ then the utility at each step will be negative, and consequently so will V^*. This automatically places an upper bound of 0 on V^*. If $0 \leq \gamma < 1$ then the utility at each step will be positive and the upper bound on V^* stays at $+\infty$. Note that the lower bound is also positive in this case. In either case, to achieve the upper bound, the consumption rate will have to be infinite, which is infeasible given a finite initial wealth.

Therefore, given state $(w, x) \in \mathcal{S}$, V^* must take some real value in $\left(\frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma, 0\right)$ if $\gamma < 0$ and in $\left(\frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma, \infty\right)$ if $0 \leq \gamma < 1$. □

Lemma A-6. *The function*

$$V_{\gamma^*}^{c,\alpha}(w, c_-) = \frac{1}{\delta}u(c) + \frac{r}{\delta}\left(\frac{1}{r^*} - \frac{1}{r}\right)^{1-\gamma^*} c^\gamma \left(\frac{w}{c} - \frac{1}{r}\right)^{\gamma^*} \tag{A-13}$$

where $\gamma^* \in (\gamma, 1)$ *is an appropriate form for the value function at any feasible state* (w, c_-) *given a stationary policy* $[c(w, c_-), \alpha(w, c_-)]$.

Consequently, the function $V_{\gamma^*}(w, c_-)$ *as defined in equation (2.19) is an appropriate form of the value function for the policy defined in equations (2.21), (2.22), (2.24), (2.25) and (2.26).*

Proof: Given any stationary policy $[c(w, c_-), \alpha(w, c_-)]$, we know that every feasible solution guarantees a consumption stream of at least c. This gives us an immediate lower bound on the value function as $\frac{1}{\delta}u(c)$, which is the first term in the definition of $V^{c,\alpha}_{\gamma^*}(w, c_-)$.

The only other source of value in our problem is the potential for an increase in the guaranteed consumption amount. The contribution from this source can only be positive. It can easily be shown that the term $\frac{r}{\delta}\left(\frac{1}{r^*} - \frac{1}{r}\right)^{1-\gamma^*} c^{\gamma}\left(\frac{w}{c} - \frac{1}{r}\right)^{\gamma^*}$ is always positive. Moreover, for any value increment in $(0, \infty)$, we can find a $\gamma^* \in (\gamma, 1)$ such that this term exactly matches the increment. Therefore, $V^{c,\alpha}_{\gamma^*}(w, c_-)$ as defined in equation (A-13) is a legitimate representation of the value function corresponding to any stationary policy $[c(w, c_-), \alpha(w, c_-)]$.

The consumption is modeled in equations (2.24), (2.25) and (2.26) as a regulated process. This is equivalent to the consumption rule

$$c = max(c_-, r^* w) \tag{A-14}$$

as utilized in the function $V_{\gamma^*}(w, c_-)$. Plugging in the above (equation (A-14)) consumption policy into the definition of $V^{c,\alpha}_{\gamma^*}(w, c_-)$ (equation (A-13)), we see that the function $V_{\gamma^*}(w, c_-)$ as defined in equation (2.19) is an appropriate form of the value function for the policy defined in equations (2.21), (2.22), (2.24), (2.25) and (2.26). □

Lemma A-7. *The function $V_{\gamma^*}(w, c_-)$ as defined in equation (2.19) can be restated in terms of the wealth w and a function $U_{\gamma^*}(x)$ of the ratio state variable x.*

Proof: Rewriting equation (2.19), we see that

$$\begin{aligned} V_{\gamma^*}(w, c_-) =& w^\gamma u\left(max\left(\frac{c_-}{w}, r^*\right)\right) \\ &+ w^\gamma \frac{r}{\delta}\left(\frac{1}{r^*} - \frac{1}{r}\right)^{1-\gamma^*}\left(max\left(\frac{c_-}{w}, r^*\right)\right)^\gamma \left(\frac{1}{max\left(\frac{c_-}{w}, r^*\right)} - \frac{1}{r}\right)^{\gamma^*} \end{aligned}$$

