NORTHWESTERN UNIVERSITY

Generation in Geometric Derived Categories

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

Yaroslav Khromenkov

EVANSTON, ILLINOIS

September 2023

ABSTRACT

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Yaroslav Khromenkov

The main topic of this thesis is generation in derived categories of coherent sheaves on smooth projective varieties. We develop a new approach that allows us to give a new proof of a recent result by Olander that powers of an ample line bundle generate the bounded derived category of coherent sheaves on a smooth projective variety X of dimension n in n steps, we also provide an effective bound on the power of the ample line bundle needed to generate the bounded derived category of coherent sheaves on X in 2n - 1 steps. We also show that for a smooth projective toric variety X of dimension n over an arbitrary algebraically closed field, the Rouquier dimension of the bounded derived category of coherent sheaves on X is less or equal than 2n - 1. We also study derived categories of coherent D-modules on smooth projective varieties. We describe the subcategory of proper objects in the bounded derived category of coherent D-modules on a smooth projective variety X, and as a consequence we obtain that several geometric invariants of X are determined by the bounded derived category of coherent D-modules on X.

Acknowledgements

I would like to thank my advisor Mihnea Popa for his invaluable help with my research, enlightening discussions, patience and kindness. I could not wish for a better advisor. Without his continuous support I would have not been able to achieve the progress that I have made. Most of the problems and projects that are presented in this thesis were proposed by him.

I express my gratitude to my co-advisor Dmitry Tamarkin. His comments and advice advanced my understanding of several research areas presented in this thesis.

I thank my teachers Alexey Bondal and Sergey Galkin from HSE University who first introduced me to the fascinating world of algebraic geometry.

Lastly, I thank my friends and family for their never ending support.

Table of Contents

CHAPTER 1

Introduction

Our main focus in this thesis is the study of the bounded derived category of coherent sheaves on a smooth projective variety. Bondal and Van den Bergh defined the notion of strong generator in a triangulated category. Informally, an object G in a triangulated category $\mathcal{T}$ is called a strong generator if there is a constant k such that every object $\mathscr{F}^\bullet \in \mathcal{T}$ can be constructed from G using no more than k cones. Rouquier introduces the notion of dimension for an abstract triangulated category $\mathcal{T}$ as the smallest number n such that there is a strong generator $G \in \mathcal{T}$ that generates $\mathcal{T}$ in n steps. One can think of the Rouquier dimension for triangulated categories as an analogue of the global dimension for abelian categories. Orlov conjectured that the Rouquier dimension of the derived category of coherent sheaves on a smooth projective variety X is equal to the dimension of X.

We prove several results on generation in derived categories of coherent sheaves on a smooth projective variety related to the Rouquier dimension and Orlov's conjecture. Our main tool is the study of the obstruction morphism that arises from a resolution of the structure sheaf of the diagonal sheaf $\mathscr{O}_\Delta$. Existence of a short resolution of $\mathscr{O}_\Delta$ by external products of sheaves is a common way of proving bounds on the Rouquier dimension of X. However, we use a different approach that does not require as much from the resolution. In the second chapter we remind several technical lemmas on derived categories of coherent sheaves and generation in triangulated categories.

In the third chapter for a fixed resolution of the diagonal sheaf by external products of sheaves and a fixed natural number k we define the obstruction morphism $\phi_{\mathscr{C}^\bullet,k}$ for every object $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$, which is an element of a vector space that we call the obstruction space of $\mathscr{C}^\bullet$. We show how to generate $\mathscr{C}^\bullet$ in $2\dim X - k$ steps given that $\phi_{\mathscr{C}^\bullet,k}$ vanishes. After that we construct several classes of coherent sheaves for which the vanishing of the obstruction morphism is easy to prove due to the vanishing of the obstruction space. Then we give a criterion for when an extension of two objects with the vanishing obstruction morphism results in an object with the vanishing obstruction morphism. This result will allow us to show that sometimes the vanishing of the obstuction morphism of a complex follows from the vanishing of the obstruction morphisms of all of its cohomology sheaves. In the final section of the third chapter we prove that powers of a fixed ample line bundle generate $\mathbf{D}^b_{coh}(X)$ in $\dim X$ steps, one can think of this theorem as a weaker version of Orlov's conjecture, where a countable collection of objects is allowed to be a generating set (instead of a finite collection in the classical case). This result was recently proved by Olander [**17**] using a different method.

In the fourth chapter we give a criterion for the vanishing of the obstruction morphism of a sheaf $\mathscr{F}$ in terms of the properties of the Harder-Narasimhan filtration of $\mathscr{F}$. We show that the required property is satisfied for pullbacks of $\mathscr{F}$ if we take a finite polarized endomorphism of sufficiently high degree. This allows us to prove that the Rouquier dimension of a toric variety X is less or equal than $2\dim X - 1$. Recently it was shown that a better bound of $2\dim X - 3$ holds for toric varieties over the field of complex numbers [**3**]. The bound for toric varieties over the complex numbers was improved to $\dim X$ [**8**] this year. Both of the mentioned results require the use of mirror symmetry and powerful theorems from symplectic geometry. Our method works over an algebraically closed field of arbitrary characteristic. Moreover, we obtain several results on generation for arbitrary smooth projective variety that do not improve the bound on the Rouquier dimension but are of interest on their own.

In the fifth chapter we study the bounded derived category of coherent D-modules on a smooth projective variety. We show that the subcategory generated by integrable connections is a derived invariant, and that this subcategory has a Serre functor. Using this result we show that the bounded derived category of coherent D-modules on X determines dimension of X and whether or not X is simply connected. Moreover, we prove that simply connected smooth projective varieties with equivalent bounded derived categories of coherent D-modules have equal Betti numbers.

CHAPTER 2

Preliminaries

2.1. Derived categories of coherent sheaves

One of the most important invariants of an algebraic variety X is the bounded derived category of coherent sheaves on X denoted by $\mathbf{D}^b_{coh}(X)$. We treat $\mathbf{D}^b_{coh}(X)$ as a triangulated category. In this section we first recall some basic definitions and well-known results about bounded derived categories of coherent sheaves. The reference for all definitions and results in this section is [**10**].

All varieties are always assumed to be defined over an algebraically closed field $\mathbb{K}$. Throughout this text for a morphism of schemes $f : X \to Y$ by f_* and f^* we always denote the corresponding derived functors of pushforward and pullback

$$f_*, f^* : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(Y).$$

For two objects $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in \mathbf{D}^b_{coh}(X)$ we always denote by $\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet$ the derived tensor product of $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$. For varieties X and Y we denote the projections $X \times Y \to X$ and $X \times Y \to Y$ by π_X and π_Y respectively, and for objects $\mathscr{C}^\bullet_X \in \mathbf{D}^b_{coh}(X), \mathscr{C}^\bullet_Y \in \mathbf{D}^b_{coh}(Y)$ we denote $\pi_X^*\mathscr{C}^\bullet_X \otimes \pi_Y^*\mathscr{C}^\bullet_Y$ by $\mathscr{C}^\bullet_X \boxtimes \mathscr{C}^\bullet_Y$ and call it the box product of $\mathscr{C}^\bullet_X$ and $\mathscr{C}^\bullet_Y$. For a variety X we denote by $\mathscr{O}_\Delta \in \mathbf{D}^b_{coh}(X \times X)$ the structure sheaf of the diagonal that we treat as a complex concentrated in degree 0. We will often use Fourier-Mukai functors between derived categories of coherent sheaves.

Definition 2.1. Given an object $\mathscr{K}^\bullet \in \mathbf{D}^b_{coh}(X \times Y)$, Fourier-Mukai functor with the kernel $\mathscr{K}^\bullet$ is a functor $\Phi_{\mathscr{K}^\bullet} : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(Y)$ given by $\Phi_{\mathscr{K}^\bullet}(\mathscr{C}^\bullet) = \pi_{Y*}(\mathscr{K}^\bullet \otimes \pi_X^* \mathscr{C}^\bullet)$.

For a Fourier-Mukai functor with the kernel that is a box product we have the following formula.

Lemma 2.2. *For $\mathscr{K}^\bullet = \mathscr{F} \boxtimes \mathscr{G}$ and $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$ we have that*

$$\Phi_{\mathscr{K}^\bullet}(\mathscr{C}^\bullet) = R\Gamma(\mathscr{C}^\bullet \otimes \mathscr{F}) \otimes \mathscr{G}.$$

Proof. Clear from the projection formula. □

The identity functor is an example of a Fourier-Mukai functor as shown by the following lemma. [**10**, Examples 5.4].

Lemma 2.3. $\Phi_{\mathscr{O}_\Delta}$ *is naturally isomorphic to the identity functor.*

We say that a complex $\mathscr{C}^\bullet$ is concentrated in degrees below d (above d) if $\mathscr{H}^i(\mathscr{C}^\bullet) = 0$ for all $i \geq d$ ($i \leq d$).

Lemma 2.4. *Given a complex $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$ concentrated in degrees below d and $\mathscr{K}^\bullet \in \mathbf{D}^b_{coh}(X \times X)$ concentrated in degrees below d' we have that $\Phi_{\mathscr{K}^\bullet}(\mathscr{C}^\bullet)$ is concentrated in degrees below $d + d' + \dim X$.*

Proof. Clear from the definition of Fourier-Mukai functors. □

Lemma 2.5. *Given $\mathscr{K}^\bullet \in \mathbf{D}^b_{coh}(X \times X)$ concentrated in degrees less or equal than d of the form*

$$\mathscr{K} = \{... \to \mathscr{F}_{d-2} \boxtimes \mathscr{G}_{d-2} \to \mathscr{F}_{d-1} \boxtimes \mathscr{G}_{d-1} \to \mathscr{F}_d \boxtimes \mathscr{F}_d\}$$

and $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$*, denote by* m *the minimum* i *such that*

$$\mathbb{H}^{k+j}(\mathscr{F}_{d-j} \otimes \mathscr{C}^\bullet) = 0$$

for all $j \geq 0$ *and* $k \geq i$*. Then* $\Phi_{\mathscr{K}^\bullet}(\mathscr{C}^\bullet)$ *is concentrated in degrees below* $m + d$*.*

Proof. Clear from the fact that $\Phi_{\mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet}(\mathscr{C}^\bullet) = R\Gamma(\mathscr{C}^\bullet \otimes \mathscr{F}^\bullet) \otimes \mathscr{G}^\bullet$. $\square$

The following well-known lemma ([**25**], Lemma 13.4.8) is often useful to show that a morphism vanishes in $\mathbf{D}^b_{coh}(X)$.

Lemma 2.6. *Consider a morphism of distinguished triangles*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi_a"] & B \arrow[r] \arrow[d, "\phi_b"] & C \arrow[r, dashed] \arrow[d, "\phi_c"] & A[1] \arrow[d, "\phi_a[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r, dashed] & A'[1] \end{tikzcd}$$

in which $\phi_a = 0$ *and* $\phi_b = 0$*. If* $\mathrm{Hom}(C, C') = \mathrm{Hom}(C, A') = 0$*, then* $\phi_b = 0$*.*

Serre functors are a useful tool in the study of triangulated categories.

Definition 2.7. An equivalence $S : \mathcal{T} \to \mathcal{T}$ is a Serre functor if there exist isomorphisms $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, S(A))^*$ functorial in A and B.

Existence of a Serre functor for the bounded derived category of coherent sheaves on a smooth projective variety is given by the following lemma [**10**, Theorem 3.12].