Using equation (2.13), we have

$$V_{\gamma^*}(w, c_-) = V_{\gamma^*}(w, x) = w^\gamma U_{\gamma^*}(x) \tag{A-15}$$

where

$$\begin{aligned} U_{\gamma^*}(x) =& \frac{1}{\delta} u\left(max\left(x, r^*\right)\right) \\ &+ \frac{r}{\delta}\left(\frac{1}{r^*} - \frac{1}{r}\right)^{1-\gamma^*}\left(max\left(x, r^*\right)\right)^\gamma \left(\frac{1}{max\left(x, r^*\right)} - \frac{1}{r}\right)^{\gamma^*} \end{aligned} \tag{A-16}$$

□

Lemma A-8. *For any given feasible state (w, c_-) (or equivalently, $(w, x) \in \mathcal{S}$), the function $V_{\gamma^*}(w, c_-)$ as defined in equation (2.19) is monotonically decreasing in γ^* for $\gamma^* \in (\gamma, 1)$.*

Proof: We know that we can rewrite $V_{\gamma^*}(w, c_-)$ as

$$V_{\gamma^*}(w, c_-) = V_{\gamma^*}(w, x) = w^\gamma U_{\gamma^*}(x)$$

where $U_{\gamma^*}(x)$ is obtained from the definition (A-16). Assuming we started with the non-trivial case of $W_0 > 0$, we will have $w > 0$ and consequently $w^\gamma > 0$ (constant, since we are given a particular state). To show that $V_{\gamma^*}(w, c_-)$ is monotonic decreasing in γ^*

for $\gamma^* \in [\gamma, 1]$, it is enough to show that this property holds for $U_{\gamma^*}(x)$. We know that $x \in [0, r]$. Consider the case where $x \in [0, r^*]$. Then

$$U_{\gamma^*}(x) = \frac{1}{\delta}\frac{(r^*)^\gamma}{\gamma} + \frac{r}{\delta}\left(\frac{1}{r^*} - \frac{1}{r}\right)^{1-\gamma^*}(r^*)^\gamma\left(\frac{1}{r^*} - \frac{1}{r}\right)^{\gamma^*}$$

Using equation (2.20), we have

$$U_{\gamma^*}(x) = \frac{1}{\delta}\frac{r^\gamma}{\gamma}(1-\gamma)^{1-\gamma}\gamma^*(\gamma^* - \gamma)^{\gamma-1}$$

Taking the partial derivative of $U_{\gamma^*}(x)$ with respect to γ^*, we obtain

$$\frac{\partial U_{\gamma^*}(x)}{\partial \gamma^*} = \frac{1}{\delta}r^\gamma(1-\gamma)^{1-\gamma}(\gamma^* - \gamma)^{\gamma-2}(\gamma^* - 1) \tag{A-17}$$

which is strictly negative for $\gamma^* \in (\gamma, 1)$. Now consider the case where $x \in (r^*, r]$. Then

$$U_{\gamma^*}(x) = \frac{1}{\delta}\frac{x^\gamma}{\gamma} + \frac{r}{\delta}\left(\frac{1}{r^*} - \frac{1}{r}\right)^{1-\gamma^*} x^\gamma\left(\frac{1}{x} - \frac{1}{r}\right)^{\gamma^*}$$

Using equation (2.20), we have

$$U_{\gamma^*}(x) = \frac{1}{\delta}x^\gamma\left[\frac{1}{\gamma} + \left(\frac{1-\gamma^*}{\gamma^* - \gamma}\right)^{1-\gamma^*}\left(\frac{r}{x} - 1\right)^{\gamma^*}\right]$$

Taking the partial derivative of $U_{\gamma^*}(x)$ with respect to γ^*, we obtain

$$\frac{\partial U_{\gamma^*}(x)}{\partial \gamma^*} = \frac{1}{\delta}x^\gamma\left(\frac{1-\gamma^*}{\gamma^* - \gamma}\right)^{1-\gamma^*}\left(\frac{r}{x} - 1\right)^{\gamma^*} ln\left[\frac{\left(\frac{r}{x} - 1\right)}{\left(\frac{1-\gamma^*}{\gamma^*-\gamma}\right)}e^{-\left(\frac{1-\gamma}{\gamma^*-\gamma}\right)}\right] \tag{A-18}$$