Theorem 2.8. *For a smooth projective variety* X *we define* $S_X : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(X)$ *by* $S(A) = \omega_X \otimes A[n]$*.* S_X *is a Serre functor for* $\mathbf{D}^b_{coh}(X)$*.*

2.2. Generators in triangulated categories

For a triangulated category $\mathcal{T}$ Bondal and Van den Bergh define the notion of a strong generator [**6**]. First, we need some preliminary definitions that will be used later.

Definition 2.9. Given a subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ we denote by $\langle\mathcal{C}\rangle$ the smallest full subcategory of $\mathcal{T}$ that contains $\mathcal{C}$ and is closed under shifts, direct summands and finite direct sums.

Definition 2.10. Given subcategories $\mathcal{C}_1$, $\mathcal{C}_2$ of a triangulated category $\mathcal{T}$ we define $\mathcal{C}_1 \star \mathcal{C}_2$ as a full subcategory consisting of objects C for which there is a distinguished triangle $C_1 \to C \to C_2$ for some $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$.

Definition 2.11. Given a subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ we define $\langle\mathcal{C}\rangle_k$ inductively as $\langle\mathcal{C}\rangle_0 = \langle C\rangle$, $\langle\mathcal{C}\rangle_{i+1} = \langle\langle\mathcal{C}\rangle_i \star \langle C\rangle\rangle$.

In other words, $\langle\mathcal{C}\rangle_k$ is a subcategory generated from $\mathcal{C}$ using finite direct sums, direct summands, shifts and by taking no more than k cones.

The following technical lemma [**6**, Lemma 2.2.1] shows that we can always assume that the operation of taking a direct summand is taken once and as the last operation.

Lemma 2.12. *For a subcategory $\mathcal{C} \subset \mathcal{T}$ we denote by $smd(\mathcal{C})$ the minimal subcategory that contains $\mathcal{C}$ and is closed under taking direct summands. For subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$ that are closed under finite direct sums we have*

$$smd(\mathcal{C}_1) \star \mathcal{C}_2 \subset smd(\mathcal{C}_1 \star \mathcal{C}_2),\ \mathcal{C}_1 \star smd(\mathcal{C}_2) \subset smd(\mathcal{C}_1 \star \mathcal{C}_2)$$

Definition 2.13. We say that $G \in \mathcal{T}$ is a strong generator for a triangulated category $\mathcal{T}$ if there is $k \in \mathbb{N}$ such that $\langle G \rangle_k = \mathcal{T}$.

Bondal and Van den Bergh [**6**] show the existence of a strong generator for the bounded derived category of coherent sheaves on a smooth projective variety.

Existence of a strong generator implies the representability of cohomological functores, result similar to the Brown representability theorem [**6**].

Theorem 2.14. *In a proper Karubian triangulated category with a strong generator every cohomological functor of finite type is representable.*

2.3. The Rouquier dimension

Based on the notion of a strong generator Rouquier defines the notion of dimension for an abstract triangulated category [**23**].

Definition 2.15. For a triangulated category $\mathcal{T}$ the Rouquier dimension $\mathrm{r.dim}\,\mathcal{T}$ is defined as the smallest m for which there is $G \in T$ such that $\langle G \rangle_m = \mathcal{T}$.

This definition is similar to the definition of the global dimension for an abelian category of right modules over a ring R. If we take G to be R, then existence of a projective resolution of length m of a module M shows that $M \in \langle G \rangle_m$. In a triangulated category instead of a resolution of length m of an object M one considers a "resolution by distinguished triangles of length m", which is just any object in $\langle G \rangle_m$. And since there is no distinguished object R one must consider all possible objecsts $G \in \mathcal{T}$ as a possible replacement for R. Discussion above shows that for a derived category of modules over a ring R we have that for every module M there is an inclusion $M \in \langle R \rangle_{gl.\,dim\,R}$. Under some mild conditions on R every complex of modules can be generated from R in $gl.\,dim\,R$ steps, which is the content of the following theorem ([**14**, Proposition 2.6.], see also [**2**]).

Theorem 2.16. *The Rouquier dimension of the bounded derived category of finitely presented right modules over a right coherent ring R is less or equal than the global dimension of R.*

Given that we now have a notion of dimension for abstract triangulated categories, the natural question to ask is "Given a variety X, what can we say about the Rouquier dimension of $\mathbf{D}^b_{coh}(X)$?". Rouquier [**23**, Proposition 7.9, Proposition 7.16] proves the following bounds

Theorem 2.17. *For a smooth quasi-projective variety X we have that*

$$\dim X \leq \mathrm{r.}\dim \mathbf{D}^b_{coh}(X) \leq 2 \dim X.$$

This is still the best known bound for a general smooth quasi-projective variety. For smooth affine varieties Roquier shows that the dimension of X and the Rouquier dimension of $\mathbf{D}^b_{coh}(X)$ coincide [**23**, Theorem 7.17].

Theorem 2.18. *For a smooth affine scheme X we have that* $\mathrm{r.}\dim \mathbf{D}^b_{coh}(X) = \dim X$.

Finally, for some special cases the same equality holds [**23**, Example 7.7, Example 7.8].

Theorem 2.19. *If X is $\mathbb{P}^n$ or a smooth quadric, then* $\mathrm{r.}\dim \mathbf{D}^b_{coh}(X) = \dim X$.

2.4. Slope stability and the Harder-Narasimhan filtration

In this section we recall the definition and basic properties of slope stability and the Harder-Narasimhan filtration for coherent sheaves. The reference is chapter 1 of [**11**]. Fix a smooth projective variety X of dimension n and an ample line bundle L on X.

Definition 2.20. The slope of a coherent sheaf $\mathscr{F}$ on X with $\operatorname{rk}\mathscr{F} > 0$ is defined as

$$\mu(\mathscr{F}) = \frac{c_1(L)^{n-1}.c_1(\mathscr{F})}{\operatorname{rk}\mathscr{F}}$$

For a torsion sheaf $\mathscr{F}$ the slope $\mu(\mathscr{F})$ is defined to be ∞.

Definition 2.21. A torsion free coherent sheaf $\mathscr{F}$ is called semistable if for all nonzero subsheaves $0 \neq \mathscr{G} \subset \mathscr{F}$ we have that $\mu(\mathscr{G}) \leq \mu(\mathscr{F})$.

Theorem 2.22. *For every torsion free coherent sheaf $\mathscr{F}$, there exists a unique filtration*

$$\mathscr{F}_0 = 0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \dots \subset \mathscr{F}_k = \mathscr{F}$$

such that all the quotients $\mathscr{F}_i/\mathscr{F}_{i-1}$ are torsion free and semistable, and slopes of the quotient sheaves are strictly decreasing $\mu(\mathscr{F}_1/\mathscr{F}_0) > \mu(\mathscr{F}_2/\mathscr{F}_1) > \dots > \mu(\mathscr{F}_k/\mathscr{F}_{k-1})$. This filtration is called the Harder-Narasimhan filtration of $\mathscr{F}$.

Definition 2.23. For a sheaf $\mathscr{F}$ consider the torsion subsheaf $\mathscr{F}_t$. From the previous theorem it follows that there is a unique filtration

$$0 = \mathscr{F}_0 \subset \mathscr{F}_t = \mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3 \subset \dots \subset \mathscr{F}_k = \mathscr{F}$$

such that all the quotients $\mathscr{F}_i/\mathscr{F}_{i-1}$ are torsion free and semistable for $i > 1$, and slopes of the quotient sheaves are strictly decreasing $\mu(\mathscr{F}_1/\mathscr{F}_0) > \mu(\mathscr{F}_2/\mathscr{F}_1) > \ldots > \mu(\mathscr{F}_k/\mathscr{F}_{k-1})$. We will be calling this filtration the Harder-Narasimhan filtration of $\mathscr{F}$.

2.5. Orlov's conjecture

Based on the results of Rouquier it is natural to conjecture that for a smooth quasi-projective variety X we have that $\mathrm{r.dim}\,\mathbf{D}^b_{coh}(X) = \dim X$. This is the conjecture posed by Orlov in [**19**].

Conjecture 2.24. *For a smooth quasi-projective variety X we have that* $\mathrm{r.dim}\,X = \dim X$.

Orlov shows that this conjecture is true for curves [**19**, Theorem 6].

Theorem 2.25. *For a smooth curve X we have that* $\mathrm{r.dim}\,\mathbf{D}^b_{coh}(X) = 1$.

We provide the sketch of the proof since some of our results are an attempt to generalize it to higher dimension. We also slightly simplify it using the notion of Castelnuovo-Mumford regularity which will be useful to us later.

Proof. We notice that every object $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$ is isomorphic to the direct sum of shifts of its cohomology sheaves $\mathscr{C}^\bullet \simeq \oplus \mathscr{H}^i(\mathscr{C}^\bullet)[-i]$. Therefore, it is enough to find $G \in \mathbf{D}^b_{coh}(X)$ such that for every coherent sheaf $\mathscr{F}$ on X we have that $\mathscr{F} \in \langle G \rangle_1$. Fix a very ample line bundle L on X. If $H^1(\mathscr{F} \otimes L^{-1+r}) = 0$, then $\mathscr{F}$ is $r-$Castelnuovo-Mumford regular with respect to L, and there is an exact sequence $\oplus_{i=0}^{a_1} L^{-1-r} \to \oplus_{i=0}^{a_0} L^{-r} \to \mathscr{F}$. Since $\mathrm{Ext}^2(\mathscr{G}_1, \mathscr{G}_2) = 0$ for all coherent sheaves $\mathscr{G}_1, \mathscr{G}_2$ on a smooth projective curve, we have that $\mathscr{F}$ is a direct summand of the complex $\oplus_{i=0}^{a_1} L^{-1-r} \to \oplus_{i=0}^{a_0} L^{-r}$ concentrated in degrees -1 and 0. Therefore, if we have that $H^1(\mathscr{F}^\vee \otimes L^{-1+r}) \simeq H^0(\mathscr{F} \otimes \omega_X \otimes L^{1-r}) = 0$, then $\mathscr{F}$ is a direct summand of a complex $\oplus_{i=0}^{a_0} L^{r} \to \oplus_{i=0}^{a_0} L^{r+1}$ concentrated in degrees 0 and 1. Assume that for a coherent sheaf $\mathscr{F}$ and integers r_+, r_- there is a short exact sequence $0 \to \mathscr{F}_+ \to \mathscr{F} \to \mathscr{F}_- \to 0$ such that $H^1(\mathscr{F}_+ \otimes L^{-1+r_+}) = 0$ and $H^0(\mathscr{F}_- \otimes \omega_X \otimes L^{1-r_-}) = 0$. Then from the octahedral axiom it follows that $\mathscr{F}$ is a direct summand of a cone of a morphism from $\oplus(L^{-r_+-1}[0] \oplus L^{r_-}[-1])$ to $\oplus(L^{-r_+}[0] \oplus L^{r_-+1}[-1])$. In particular, in this case we have that

$$\mathscr{F} \in \langle L^{-r_+} \oplus L^{-r_+-1} \oplus L^{r_-} \oplus L^{r_-+1} \rangle_1.$$

Therefore, it is enough to find $r_+, r_- \in \mathbb{Z}$ such that for every coherent sheaf $\mathscr{F}$, there is an exact sequence $0 \to \mathscr{F}_+ \to \mathscr{F} \to \mathscr{F}_- \to 0$ such that $H^1(\mathscr{F}_+ \otimes L^{-1+r_+}) = 0$ and $H^0(\mathscr{F}_- \otimes \omega_X \otimes L^{1-r_-}) = 0$. For a coherent sheaf $\mathscr{F}$ consider the Harder-Narasimhan filtration of $\mathscr{F}$

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset ... \subset \mathscr{F}_k = \mathscr{F}.$$

Take m to be the maximum i such that $\mu(\mathscr{F}_i/\mathscr{F}_{i-1}) > 0$ or 0 if there is no such i. We define $\mathscr{F}_+ = \mathscr{F}_m$, $\mathscr{F}_- = \mathscr{F}/\mathscr{F}_m$. Take some $r > 1 + |deg(\omega_X)|$, then $H^1(\mathscr{F}_+ \otimes L^{-1+r}) \simeq \mathrm{Hom}(\mathscr{F}_+, L^{1-r} \otimes \omega_X) = 0$ since $\mu(L^{1-r} \otimes \omega_X) < 0$. Similarly, $H^0(\mathscr{F}_- \otimes \omega_X \otimes L^{1-r}) \simeq \mathrm{Hom}(L^{-1+r} \otimes \omega_X^\vee, \mathscr{F}_-) = 0$ since $\mu(L^{-1+r} \otimes \omega_X^\vee) > 0$. Therefore, one can take $r_+ = r_- = r$. $\square$

2.6. Technical lemmas on generation

In this section we give several technical definitions and lemmas on generation that we will need later. Results in this sections are simple and well-known. Proofs are provided for the reader's convenience.

Definition 2.26. We say that $G \in \mathcal{T}$ generates a (not necessarily triangulated) subcategory $\mathcal{C} \subset \mathcal{T}$ in n steps if $\mathcal{C} \subset \langle G \rangle_n$. The generation time of a (not necessarily triangulated) subcategory $\mathcal{C} \subset \mathcal{T}$ with respect to $G \in \mathcal{T}$ is the minimal n such that G generates $\mathcal{C}$ in n steps and ∞ if there is no such n.

Definition 2.27. We say that a sequence of objects $G_i \in \mathcal{T}$ generates a (not necessarily triangulated) subcategory $\mathcal{C} \subset \mathcal{T}$ in n steps if $\mathcal{C} \subset \cup_{d \in \mathbb{N}} \langle \oplus_{i=1}^{i=d} G_i \rangle_n$. The generation time of a (not necessarily triangulated) subcategory $\mathcal{C} \subset \mathcal{T}$ with respect to a sequence $G_i \in \mathcal{T}$ is the minimal n s.t. the sequence G_i generates $\mathcal{C}$ in n steps and ∞ if there is no such n.

Sometimes it is interesting to study how many elements of the sequence G_i we have to take to generate an object X in m steps.

Definition 2.28. Assume that a sequence a sequence of objects $G_i \in \mathcal{T}$ generates $\mathcal{T}$ in m steps. We define the $m - depth$ of an object $\mathscr{C}^\bullet \in \mathcal{T}$ with respect to the sequence $\{G_i\}$ as the minimal d s.t. $\mathscr{C}^\bullet \in \langle \oplus_{i=1}^{i=d} G_i \rangle_m$.

In the following few lemmas we relate the generation time of $\mathscr{F}$ and $f^*\mathscr{F}$ for a morphism f.

Lemma 2.29. *Consider a finite morphism $\phi : X \to Y$ and a subcategory $\mathcal{C} \subset \mathbf{D}^b_{coh}(Y)$. Assume that $\mathscr{O}_Y \to \phi_* \mathscr{O}_X$ splits (which is always true in characteristic 0). If $\phi^* \mathcal{C} \subset \mathbf{D}^b_{coh}(X)$ is generated from $G \in \mathbf{D}^b_{coh}(X)$ (a sequence $G_i \in \mathbf{D}^b_{coh}(X)$) in k steps we have that C is generated from $\phi_* G$ (generated from the sequence $\phi_* G_i$) in k steps.*

Proof. Since $\phi^* \mathcal{C} \subset \mathbf{D}^b_{coh}(X)$ is generated from $G \in \mathbf{D}^b_{coh}(X)$ in k steps it is obvious that $\phi_* \phi^* \mathcal{C} \subset \mathbf{D}^b_{coh}(X)$ is generated from $\phi_* G$ since $\phi_*(\langle G \rangle_n) \subset \langle \phi_* G \rangle_n$. But every object $\mathscr{C}^\bullet \in \mathcal{C}$ is a direct summand of $\phi_* \mathscr{O}_X \otimes \mathscr{C}^\bullet \simeq \phi_* \phi^* \mathscr{C}^\bullet \in \phi_* \phi^* \mathcal{C}$. $\square$

Lemma 2.30. *Consider the blow-up of X at a smooth subvariety $\phi : \tilde{X} \to X$ and a subcategory $\mathcal{C} \subset \mathbf{D}^b_{coh}(Y)$. If $\phi^* \mathcal{C} \subset \mathbf{D}^b_{coh}(X)$ is generated from $G \in \mathbf{D}^b_{coh}(X)$ (sequence $G_i \in \mathbf{D}^b_{coh}(X)$) in k steps, then $\mathcal{C}$ is generated from $\phi_* G$ (generated from a sequence $\phi_* G_i$) in k steps.*

Proof. We proceed as in the proof of the previous lemma and use that $\phi_* \phi^* \mathscr{C}^\bullet \simeq \mathscr{C}^\bullet$. $\square$

It is also clear that the $m - depth$ of any object $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(Y)$ with respect to $\{\phi_* G_i\}$ is less or equal than the $m - depth$ of $\phi^* \mathcal{C}$ with respect to $\{G_i\}$ both for finite maps and blow-ups.

Lemma 2.31. *If $\mathscr{K}^\bullet \in \langle G \rangle_k$, where $G = \mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet$, then $\Phi_{\mathscr{K}^\bullet}(\mathscr{C}^\bullet) \in \langle \mathscr{G}^\bullet \rangle_k$ for every $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$. In particular, if there is a resolution*

$$0 \to \mathscr{F}_k \boxtimes \mathscr{G}_k \to ... \to \mathscr{F}_2 \boxtimes \mathscr{G}_2 \to \mathscr{F}_1 \boxtimes \mathscr{G}_1 \to \mathscr{F}_0 \boxtimes \mathscr{G}_0 \to \mathscr{O}_\Delta \to 0$$

then $\mathbf{D}^b_{coh}(X) = \langle \oplus_{i=0}^{i=k} \mathscr{G}_i \rangle_k$.

Proof. We know that $\Phi_{\mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet}(\mathscr{C}^\bullet) = R\Gamma(\mathscr{C}^\bullet \otimes \mathscr{F}^\bullet) \otimes \mathscr{G}^\bullet$. The statement is then clear by induction on k since if there is a distinguished triangle $\mathscr{K}_1^\bullet \to \mathscr{K}^\bullet \to \mathscr{K}_2^\bullet$, then for every $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$ there is a distinguished triangle

$$\Phi_{\mathscr{K}_1^\bullet}(\mathscr{C}^\bullet) \to \Phi_{\mathscr{K}^\bullet}(\mathscr{C}^\bullet) \to \Phi_{\mathscr{K}_2^\bullet}(\mathscr{C}^\bullet).$$

□

This lemma is how most of the first results on the Rouquier dimension of smooth projective varieties were proved. For example, if X is $\mathbb{P}^n$ or a smooth quadric, then there is a resolution of the diagonal sheaf by box-products of length $\dim X$ [**4**] [**12**].

However, in general the length of the shortest resolution of $\mathscr{O}_X$ by box-products could be greater than the Rouquier dimension of $\mathbf{D}^b_{coh}(X)$. One defines the diagonal dimension of X as the smallest k such that $\mathscr{O}_\Delta \in \langle G \boxtimes G \rangle_k$ for some $G \in \mathbf{D}^b_{coh}(X)$ (this notion at least apriori depends not only on $\mathbf{D}^b_{coh}(X)$ but also on X itself). We see that the Rouquier dimension of $\mathbf{D}^b_{coh}(X)$ is less or equal than the diagonal dimension of X. The inequality could be strict as shown by the following theorem by Olander [**16**].

Theorem 2.32. *The diagonal dimension of a smooth projective curve of genus* ≥ 1 *is* 2.

However, it is obvious that there is always an infinite resolution of $\mathscr{O}_\Delta$ by direct sums of $L^{-i} \boxtimes L^{-i}$, where L is an ample line bundle.

Lemma 2.33. *Given an ample line bundle L there exists a resolution of $\mathscr{O}_\Delta$ of the form*

$$... \to \oplus L^{-a_2} \boxtimes L^{-a_2} \to \oplus L^{-a_1} \boxtimes L^{-a_1} \to \oplus L^{-a_0} \boxtimes L^{-a_0} \to \mathscr{O}_\Delta \to 0,$$

where every direct sum is a finite direct sum.

A useful trick is that given a resolution as in the previous lemma we can tensor it by $L^a \boxtimes L^{-a}$ to obtain the resolution of the form

$$\to \oplus L^{-a_2-a} \boxtimes L^{-a_2+a} \to \oplus L^{-a_1-a} \boxtimes L^{-a_1+a} \to \oplus L^{-a_0-a} \boxtimes L^{-a_0+a} \to \mathscr{O}_\Delta \to 0$$

since $(L^a \boxtimes L^{-a}) \otimes \mathscr{O}_\Delta \simeq \mathscr{O}_\Delta$.

CHAPTER 3

Obstruction morphisms

One of the issues that appear in dimensions greater than 1 is that there are objects in $\mathbf{D}^b_{coh}(X)$ that are not isomorphic to a direct sum of shifted sheaves. Therefore, we do not obtain a bound on the Rouquier dimension of $\mathbf{D}^b_{coh}(X)$ just from the bound on the generation time of coherent sheaves. We need to find some way to glue "resolutions" of $\mathscr{H}^i(\mathscr{C}^\bullet)$ into a "resolution" of $\mathscr{C}^\bullet$. But in Orlov's proof for curves even for the sheaves a certain gluing of "resolutions" of a negative part of the sheaf and a positive part of the sheaf takes place, which is only possible due to the vanishing of $\mathrm{Ext}^i(\mathscr{F},\mathscr{G})$ for $i > 1$.
One way to deal with the mentioned issues is to consider "resolutions" that are more functorial. We consider the canonical truncation of a resolution of the diagonal sheaf $\mathfrak{t}_{\geq -2n+k}\mathscr{C}^\bullet$. Even if $\mathscr{O}_\Delta$ does not split off from the truncated resolution (in such case we obtain a global bound on the diagonal dimension of $\mathbf{D}^b_{coh}(X)$) it can still happen that $\mathcal{F}^\bullet$ splits from $\Phi_{\mathfrak{t}_{\geq -2n+k}\mathscr{C}^\bullet}(\mathcal{F}^\bullet)$ in which case we obtain a bound on the generation time of $\mathcal{F}^\bullet$. This splitting is equivalent to the vanishing of a certain morphism that we call the obstruction morphism $\phi_{\mathscr{F}^\bullet,k}$ of $\mathscr{F}^\bullet$ with respect to the resolution $\mathscr{C}^\bullet$ and truncation at k. The obstruction morphism behaves functorially and we show that under some restrictions for a distinguished triangle $A \to B \to C$ the vanishing of ϕ_B can be deduced from the vanishing of ϕ_A and ϕ_C .
Let's fix the notation for this chapter. From now on we always assume X to be a smooth projective variety of dimension n. We consider a resolution of the diagonal sheaf of the

following form

$$\dots \to L_2 \boxtimes R_2 \to L_1 \boxtimes R_1 \to L_0 \boxtimes R_0 \to \mathscr{O}_\Delta \to 0, \tag{3.1}$$

where L_i, R_i are vector bundles on X. We showed in the previous chapter that such resolution exists with $L_i = R_i = \oplus_{j=0}^{j=k_i} L^{-a_i}$, where L is an ample line bundle. For the resolution (3.1) we define truncation complexes

$$\mathscr{C}_k^\bullet := L_{2n-k} \boxtimes R_{2n-k} \to L_{2n-k-1} \boxtimes R_{2n-k-1} \to \dots \to L_1 \boxtimes R_1 \to L_0 \boxtimes R_0 \tag{3.2}$$