Now, since $x \in (r^*, r]$

$$\frac{1}{\delta} x^\gamma \left(\frac{1-\gamma^*}{\gamma^*-\gamma}\right)^{1-\gamma^*} \left(\frac{r}{x}-1\right)^{\gamma^*}$$

is strictly positive for $\gamma^* \in (\gamma, 1)$, and

$$\sup_{x\in(r^*,r]} \; ln\left[\frac{\left(\frac{r}{x}-1\right)}{\left(\frac{1-\gamma^*}{\gamma^*-\gamma}\right)} e^{-\left(\frac{1-\gamma}{\gamma^*-\gamma}\right)}\right] = ln\left[\frac{\left(\frac{r}{r^*}-1\right)}{\left(\frac{1-\gamma^*}{\gamma^*-\gamma}\right)} e^{-\left(\frac{1-\gamma}{\gamma^*-\gamma}\right)}\right] = -\left(\frac{1-\gamma}{\gamma^*-\gamma}\right)$$

which is strictly negative for $\gamma^* \in (\gamma, 1)$. Thus, for all $x \in (r^*, r]$ we have $\frac{\partial U_{\gamma^*}(x)}{\partial \gamma^*} < 0$ where $\gamma^* \in (\gamma, 1)$ and consequently, both $U_{\gamma^*}(x)$ and $V_{\gamma^*}(w, c_-)$ are monotonically decreasing in γ^* for $\gamma^* \in (\gamma, 1)$ □

Lemma A-9. *If $\gamma < 0$, for any possible value of V^* in $\left(\frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma, 0\right)$, there is a* unique *$\gamma^* \in (0, 1)$ such that the proposed form of the value function $V_{\gamma^*}(w, c_-)$ takes the same value as V^*.*

If $0 \leq \gamma < 1$, for any possible value of V^* in $\left(\frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma, \infty\right)$, there is a unique *$\gamma^* \in (\gamma, 1)$ such that the proposed form of the value function $V_{\gamma^*}(w, c_-)$ takes the same value as V^*.*

Moreover, $\gamma^*(w, c_-)$ is continuous in the current state variables (w, c_-).

Proof: ¿From equation (2.19), we can see that for a given feasible state (w, c_-), $V_{\gamma^*}(w, c_-)$ is continuous in γ^* for $\gamma^* \in (\gamma, 1)$.

We have also shown (Lemma A-8) that for any feasible state (w, c_-), $V_{\gamma^*}(w, c_-)$ is monotonically decreasing in γ^* for $\gamma^* \in (\gamma, 1)$.

Finally, we have

$$\lim_{\gamma^* \downarrow \gamma} V_{\gamma^*}(w, c_-) = +\infty \tag{A-19}$$

$$\lim_{\gamma^* \uparrow 1} V_{\gamma^*}(w, c_-) = \frac{1}{\delta}\frac{1}{\gamma}(rw)^\gamma \tag{A-20}$$

and

$$\lim_{\gamma^* \to 0} V_{\gamma^*}(w, c_-) = 0 \tag{A-21}$$

These limits are the same as the bounds established on V^* in Lemma A-5 for the cases $0 \leq \gamma < 1$ and $\gamma < 0$.

The continuity and monotonicity of $V_{\gamma^*}(w, c_-)$ in γ^* along with the above limits give us the existence of a unique γ^* that matches V^* and V_{γ^*} for every possible value of V^*.