We consider $\mathscr{C}_k^\bullet$ as an object in $\mathbf{D}^b_{coh}(X \times X)$ concentrated in degrees from $-2n+k$ to 0. From (3.1) we obtain the following distinguished triangle of objects in $\mathbf{D}^b_{coh}(X \times X)$

$$\mathscr{K}_k[2n-k] \to \mathscr{C}_k^\bullet \to \mathscr{O}_\Delta, \tag{3.3}$$

where $\mathscr{K}_k$ is a coherent sheaf on $X \times X$. Moreover, there is a resolution for $\mathscr{K}_k$ from (3.1) that gives a quasi-isomorphism

$$\{\dots \to L_{2n-k+2} \boxtimes R_{2n-k+2} \to L_{2n-k+1} \boxtimes R_{2n-k+1}\} \simeq \mathscr{K}_k[2n-k], \tag{3.4}$$

where the complex on the left is concentrated in degrees up to $-2n+k$. Now we see that there is a triplet of Fourier-Mukai functors with natural transformations between them

$$\Phi_{\mathscr{K}_k}[2n-k] \to \Phi_{\mathscr{C}_k^\bullet} \to \Phi_{\mathscr{O}_\Delta} = \mathrm{id}\,. \tag{3.5}$$

Therefore, for every object $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(X)$ there is a distinguished triangle obtained from (3.5)

$$\Phi_{\mathscr{K}_k}(\mathscr{F}^\bullet)[2n-k] \to \Phi_{\mathscr{C}^\bullet_k}(\mathscr{F}^\bullet) \to \mathscr{F}^\bullet \xrightarrow{\phi_{\mathscr{F}^\bullet,k}} \Phi_{\mathscr{K}_k}(\mathscr{F}^\bullet)[2n-k+1]. \tag{3.6}$$

Definition 3.7. We call $\phi_{\mathscr{F}^\bullet,k}$ the obstruction morphism of $\mathscr{F}^\bullet$ with respect to the resolution (3.1) truncated at k.

Usually the resolution and k are clear from the context and we just call it the obstruction morphism of $\mathscr{F}^\bullet$ and denote it by $\phi_{\mathscr{F}^\bullet}$. $\phi_{\mathscr{F}^\bullet,k}$ is the obstruction to the splitting of $\mathscr{F}^\bullet$ from $\Phi_{\mathscr{C}^\bullet_k}(\mathscr{F}^\bullet)$ as a direct summand. Therefore, using lemma 2.31 we obtain the following lemma

Lemma 3.8. *If $\phi_{\mathscr{F}^\bullet,k} = 0$ for $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(X)$, then $\mathscr{F}^\bullet$ can be generated from $\bigoplus_{i=0}^{2n-k} R_i$ in $2n-k$ steps.*

We can often derive the vanishing of the obstruction morphism of $\mathscr{F}$ from the vanishing of the vector space $\mathrm{Hom}(\mathscr{F}^\bullet, \Phi_{\mathscr{K}_k}(\mathscr{F}^\bullet)[2n-k+1])$ which we call the obstruction space of $\mathscr{F}^\bullet$.

Example 3.9. If for a coherent sheaf $\mathscr{F}$ we have that $\mathscr{H}^i(\Phi_{\mathscr{K}_k}(\mathscr{F})) = 0$ for all $i \geq n-k+1$, then $\mathrm{Hom}(\mathscr{F}, \Phi_{\mathscr{K}_k}(\mathscr{F})[2n-k+1]) = 0$ and $\phi_{\mathscr{F}} = 0$ with respect to the truncation at k.

Proof. We have that $\Phi_{\mathscr{K}_k}(\mathscr{F}^\bullet)[2n-k+1]$ is concentrated in degrees below $-n$. Therefore, the vanishing of $\mathrm{Hom}(\mathscr{F}, \Phi_{\mathscr{K}_k}(\mathscr{F})[2n-k+1])$ follows from the fact that for a smooth variety X we have that $\mathrm{Ext}^i(\mathscr{F}_1, \mathscr{F}_2) = 0$ for $i > \dim X$ and coherent sheaves $\mathscr{F}_1, \mathscr{F}_2$ on X. $\square$

The following example is a special case of the previous one.

Example 3.10. Consider a coherent sheaf $\mathscr{F}$ such that $\mathrm{H}^i(L_{2n-k+1+j} \otimes \mathscr{F}) = 0$ for all $j \geq 0, i \geq n-k+1+j$. Then we have that $\mathrm{Hom}(\mathscr{F}, \Phi_{\mathscr{K}_k}(\mathscr{F})[2n-k+1]) = 0$ and $\phi_{\mathscr{F}} = 0$ with respect to the truncation at k.

Proof. We have that $\mathscr{H}^i(\Phi_{\mathscr{K}_k}(\mathscr{F})) = 0$ for $i \geq n-k+1$ from lemma 2.5 in the preliminaries. $\square$

Now we study the behavior of obstruction morphisms in distinguished triangles.

Lemma 3.11. *Given a distinguished triangle* $\mathscr{F}_+^\bullet \to \mathscr{F}^\bullet \to \mathscr{F}_-^\bullet$ *such that* $\phi_{\mathcal{F}_-^\bullet} = 0$*, the obstruction space of* $\mathscr{F}_-^\bullet$ *is* 0 *and* $\mathrm{Hom}(\mathcal{F}_-^\bullet, \Phi_{\mathscr{K}_k}(\mathscr{F}_+^\bullet)[2n-k+1]) = 0$ *we have that* $\phi_{\mathcal{F}^\bullet} = 0$.

Proof. From (3.5) we have the following morphism of distinguished triangles

$$\begin{tikzcd}
\mathcal{F}_+^\bullet \arrow[r] \arrow[d, "0"] & \mathcal{F}^\bullet \arrow[r] \arrow[d, "\phi_{\mathcal{F}^\bullet}"] & \mathcal{F}_-^\bullet \arrow[d, "0"] \arrow[dll, dashed] \\
\Phi_{\mathcal{K}_k}(\mathcal{F}_+^\bullet)[2n-k+1] \arrow[r] & \Phi_{\mathcal{K}_k}(\mathcal{F}^\bullet)[2n-k+1] \arrow[r] & \Phi_{\mathcal{K}_k}(\mathcal{F}_-^\bullet)[2n-k+1]
\end{tikzcd}$$

The lemma now is a direct consequence of 2.6 from the preliminaries. $\square$

Using this lemma we can sometimes deduce the vanishing of $\phi_{\mathscr{F}^\bullet}$ from the vanishings of $\phi_{\mathscr{H}^i(\mathscr{F}^\bullet)}$ for all i. The first application is a generalization of 3.9.

Lemma 3.12. *Assume that there is* $0 \leq k \leq n$ *such that* $\mathscr{H}^i(\Phi_{\mathscr{K}_k}(\mathscr{H}^j(\mathscr{F}^\bullet))) = 0$ *for* $i \geq n-k+1$ *for all* $j \in \mathbb{Z}$ *(i.e. all cohomology sheaves of* $\mathscr{F}^\bullet$ *satisfy the assumptions from 3.9). Then for a truncation at* k *we have* $\phi_{\mathscr{F}^\bullet} = 0$.

Proof. We prove it by induction on the number N of indices j such that $\mathscr{H}^j(\mathscr{F}^\bullet) \neq 0$. If $N = 1$, then $\mathscr{F}^\bullet$ is a shifted sheaf and the lemma follows from the example 3.9. For the induction step take m to be the maximal j such that $\mathscr{H}^j(\mathscr{F}^\bullet) \neq 0$ and consider the canonical truncation $\mathfrak{t}_{\leq m-1}\mathscr{F}^\bullet$. We have a distinguished triangle

$$\mathfrak{t}_{\leq m-1}\mathscr{F}^\bullet \to \mathscr{F}^\bullet \to \mathscr{H}^m(\mathscr{F}^\bullet)[-m]$$

From the induction assumption it follows that $\phi_{\mathfrak{t}_{\leq m-1}\mathscr{F}^\bullet} = 0$ and the obstruction space of $\mathscr{H}^m(\mathscr{F}^\bullet)[-m]$ is 0 from 3.9. The vanishing of $\operatorname{Hom}(\mathscr{H}^m(\mathscr{F}^\bullet)[-m], \Phi_{\mathscr{K}_k}(\mathscr{F}^\bullet_+)[2n-k+1])$ follows from the fact that $\Phi_{\mathscr{K}_k}(\mathfrak{t}_{\leq m-1}\mathscr{F}^\bullet)[2n-k+1])$ is concentrated in degrees $\leq m-n-1$. $\square$

Remark 1. Condition $\operatorname{Hom}(\mathscr{F}^\bullet, \Phi_{\mathscr{G}}(\mathscr{F}^\bullet)[2n-k+1]) = 0$ is almost never satisfied for complexes with several nonzero cohomology sheaves. Therefore it is important to consider the vanishing of the obstruction morphism itself instead of the vanishing of the obstruction space.

The following is the main technical result of this section.

Corollary 3.13. *Fix an ample line bundle* L. *For every* $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(X)$, *there is* $C \in \mathbb{Z}$ *s.t. for all* $k \geq C$ *we have that* $\phi_{\mathscr{F}^\bullet \otimes L^k, n} = 0$.

Proof. For sufficiently large k the assumptions of 3.12 are satisfied for $\mathscr{F}^\bullet \otimes L^k$ due to Serre's vanishing. $\square$

From this corollary and lemma 3.8 we obtain the main theorem of this section.

Theorem 3.14. *Fix an ample line bundle L. The sequence $\{L^{-i} \mid i \in \mathbb{N}\}$ generates $\mathbf{D}^b_{coh}(X)$ in n steps.*

Proof. Take a resolution of the diagonal with $R_i = L^{-a_i}$. From 3.8 and 3.13 we obtain that for every coherent sheaf $\mathscr{F}^\bullet$, there is k such that $\mathscr{F}^\bullet \otimes L^k$ is generated from $\bigoplus_{i=0}^{n} L^{-a_i}$ in n steps. It is clear then that $\mathscr{F}^\bullet$ can be generated from $\bigoplus_{i=0}^{n} L^{-i-k}$ in n steps. $\square$

Orlov's conjecture predicts that there is an object $G \in \mathbf{D}^b_{coh}(X)$ such that G generates $\mathbf{D}^b_{coh}(X)$ in n steps. There is no difference in taking just one object or finitely many objects as a generating set since one can just take a direct sum of finitely many objects (and taking a direct summand is a free operation). One could ask what happens if countably many objects are allowed as a generating set. Theorem 3.14 implies that for a smooth projective variety there is always a countable set of objects in $\mathbf{D}^b_{coh}(X)$ that generates $\mathbf{D}^b_{coh}(X)$ in $\dim X$ steps. Recently this result was obtained by Olander [**18**] using a different method.

Remark 2. It is clear that positive powers of an ample line bundle L generate $\mathbf{D}^b_{coh}(X)$ in n steps as well since there is the duality equivalence $\mathbb{D} : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(X)^{opp}$. It is also clear from the duality that an analogue of the corollary 3.13 for tensoring with a sufficiently negative power of L is true. However, even if we wanted to only show that sheaves can be generated in n steps from positive powers of an ample line bundle (a statement which is obvious for negative powers), it will require us to work with complexes of sheaves since $\mathbb{D}\mathscr{F}$ can have nonzero cohomology sheaves in several degrees.

Similar results are true if we tensor with a $q-ample$ line bundle for the generation in $n+q$ steps.

Definition 3.15. We say that a line bundle $\mathscr{L}$ on X is $q-ample$ if for every coherent sheaf $\mathscr{F}$ on X, there is $n \in \mathbb{N}$ such that $H^i(\mathscr{F} \otimes \mathscr{L}^j) = 0$ for all $i > q, j > n$.

Sometimes line bundles that are $q-ample$ in the sense of this definition are called naively $q-ample$ in the literature. For alternative definitions and basic properties of $q-ample$ line bundles we refer to [**27**].

Lemma 3.16. *Fix a $q-$ample line bundle L and a resolution of the diagonal of the form 3.1. For every $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(X)$, there is $C \in \mathbb{Z}$ such that for all $k \geq C$ we have that $\phi_{\mathscr{F}^\bullet \otimes \mathscr{L}^k, n-q} = 0$.*

Proof. The proof is identical to that of 3.13, but we use the definition of $q-$ample line bundles instead of Serre's vanishing. □

As a corollary we obtain the following theorem.

Theorem 3.17. *For a resolution of the form 3.1 and a $q-ample$ line bundle L we have that the sequence $\{L^{-i} \otimes R | i \in \mathbb{N}\}$ generates $\mathbf{D}^b_{coh}(X)$ in $n+q$ steps, where $R = \bigoplus_{0 \leq j \leq n+q} R_j$.*

We can apply lemma 3.16 to the blow-up of X at some point $pt \in X$ with $L = \mathscr{O}(E)$, where E is the exceptional divisor since $\mathscr{O}(E)$ is $n-1$ ample. Using 2.30 we obtain the following lemma.

Lemma 3.18. *Fix a resolution of the diagonal of the form 3.1 and a point $pt \in X$. For every $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(X)$, there is $C \in \mathbb{Z}$ s.t. for all $k \geq C$ we have that $\phi_{\mathscr{F}^\bullet \otimes \mathcal{I}^k_{pt},1} = 0$.*

Applying lemma 3.8 we obtain the following theorem.

Theorem 3.19. *Fix a resolution of the form 3.1 and a point $pt \in X$. We have that the sequence $\{\mathcal{I}^i_{pt} \otimes R | i \in \mathbb{N}\}$ generates $\mathbf{D}^b_{coh}(X)$ in $2n-1$ steps, where $R = \bigoplus_{0 \leq j \leq n+q} R_j$.*

3.1. Truncating at k = 1

In this section we always assume that the truncation is at $k = 1$. The obstruction space in this case is easy to study since we have the following lemma

Lemma 3.20. *The obstruction space of a coherent sheaf $\mathscr{F}$ is a quotient of*

$$H^n(\mathscr{F} \otimes L_{2n})^* \otimes \mathrm{Hom}(R_{2n} \otimes \omega_X^{-1}, \mathscr{F})^*. \tag{3.21}$$

Proof. From the distinguished triangle

$$L_{2n} \boxtimes R_{2n}[2n] \to \mathscr{K}_1[2n] \to \mathscr{C}^\bullet,$$

where

$$\mathscr{C}^\bullet = \{... \to L_{2n+2} \boxtimes R_{2n+2} \to L_{2n+1} \boxtimes R_{2n+1}\}$$

we obtain an exact sequence

$$\mathrm{Hom}(\mathscr{F}, \Phi_{L_{2n} \boxtimes R_{2n}}(\mathscr{F})[2n]) \to \mathrm{Hom}(\mathscr{F}, \Phi_{\mathscr{K}_1}(\mathscr{F})[2n]) \to \mathrm{Hom}(\mathscr{F}, \Phi_{\mathscr{C}^\bullet}(\mathscr{F})).$$

But since $\mathscr{H}^i(\mathscr{C}^\bullet) = 0$ for $i \geq -2n$ we have that $\mathscr{H}^i(\Phi_{\mathscr{C}^\bullet}(\mathscr{F})) = 0$ for $i \geq -n$ by 2.5. Therefore, $\mathrm{Hom}(\mathscr{F}, \Phi_{\mathscr{C}^\bullet}(\mathscr{F})) = 0$ for dimension reasons and $\mathrm{Hom}(\mathscr{F}, \Phi_{\mathscr{K}_1}(\mathscr{F})[2n])$ is a quotient of

$$\begin{aligned}
\mathrm{Hom}(\mathscr{F}, \Phi_{L_{2n}\boxtimes R_{2n}}(\mathscr{F})[2n]) &\simeq \oplus_i \mathrm{Hom}(\mathscr{F}, H^i(\mathscr{F} \otimes L_{2n}) \otimes R_{2n}[2n-i]) \\
&\simeq \mathrm{Hom}(\mathscr{F}, H^n(\mathscr{F} \otimes L_{2n}) \otimes R_{2n}[n]) \\
&\simeq H^n(\mathscr{F} \otimes L_{2n})^* \otimes \mathrm{Hom}(R_{2n} \otimes \omega_X^{-1}, \mathscr{F})^*.
\end{aligned}$$

□

Definition 3.22. We say that a coherent sheaf $\mathscr{F}$ is 1-positive if $H^n(\mathscr{F} \otimes L_{2n}) = 0$. If $\mathrm{Hom}(R_{2n} \otimes \omega_X^{-1}, \mathscr{F}) = 0$, then we say that $\mathscr{F}$ is 1-negative.

It is clear that for a 1-positive sheaf $\mathscr{F}$ we have that $\Phi_{\mathscr{K}_1}(\mathscr{F})$ is concentrated in degrees $\leq n-1$ by 2.5. It is also obvious that an extension of two 1-positive sheaves is 1-positive and an extension of two 1-negative sheaves is 1-negative.

We know that 1-negative and 1-positive sheaves can be generated from $\oplus_{0\leq i\leq 2n} R_i$ in $2n-1$ steps due to the vanishing of the obstruction spaces and lemma 3.8. Now we want to explain how to glue "resolutions" for them.

We prove a criterion for the vanishing of the obstruction space of an extension of two objects in $\mathbf{D}^b_{coh}(X)$.

Lemma 3.23. *Given a distinguished triangle $\mathscr{C}_+^\bullet \to \mathscr{C}^\bullet \to \mathscr{C}_-^\bullet$ such that $\phi_{\mathscr{C}^\bullet} = 0$, the obstruction space of $\mathscr{C}_-^\bullet$ is* 0 *and* $\mathrm{Hom}(\mathscr{C}_-^\bullet, \Phi_{\mathcal{K}}(\mathscr{C}_+^\bullet)[2n]) = 0$ *we have that* $\phi_{\mathscr{C}^\bullet} = 0$.

Proof. Consider the following morphism of triangles

$$\begin{tikzcd}
\mathscr{C}^\bullet_+ \arrow[r] \arrow[d, "0"] & \mathscr{C}^\bullet \arrow[r] \arrow[d, "\phi_{\mathscr{C}^\bullet}"] & \mathscr{C}^\bullet_- \arrow[d, "0"] \arrow[dll, dashed] \\
\Phi_{\mathcal{K}}(\mathscr{C}^\bullet_+)[2n] \arrow[r] & \Phi_{\mathcal{K}}(\mathscr{C}^\bullet)[2n] \arrow[r] & \Phi_{\mathcal{K}}(\mathscr{C}^\bullet_-)[2n]
\end{tikzcd}$$

We have that $\mathrm{Hom}(\mathscr{C}^\bullet_-, \Phi_{\mathcal{K}}(\mathscr{C}^\bullet_+)[2n]) = 0$ by the assumption, and $\mathrm{Hom}(\mathscr{C}^\bullet_-, \Phi_{\mathcal{K}}(\mathscr{C}^\bullet_-)[2n]) = 0$ since $\mathscr{C}^\bullet_-$ has the vanishing obstruction space. Therefore, the result follows from 2.6. $\square$

Corollary 3.24. *Given a short exact sequence of sheaves $0 \to \mathscr{F}_+ \to \mathscr{F} \to \mathscr{F}_- \to 0$, where $\mathscr{F}_+$ is 1-positive and $\mathscr{F}_-$ is 1-negative we have that $\phi_{\mathscr{F}} = 0$.*

Proof. We have that $\mathrm{Hom}(\mathcal{F}_-, \Phi_{\mathcal{K}}(\mathcal{F}_+)[2n]) = 0$ because the cohomology sheaves of $\Phi_{\mathcal{K}}(\mathcal{F}_+)$ are concentrated in degrees up to $-n-1$. Since $\mathcal{F}_-$ is 1-negative, it has the vanishing obstruction space, the result then follows from 3.23. $\square$

We can prove a criterion for the vanishing of the obstruction morphism of a complex of sheaves which is the main technical result of this section.

Lemma 3.25. *Consider an object $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(X)$. Assume that for every $i \in \mathbb{Z}$ there is a short exact sequence $0 \to \mathscr{H}^i_+(\mathscr{F}^\bullet) \to \mathscr{H}^i(\mathscr{F}^\bullet) \to \mathscr{H}^i_-(\mathscr{F}^\bullet) \to 0$, where $\mathscr{H}^i_+(\mathscr{F}^\bullet)$ is 1-positive and $\mathscr{H}^i_-(\mathscr{F}^\bullet)$ is 1-negative. Then $\phi_{\mathscr{F}^\bullet} = 0$.*

Proof. Similarly to the proof of 3.12 we proceed by induction on the number N of indices j such that $\mathscr{H}^j(\mathscr{F}^\bullet) \neq 0$. If $N = 1$ then $\mathscr{F}^\bullet$ is a shifted sheaf and the lemma follows from 3.24. For the induction step take m to be the maximal j such that $\mathscr{H}^j(\mathscr{F}^\bullet) \neq 0$. We have two distinguished triangles.

$$\mathscr{F}^\bullet_{\leq m-1} \to \mathscr{F}^\bullet_+ \to \mathscr{H}^m_+(\mathscr{F}^\bullet)[-m]$$

$$\mathscr{F}^\bullet_+ \to \mathscr{F}^\bullet \to \mathscr{H}^m_-(\mathscr{F}^\bullet)[-m]$$

First, we show that $\phi_{\mathscr{F}^\bullet_+} = 0$. From the induction assumption it follows that $\phi_{\mathscr{F}^\bullet_{\leq m-1}} = 0$, and the obstruction space of $\mathscr{H}^m_+(\mathscr{F}^\bullet)[-m]$ is 0 since $\mathscr{H}^m_+(\mathscr{F}^\bullet)$ is 1-positive. Since cohomology sheaves of $\Phi_{\mathcal{K}}(\mathscr{F}^\bullet_{\leq m-1})[2n]$ are concentrated in degrees up to $-n-1+m$ we have that $\mathrm{Hom}(\mathscr{H}^m_+(\mathscr{F}^\bullet)[-m], \Phi_{\mathcal{K}}(\mathscr{F}^\bullet_{\leq m-1})[2n]) = 0$. Therefore, the vanishing of $\phi_{\mathscr{F}^\bullet_+} = 0$ follows from lemma 3.23 applied to the first distinguished triangle. Now we show that $\phi_{\mathscr{F}^\bullet_-} = 0$ using the second distinguished triangle in the same way. The only change is that we need to check that cohomology sheaves of $\Phi_{\mathcal{K}}(\mathcal{F}_+)$ are concentrated in degrees up to $-n-1+m$ which follows from the first distinguished triangle and the fact that both $\Phi_{\mathcal{K}}(\mathscr{F}^\bullet_{\leq m-1})[2n]$ and $\Phi_{\mathcal{K}}(\mathscr{H}^m_+(\mathscr{F}^\bullet)[-m])[2n]$ have cohomology sheaves concentrated in degrees up to $-n-1+m$. $\square$

Combining this result with lemma 3.8, we obtain the following corollary.