The consumption rule (A-14) is in a continuous form. The continuity of $\gamma^*(w, c_-)$ follows from the resulting continuity of $V^*(w, c_-)$ (as shown in Lemma A-4), and the continuity of V_{γ^*} in γ^*. $\square$

Proof of Theorem 1: In Lemma A-6, we have shown that $V_{\gamma^*}(w, c_-)$ is a legitimate form of the value function. Lemma A-2 and Lemma A-9 together show the existence of a function $\gamma^*(\cdot)$ that satisfies the Bellman equation (2.18). This specific form of the Bellman equation is obtained in Lemma A-2. The range of values taken by $\gamma^*(\cdot)$ are established in Lemma (A-9).

As noted in Lemma A-6, the consumption is modeled in equations (2.24), (2.25) and (2.26) as a regulated process. This is equivalent to the consumption rule,

$$c_n = max(c_{n-1}, r^*W_n) \tag{A-22}$$

as utilized in the value function. Once we have computed the $\gamma^*(\cdot)$ function, equations (2.20) and (2.22) provides us with the equivalent $r^*(\cdot)$ and $\hat{\alpha}(\cdot)$ functions over the feasible values of $\{X_n\}$, i.e., $[0, r]$.

Given a particular value of X_n at a particular decision epoch, we can obtain the appropriate values of γ^*, r^* and $\hat{\alpha}$ for that decision epoch. Equation (2.21) then allows to obtain the corresponding value of α.

Thus, the solution to the discrete-time problem (Problem 1) is of the form proposed in Theorem 1. □

Proof of Theorem 2: Using equations (2.18) and (A-15), the Bellman equation can be written as

$$w^\gamma U_{\gamma^*}(x) = \frac{1}{\gamma}\frac{1-e^{-\delta\tau}}{\delta}(max(c_-, r^*w))^\gamma + \Delta^{-1}\mathbb{E}_{\tilde{R}}\left[w_+^\gamma U_{\gamma^*}(x_+)\middle| w, x\right]$$

Since w is known, dividing the above equation by w^γ gives us

$$U_{\gamma^*}(x) = \frac{1}{\gamma}\frac{1-e^{-\delta\tau}}{\delta}\left(max\left(\frac{c_-}{w}, r^*\right)\right)^\gamma + \Delta^{-1}\mathbb{E}_{\tilde{R}}\left[\left(\frac{w_+}{w}\right)^\gamma U_{\gamma^*}(x_+)\middle| w, x\right] \tag{A-23}$$

Also, from equation (2.11) and $c = max(c_-, r^*w) = w max\left(\frac{c_-}{w}, r^*\right)$, we obtain

$$\frac{w_+}{w} = \left(1 - max\left(\frac{c_-}{w}, r^*\right)\right)\left[(1-\alpha)R + \alpha\tilde{R}\right] \tag{A-24}$$

Putting equations (A-23) and (A-24) together and using $x = \frac{c_-}{w}$, we can write the Bellman equation as

$$\begin{aligned} U_{\gamma^*}(x) = \ & \frac{1}{\gamma}\frac{1-e^{-\delta\tau}}{\delta}\left(max(x,r^*)\right)^{\gamma} + \\ & \Delta^{-1}\mathbb{E}_{\tilde{R}}\left[\Big((1-max(x,r^*))\left[(1-\alpha)R+\alpha\tilde{R}\right]\Big)^{\gamma}U_{\gamma^*}(x_+)\Big|w,x\right] \end{aligned} \tag{A-25}$$

Finally, we show that the dynamics of x are independent of the wealth w . Applying equation (2.27) to the state (w, c_-) we obtain

$$w_+ = \left(w - \frac{c}{r}\right)\left[(1-\hat{\alpha}(w,c_-))R + \hat{\alpha}(w,c_-)\tilde{R}\right] + \frac{c}{r} \tag{A-26}$$

Dividing by c, and using $c = max(c_-, r^* w)$, $x = \frac{c_-}{w}$, we obtain

$$x_+ = \frac{1}{\left(\frac{1}{max(x,r^*)} - \frac{1}{r}\right)\left[(1-\hat{\alpha})R + \hat{\alpha}\tilde{R}\right] + \frac{1}{r}} \tag{A-27}$$

We see that the Bellman equation can be stated purely in terms of x as in equation (A-25), and the dynamics of x are independent of the wealth process. □