Corollary 3.26. *Consider an object $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$. Assume that for every $i \in \mathbb{Z}$ there is a short exact sequence $0 \to \mathscr{H}^i_+(\mathscr{C}^\bullet) \to \mathscr{H}^i(\mathscr{C}^\bullet) \to \mathscr{H}^i_-(\mathscr{C}^\bullet) \to 0$, where $\mathscr{H}^i_+(\mathscr{C}^\bullet)$ is 1-positive and $\mathscr{H}^i_-(\mathscr{C}^\bullet)$ is 1-negative. Then $\mathscr{C}^\bullet$ is generated by from $\oplus_{i=0}^{i=2n-1} R_i$ in $2n-1$ steps.*

CHAPTER 4

Theorems on generation using the Harder-Narasimhan filtration

4.1. Positive and negative classes using Harder-Narasimhan filtrations

As in the previous chapter we always assume X to be a smooth projective variety of dimension n. We fix an ample line bundle L with respect to which we consider slope-stability and Harder-Narasimhan filtrations of all sheaves. We fix a resolution of $\mathscr{O}_\Delta$ of the form 3.1 with $L_i = \oplus_{j=0}^{k_i} L^{a_i}, R_i = L^{b_i}$. We always consider obstruction morphisms and obstruction spaces with respect to this chosen resolution. All definitions given in this chapter are with respect to this chosen resolution. In this chapter we are only concerned with generation in $2\dim X - 1$ steps. Therefore, we always consider truncation at $k = 1$ and the corresponding obstruction morphism. In this section we describe how to obtain the vanishing of the obstruction morphism $\phi_{\mathscr{F}}$ from the properties of the Harder-Narasimhan filtration of $\mathscr{F}$.

Given a coherent sheaf $\mathscr{F}$, our goal now is to construct a decomposition

$$0 \to \mathscr{F}_+ \to \mathscr{F} \to \mathscr{F}_- \to 0$$

that satisfies the requirements of 3.24. The idea is to truncate a certain filtration of $\mathscr{F}$. One natural choice choice is the Harder-Narasimhan filtration as in Orlov's proof for curves. Unfortunately, the existence of a decomposition of the needed form depends on the existence of a large gap in the Harder-Narasimhan filtration. We will overcome this difficulty by taking a pullback of $\mathscr{F}$ for a finite polarized endomorphism of sufficiently high degree.

Lemma 4.1. *There exist $a, b \in \mathbb{R}$ such that all slope-semistable sheaves $\mathcal{F}$ with $\mu(\mathscr{F}) > b$ are 1-positive and all slope-semistable sheaves $\mathcal{F}$ with $\mu(\mathscr{F}) < a$ are 1-negative.*

Proof. The condition of being 1-positive can be rewritten as

$$\mathrm{Hom}(\mathscr{F} \otimes \omega_X^{-1}, L_{2n}^\vee) = 0$$

which is satisfied if $\mu(\mathscr{F}) > \mu(L_{2n}^\vee) + \mu(\omega_X) = \mu(\omega_X) - \mu(L_{2n}) = b$ (we use that $L_{2n} = \oplus_{j=0}^{j=k_{2n}} L^{a_{2n}}$ is semistable). Similarly, the condition of being 1-negative

$$\mathrm{Hom}(R_{2n} \otimes \omega_X^{-1}, \mathscr{F}) = 0$$

follows from $\mu(\mathscr{F}) < \mu(R_{2n}) - \mu(\omega_X) = a$. □

It will be convenient to assume that both $a = \mu(\omega_X) - \mu(L_{2n})$ and $b = \mu(R_{2n}) - \mu(\omega_X)$ in the previous lemma are positive. This can always be achieved by modifying the chosen resolution of $\mathscr{O}_\Delta$ by tensoring it with $L^m \boxtimes L^{-m}$ for sufficiently large m. From now on we assume that $\mu(\omega_X) - \mu(L_{2n})$ and $\mu(R_{2n}) - \mu(\omega_X)$ are positive for the chosen resolution of $\mathscr{O}_\Delta$.

Lemma 4.2. *Consider the Harder-Narasimhan filtration $F_i\mathscr{F}$ of a coherent sheaf $\mathscr{F}$ with slopes*

$$\mu(F_i\mathscr{F}/F_{i-1}\mathscr{F}) = \mu_i.$$

If there is no i such that $\mu_i \in [a,b]$, where b,a are the constants from the previous lemma then there is a short exact sequence $0 \to \mathscr{F}_+ \to \mathscr{F} \to \mathscr{F}_- \to 0$, where $\mathscr{F}_+$ is $1-$positive and $\mathscr{F}_-$ is $1-$negative. In particular, $\phi_{\mathscr{F}} = 0$. The condition is always satisfied if $b < a$.

Proof. Consider $k = max\{i \mid \mu_i > b\}$. Then, $\mu_i > b$ for all $i \leq k$ and $\mu_i < a$ for all $i \geq k+1$. For $\mathscr{F}_+ = \mathscr{F}_k, \mathscr{F}_- = \mathscr{F}/\mathscr{F}_k$ we have a short exact sequence

$$0 \to \mathscr{F}_+ \to \mathscr{F} \to \mathscr{F}_- \to 0.$$

We have that $\mathscr{F}_+$ is $1-$positive from the previous lemma since all quotient sheaves in the Harder-Narasimhan filtration of $\mathscr{F}_+$ have the slope greater than b, and an extension of two $1-$positive sheaves is clearly $1-$positive. Similarly, $\mathscr{F}_-$ is $1-$negative. The vanishing of $\phi_{\mathscr{F}}$ follows from lemma 3.24. $\square$

4.2. The Rouquier dimension of toric varieties

Now we want to show that given a finite morphism $\phi : X \to X$ such that $\phi^* L = L^m$ for some $m > 1$ and a coherent sheaf $\mathscr{F}$, we have that $\phi_k^* \mathscr{F}$ satisfies the conditions of 4.2 for k large enough, where ϕ_k is the composition ϕ with itself k times. Consider the Harder-Narasimhan filtration $F_\bullet \phi_k^* \mathscr{F}$ of $\phi_k^* \mathscr{F}$ and the Harder-Narasimhan filtration $F_\bullet \mathscr{F}$ of $\mathscr{F}$. We have that $F_i \phi_k^* \mathscr{F} = \phi_k^* F_i \mathscr{F}$ for all i in characteristic 0. Therefore,

$$\mu(F_i(\phi_k^* \mathscr{F})/F_{i-1}(\phi_k^* \mathscr{F})) = m^k \mu(F_i \mathscr{F}/F_{i-1}\mathscr{F}).$$

Therefore, for k large enough all negative slopes of the quotient sheaves $F_i(\phi_k^* \mathscr{F})/F_{i-1}(\phi_k^* \mathscr{F})$ become less than a, and all positive slopes of the quotient sheaves $F_i(\phi_k^* \mathscr{F})/F_{i-1}(\phi_k^* \mathscr{F})$ become greater than b, where a and b are the constants from lemma 4.2. We assumed that the resolution of $\mathscr{O}_\Delta$ was chosen so that both a and b are positive. Therefore, quotient sheaves with the slope 0 also have slope less than a. Combining lemmas 4.2, 3.25 and 3.8, we obtain the following lemma.

Lemma 4.3. *Consider a smooth projective variety X over a field of characteristic 0 and a finite endomorphism $\phi : X \to X$ such that $\phi^* L = L^m$ for some $m > 1$. For every $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$, there is N such that for all $k > N$ we have that $\phi_k^* \mathscr{C}^\bullet$ is generated by $\oplus_{i=0}^{i=2n-1} R_i$ in $2n-1$ steps.*

Using lemma 2.29 we obtain the following corollary.

Corollary 4.4. *Consider a smooth projective variety X over a field of characteristic 0 and a finite endomorphism $\phi : X \to X$ such that $\phi^* L = L^m$ for some $m > 1$. Then the sequence $\{(\phi_k)_*(\oplus_{i=0}^{i=2n-1} R_i) \mid k \in \mathbb{N}\}$ generates $\mathbf{D}^b_{coh}(X)$ in $2n-1$ steps.*

In the case of finite characteristic we additionally assume that ϕ is either an arbitrary separable morphism or the Frobenius morphism. We denote by F the absolute Frobenius morphism and by F_k the composition of F with itself k times.

A semistable sheaf $\mathscr{F}$ on X is called strongly semistable if for every $k > 0$ we have that $F_k^* \mathscr{F}$ is semistable. For the case of finite characteristic we will need the following theorem by Langer [**15**, Theorem 2.7.] that shows that the Harder-Narasimhan filtration of Frobenius pullbacks eventually stabilizes.

Theorem. *Consider a smooth projective variety X. For every coherent sheaf $\mathscr{F}$ there exists k_0 such that all quotient sheaves in the Harder-Narasimhan filtration of $F_{k_0}^* \mathscr{F}$ are strongly semistable.*

Lemma 4.5. *Consider a smooth projective variety X over a field of characteristic p and the Frobenius morthism $F : X \to X$. For every $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$, there is N such that for all $k > N$ we have that $F_k^*\mathscr{C}^\bullet$ is generated by $\oplus_{i=0}^{i=2n-1} R_i$ in $2n-1$ steps.*

Proof. Take k_0 such that all quotient sheaves in the Harder-Narasimhan filtration of $\mathscr{E} = F_{k_0}^*\mathscr{F}$ are strongly semistable. Then we can proceed as in the proof of 4.3 in characteristic 0. $\square$

Combining cases of finite characteristic and characteristic 0, we obtain the following lemma.

Lemma 4.6. *Consider a smooth projective variety X and a finite endomorphism $\phi : X \to X$ such that $\phi^*L = L^m$ for some $m > 1$. If the characteristic of the field is finite, we additionally assume that ϕ is either a separable morphism or the Frobenius morphism. For every $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$, there is N such that for all $k > N$ we have that $\phi_k^*\mathscr{C}^\bullet$ is generated by $\oplus_{i=0}^{i=2n-1} R_i$ in $2n-1$ steps.*

Using 2.29 we obtain the following corollary.

Corollary 4.7. *Consider a smooth projective variety X and a finite endomorphism $\phi : X \to X$ such that $\phi^*L = L^m$ for some $m > 1$. If the characteristic of the field is finite we additionally assume that ϕ is either separable or the Frobenius morphism and that $\mathscr{O}_X \to \phi_*\mathscr{O}_X$ splits. Then the sequence $\{(\phi_k)_*(\oplus_{i=0}^{i=2n-1} R_i) \mid k \in \mathbb{N}\}$ generates $\mathbf{D}^b_{coh}(X)$ in $2n-1$ steps.*

Now we can prove the bound of $2n-1$ on the Rouquier dimension of toric varieties over an arbitrary algebraically closed field. We will use the toric Frobenius morphism

$\bar{m}: X \to X$, which is a unique extension of a morphism given by $(x_i) \to (x_i^m)$ on the embedded torus. We will need the following result on direct images of line bundles under the toric Frobenius morphism [**26**, Proposition 1] (see also [**1**]).

Theorem. *Fix a line bundle $\mathscr{L}$ on a toric variety X. We have that $\bar{m}_*\mathscr{L}$ is a direct sum of line bundles for all $m \in \mathbb{N}$. Moreover, there are only finitely many line bundles that can appear as a direct summand of $\bar{m}_*\mathscr{L}$ for some $m \in \mathbb{N}$.*

We denote by $Frob(X, L)$ the direct sum of all line bundles that appear as a direct summand of $\bar{m}_*L$ for some $m \in \mathbb{N}$.

Theorem 4.8. *The Rouquier dimension of a smooth projective toric variety X over an arbitrary algebraically closed field is less or equal than $2\dim X - 1$.*

Proof. Applying 4.7 to $\bar{m}_*$ for some $m > 1$ we obtain that the sequence

$$\{(\phi_k)_*(\oplus_{i=0}^{i=2n-1} R_i) \mid k \in \mathbb{N}\}$$

generates $\mathbf{D}^b_{coh}(X)$ in $2n-1$ steps. We assumed that $R_i = \oplus_{j=0}^{j=k_i} L^{b_i}$. Therefore,

$$\begin{aligned}\mathbf{D}^b_{coh}(X) &= \cup_k \langle (\phi_k)_*(\oplus_{i=0}^{i=2n-1} R_i)\rangle_{2n-1} = \cup_k \langle (\phi_k)_*(\oplus_{i=0}^{i=2n-1} L^{b_i})\rangle_{2n-1} \\ &= \langle \oplus_{i=0}^{i=2n-1} Frob(X, L^{b_i})\rangle_{2n-1}.\end{aligned}$$

We see that $\oplus_{i=0}^{i=2n-1} Frob(X, L^{b_i})$ generates $\mathbf{D}^b_{coh}(X)$ in $2n-1$ steps. $\square$

Remark 3. For a toric variety over a field of characteristic p we could just take m to be relatively prime with p instead of invoking the theorem of Langer.

4.3. Generation of complexes with a bound on the total rank

We know that for an arbitrary smooth projective variety we have that $\mathbf{D}^b_{coh}(X)$ is generated by the sequence $\{L^i | i \in \mathbb{N}\}$ in n steps. In particular, for every $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$ there is $k \in \mathbb{N}$ such that $\mathscr{C}^\bullet \in \langle \oplus_{i=0}^{i=k} L^i \rangle_{2n-1}$, in this section we give an effective upper bound on k in terms of ranks of cohomology sheaves of $\mathscr{C}^\bullet$.

First, we assume that in the resolution of $\mathscr{O}_\Delta$ we have that $L_i = \oplus_{j=0}^{j=k_i} L^{a_i}$ and $R_i = \oplus_{j=0}^{j=k_i} L^{b_i}$. By tensoring the resolution with $L^{-m} \boxtimes L^m$ we obtain a resolution of $\mathscr{O}_\Delta$ with $L_i = \oplus_{j=0}^{j=k_i} L^{a_i - m}$ and $R_i = \oplus_{j=0}^{j=k_i} L^{b_i + m}$, we call it the m-resolution. The idea is that given $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$ we choose m in such a way that the obstruction morphism of $\mathscr{C}^\bullet$ with respect to the m-resolution vanishes.

We define $\nu = 2\mu(\omega_X) - \mu(L^{a_{2n}}) - \mu(L^{b_{2n}})$. It follows from the proof of 4.2 that if $\nu < 0$ then $\mathbf{D}^b_{coh}(X) = \langle \oplus_{i=0}^{i=2n-1} L^{b_i} \rangle_{2n-1}$. Therefore, in this case the bound on k does not depend on $\mathscr{C}^\bullet$ at all. Therefore, we assume that $\nu \geq 0$. We define $\nu^+ = 2\nu + \mu(L) + 1$.

Definition 4.9. Consider the Harder-Narasimhan filtration $F_i\mathscr{F}$ of a coherent sheaf $\mathscr{F}$ with slopes

$$\mu(F_i\mathscr{F}/F_{i-1}\mathscr{F}) = \mu_i$$

We say that there is a jump at $k \in \mathbb{N}$ if there is i such that $k\nu^+ \leq \mu_i < (k+1)\nu^+$.

Lemma 4.10. *For a coherent sheaf $\mathscr{F}$ we have that $\operatorname{rk}\mathscr{F}$ is greater or equal than the number of jumps of $\mathscr{F}$.*

Proof. We have that $\operatorname{rk}\mathscr{F} = \sum \operatorname{rk}(F_i\mathscr{F}/F_{i-1}\mathscr{F})$. All jumps correspond to distinct $F_i\mathscr{F}/F_{i-1}\mathscr{F}$ with positive ranks. $\square$

Lemma 4.11. *Consider a complex $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$. We have that $\sum_i \operatorname{rk} \mathscr{H}^i(\mathscr{C}^\bullet)$ is greater or equal than the sum of numbers of jumps of $\mathscr{H}^i(\mathscr{C}^\bullet)$.*

Proof. Clear from the previous lemma. $\square$

Lemma 4.12. *For every complex $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$, there is $0 \leq k \leq \sum_i \operatorname{rk} \mathscr{H}^i(\mathscr{C}^\bullet)$ such that there is no jump at k for $\mathscr{H}^i(\mathscr{C}^\bullet)$ for all $i \in \mathbb{Z}$.*

Proof. Clear from the previous lemma. $\square$

For a given $k \in \mathbb{N}$ we define

$$m_k = min\{i \in \mathbb{N} | i\mu(L) + \mu(L^{b_{2n}}) - \mu(\omega_X) > k\nu^+\}$$

It is clear that there is a constant C that depends only on X and L such that

$$m_k \leq Ck$$

for all $k \in \mathbb{N}$.

Lemma 4.13. *If there is no jump at k for a coherent sheaf $\mathscr{F}$, then there is a short exact sequence $0 \to \mathscr{F}_+ \to \mathscr{F} \to \mathscr{F}_- \to 0$, where $\mathscr{F}_+$ is 1-positive with respect to the m_k-resolution and $\mathscr{F}_+$ is 1-negative with respect to the m_k-resolution. In particular, the obstruction morphism of $\mathscr{F}$ with respect to the m_k-resolution vanishes.*

Proof. We have the following inequalities

$$k\nu^+ < \mu(L^{b_{2n}+m_k}) - \mu(\omega_X) < -\mu(L^{a_{2n}-m_k}) + \mu(\omega_X) < (k+1)\nu^+$$

Therefore, conditions of 4.2 are satisfied if we consider the m_k- resolution since in this case $a = \mu(L^{b_{2n}+m_k}) - \mu(\omega_X)$, $b = -\mu(L^{a_{2n}-m_k}) + \mu(\omega_X)$, and there is no i such that $k\nu^+ \leq \mu_i < (k+1)\nu^+$. $\square$

Lemma 4.14. *Consider a complex $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$. If there is no jump at k for $\mathscr{H}^i(\mathscr{C}^\bullet)$ for every $i \in \mathbb{Z}$, then the obstruction morphism of $\mathscr{C}^\bullet$ with respect to the m_k-resolution vanishes.*

Proof. Direct consequence of the previous lemma and 3.25. $\square$

Now we can prove the main result of this section.

Theorem 4.15. *There is a constant K that depends only on X and L such that for every $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X)$ we have that $\mathscr{C}^\bullet \in \langle \oplus_{i=0}^{i=k} L^i \rangle_{2n-1}$, where $k = K \sum_i \operatorname{rk} \mathscr{H}^i(\mathscr{C}^\bullet)$.*

Proof. We combine lemmas 4.14, 4.12 and 3.8. $\square$

We define subcategories of complexes with a bound on the total rank

$$\mathcal{T}_N = \{\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X) | \sum_i \operatorname{rk} \mathscr{H}^i(\mathscr{C}^\bullet) < N\}$$

Corollary 4.16. *For every $N \in \mathbb{N}$, there is $G \in \mathbf{D}^b_{coh}(X)$ such that G generates $\mathcal{T}_N$ in $2n-1$ steps.*

Proof. We can take $G = \oplus_{i=0}^{i=KN} L^i$, where K is the constant from the previous theorem. $\square$

CHAPTER 5

Derived categories of coherent D-modules

For a smooth projective variety X, the derived category of coherent sheaves on X is an important and well-studied invariant. One of the most basic and important questions one can ask is the following. Given two varieties X and Y such that there is an exact equivalence $F : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(Y)$, is it true that a certain property must be the same for X and Y. In such a case we say that the property is a derived invariant. Turns out that $\mathbf{D}^b_{coh}(X)$ contains a lot of information about the geometry of X, and there are a lot of theorems of this form that are known. We list some of them here, in all of the theorems the assumptions are that X and Y are smooth projective varieties such that there is an exact equivalence $F : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(Y)$.

Theorem (Kawamata, [**13**])**.** $\dim(X) = \dim(Y)$

Theorem (Bondal-Orlov, [**5**])**.** *If the (anti-)canonical bundle of X is ample then X and Y are isomorphic.*

Theorem (Orlov, [**20**])**.** $Ext^i_{X\times X}(\delta_{X*}\mathcal{O}_X, \delta_{X*}\omega^l_X) = Ext^i_{Y\times Y}(\delta_{Y*}\mathcal{O}_Y, \delta_{Y*}\omega^l_Y)$ *for all* $i, l \in \mathbb{Z}$*, where* δ_X, δ_Y *are diagonal embeddings. As an easy corollary we obtain that*

$$\sum_{p-q=i} h^{p,q}(X) = \sum_{p-q=i} h^{p,q}(Y)$$

for all i*.*

Theorem (Rouquier, [**24**]). *$Pic^0(X) \times Aut^0(X) \simeq Pic^0(Y) \times Aut^0(Y)$*

Theorem (Popa-Schnell, [**21**]). *$h^0(X, \Omega^1_X) = h^0(Y, \Omega^1_Y)$ and $h^0(X, T_X) = h^0(Y, T_Y)$.*

We are interested in the situation when $\mathbf{D}^b_{coh}(X)$ is replaced with $\mathbf{D}^b_{coh}(\mathscr{D}_X)$, the bounded derived category of coherent $\mathscr{D}$-modules on X. The only similar theorem in this setting that we were able to find in the literature is the following.

Theorem (Favero-Arinkin, [**7**]). *An abelian variety A can be reconstructed from its derived category of coherent $\mathscr{D}$-modules. If two abelian varieties A and B have equivalent derived categories of coherent $\mathscr{D}$-modules then $A \simeq B$.*

It is stated in [**7**] that Orlov conjectured that the same is true for all varieties. For an abelian variety A we denote the moduli space of line bundles on A equipped with an integrable connection by $A^\sharp$ (see [**22**]). The theorem follows from the equivalence $\mathbf{D}^b_{coh}(\mathscr{D}_A) \simeq \mathbf{D}^b_{coh}(A^\sharp)$ and the classification of objects in $\mathbf{D}^b_{coh}(A^\sharp)$ with proper support. We say that an object $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(\mathscr{D}_X)$ is proper if for every $\mathscr{G}^\bullet \in \mathbf{D}^b_{coh}(\mathscr{D}_X)$ we have that $\mathrm{Hom}(\mathscr{G}^\bullet, \mathscr{F}^\bullet)$ and $\mathrm{Hom}(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$ are finite dimensional. We denote the subcategory of proper objects by $\mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X))$. Motivated by the result of Favero and Arinkin we study $\mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X))$ and prove the following theorem

Theorem 5.1. *For a smooth projective variety X we have that*

$$\mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X)) = \{\mathscr{C}^\bullet \mid \mathscr{H}^i(\mathscr{C}^\bullet) \text{ is a vector bundle with a flat connection for all } i \in \mathbb{Z}\}.$$

We also show that $\mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X))$ is equipped with a Serre functor given by $\mathscr{C}^\bullet \to \mathscr{C}^\bullet[2\dim X]$ and obtain the following theorem as a corollary.

Theorem 5.2. *Assume that X and Y are smooth projective varieties such that there is an equivalence $F : \mathbf{D}^b_{coh}(\mathscr{D}_X) \to \mathbf{D}^b_{coh}(\mathscr{D}_Y)$. Then $\dim X = \dim Y$. Assume additionally that X is simply connected. Then Y is also simply connected, and we have there is an isomorphism $H^i(X, \mathbb{C}) \simeq H^i(Y, \mathbb{C})$ induced by F for all i.*

It is somewhat surprising that even the equality of dimensions is not obvious. We were not able to find this result mentioned in the literature or find a proof that is more elementary than the one presented in this chapter.

We also note that the equality of Betti numbers for varieties with equivalent derived categories of coherent sheaves is only conjectured. And while it is known for surfaces and threefolds, there is no natural isomorphism of vector spaces between $H^i(X, \mathbb{C})$ and $H^i(Y, \mathbb{C})$ induced by the equivalence of derived categories of coherent sheaves of X and Y.

5.1. Notation for derived categories D-modules

We use the notation from [**9**] for functors between derived categories of coherent $D-$modules that we recall here. For a smooth variety X there is the duality functor

$$\mathbb{D} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$$

For a smooth proper morphism $f : X \to Y$ of smooth varieties we have functors

$$\int_f : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$$

$$f^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$$

$$f^\dagger = f^*[\dim X - \dim Y]$$

$$f^\star = f^*[\dim Y - \dim X]$$

We have that $f^\star$ is left-adjoint to $\int_f$ and $f^\dagger$ is right-adjoint to $\int_f$ for holonomic D-modules [**9**, Corollary 3.2.15].

5.2. Proper complexes of D-modules

Definition 5.3. Let X be a smooth projective variety. We say that an object $\mathscr{F}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ is right-proper if $\mathrm{Hom}(\mathscr{G}^\bullet, \mathscr{F}^\bullet)$ is a finite dimensional vector space for every $\mathscr{G}^\bullet \in \mathbf{D}^b_{coh}(\mathscr{D}_X)$. The subcategory of right-proper objects is denoted by $\mathfrak{Prop}_r(\mathbf{D}^b_{coh}(\mathscr{D}_X))$.

Similarly, we say that an object $\mathscr{F}^\bullet \in \mathbf{D}^b_{coh}(\mathscr{D}_X)$ is left-proper if $\mathrm{Hom}(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$ is a finite dimensional vector space for every $\mathscr{G}^\bullet \in \mathbf{D}^b_{coh}(\mathscr{D}_X)$. The subcategory of left-proper objects is denoted by $\mathfrak{Prop}_l(\mathbf{D}^b_{coh}(\mathscr{D}_X))$. It is clear that an exact equivalence $\mathbf{D}^b_{coh}(\mathscr{D}_X) \simeq \mathbf{D}^b_{coh}(\mathscr{D}_Y)$ induces equivalences

$$\mathfrak{Prop}_r(\mathbf{D}^b_{coh}(\mathscr{D}_X)) \simeq \mathfrak{Prop}_r(\mathbf{D}^b_{coh}(\mathscr{D}_Y)),\ \mathfrak{Prop}_l(\mathbf{D}^b_{coh}(\mathscr{D}_X)) \simeq \mathfrak{Prop}_l(\mathbf{D}^b_{coh}(\mathscr{D}_Y)).$$

It is also clear that the duality functor on $\mathbf{D}^b_{coh}(\mathscr{D}_X)$ interchanges right-proper objects and left-proper objects $\mathbb{D}(\mathfrak{Prop}_r(\mathbf{D}^b_{coh}(\mathscr{D}_X))) = \mathfrak{Prop}_l(\mathbf{D}^b_{coh}(\mathscr{D}_X))$.

Theorem. *For a smooth projective variety X we have that*

$$\mathfrak{Prop}_r(\mathbf{D}^b_{coh}(\mathscr{D}_X)) = \{\mathscr{C}^\bullet \mid \mathscr{H}^i(\mathscr{C}^\bullet) \text{ is a vector bundle with a flat connection for all } i \in \mathbb{Z}\}.$$

Proof. Fix an ample line bundle L on X. Given a right-proper object $\mathscr{F}^\bullet$ we have that

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{D}_X \otimes_{\mathscr{O}_X} L^i, \mathscr{F}^\bullet[j]) \simeq \mathrm{Hom}_{\mathscr{O}_X}(L^i, \mathscr{F}^\bullet[j]) \simeq \mathbb{H}^j(\mathscr{F}^\bullet \otimes L^{-i})$$

is a finite dimensional vector space for all $i, j \in \mathbb{Z}$. Consider a resolution over $\mathscr{O}_X$ of the structure sheaf of the diagonal $\mathscr{O}_\Delta \in Coh(X \times X)$ of the following form

$$... \to \oplus L^{-b_2} \boxtimes L^{-a_2} \to \oplus L^{-b_1} \boxtimes L^{-a_1} \to \oplus L^{-b_0} \boxtimes L^{-a_0} \to \mathscr{O}_\Delta \to 0,$$

where every direct sum is finite. Since $\mathrm{Ext}^i(\mathscr{F}, \mathscr{G}) = 0$ for $i > 4n$ and coherent D-modules $\mathscr{F}, \mathscr{G}$ on $X \times X$, truncating the resolution at $-4n = -4 \dim X$ we obtain that $\mathscr{O}_\Delta$ is a direct summand of

$$\mathscr{C}^\bullet = \{0 \to L^{-b_{4n}} \boxtimes L^{-a_{4n}} \to ... \to \oplus L^{-b_2} \boxtimes L^{-a_2} \to \oplus L^{-b_1} \boxtimes L^{-a_1} \to \oplus L^{-b_0} \boxtimes L^{-a_0} \to 0\},$$

where we consider $\mathscr{C}^\bullet \in \mathbf{D}^b_{coh}(X \times X)$ as a complex concentrated in degrees from $-4n$ to 0. Therefore, $\mathscr{F}^\bullet = \Phi_{\mathscr{O}_\Delta}(\mathscr{F}^\bullet)$ is a direct summand of $\Phi_{\mathscr{C}^\bullet}(\mathscr{F}^\bullet)$, where by $\Phi_{\mathscr{K}} : \mathbf{D}^b_{qcoh}(X) \to \mathbf{D}^b_{qcoh}(X)$ we denote the Fourier-Mukai functor with the kernel $\mathscr{K}$. But $\Phi_{\mathscr{C}^\bullet}(\mathscr{F}^\bullet)$ is constructed from finite direct sums of $R\Gamma(L^{-b_i} \otimes \mathscr{F}^\bullet) \otimes L^{-a_i}$ by taking $4 \dim X$ cones. Since $R\Gamma(L^{-b_i} \otimes \mathscr{F}^\bullet)$ is finite dimensional for all i, it follows that $\mathscr{F}^\bullet$ belongs to $\mathbf{D}^b_{coh}(X) \subset \mathbf{D}^b_{qcoh}(X)$ and $\mathscr{H}^i(\mathscr{F}^\bullet)$ are coherent over $\mathscr{O}_X$ for all $i \in \mathbb{Z}$. But $\mathscr{D}$-modules coherent over $\mathscr{O}_X$ are integrable connections. The other direction follows from [**9**][Lemma D.2.4.]. $\square$

As a consequence we see that $\mathfrak{Prop}_r(\mathbf{D}^b_{coh}(\mathscr{D}_X)) = \mathfrak{Prop}_l(\mathbf{D}^b_{coh}(\mathscr{D}_X))$ since the duality functor maps vector bundles with a flat connections to vector bundles with a flat connection. From now on we denote $\mathfrak{Prop}_r(\mathbf{D}^b_{coh}(\mathscr{D}_X))$ by just $\mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X))$.

Lemma 5.4. *For a smooth projective variety X we have that $\mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X))$ is equipped with a Serre functor given by $S_X : \mathscr{C}^\bullet \to \mathscr{C}^\bullet[2\dim X]$.*

Proof. Consider the projection $\pi : X \to pt$ and $\mathscr{F}^\bullet \in \mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X))$. We have that

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{F}^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}_X}(\pi^\star \mathscr{O}_{pt}[\dim X], \mathscr{F}^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}_{pt}}(\mathscr{O}_{pt}[\dim X], \int_\pi \mathscr{F}^\bullet) \simeq$$

$$\mathrm{Hom}_{\mathscr{D}_{pt}}(\int_\pi \mathscr{F}^\bullet, \mathscr{O}_{pt}[dimX])^* \simeq Hom_{\mathscr{D}_X}(\mathscr{F}^\bullet, \pi^\dagger \mathscr{O}_{pt}[\dim X])^* \simeq$$

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{F}^\bullet, \mathscr{O}_X[2\dim X])^*.$$

Therefore, for $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in \mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X))$ we have

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{F}^\bullet, \mathscr{G}^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{O}_X, \mathbb{D}\mathscr{F}^\bullet \otimes_{\mathscr{O}_X} \mathscr{G}^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}_X}(\mathbb{D}\mathscr{F}^\bullet \otimes_{\mathscr{O}_X} \mathscr{G}^\bullet, \mathscr{O}_X[2\dim X])^* \simeq$$

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{F}^\bullet \otimes_{\mathscr{O}_X} \mathbb{D}\mathscr{G}^\bullet[2\dim X])^* \simeq \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{G}^\bullet, \mathscr{F}^\bullet[2\dim X])^*,$$

where we used that $\mathbb{D}\mathscr{E} \simeq \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$ for an integrable connection $\mathcal{E}$. See [**9**][Corollary 2.6.15, Theorem 3.2.14] $\square$

Corollary 5.5. *Let X and Y be smooth projective varieties such that there is an equivalence $F : \mathbf{D}^b_{coh}(\mathscr{D}_X) \to \mathbf{D}^b_{coh}(\mathscr{D}_Y)$. Then $\dim X = \dim Y$.*

Proof. We have that F induces the exact equivalence of subcategories

$$F : \mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_X)) \to \mathfrak{Prop}(\mathbf{D}^b_{coh}(\mathscr{D}_Y)).$$

Since exact equivalences commute with Serre functors we have

$$F(\mathscr{O}_X)[2\dim X] \simeq F(\mathscr{O}_X[2\dim X]) \simeq F(S_X(\mathscr{O}_X)) \simeq$$

$$S_Y(F(\mathscr{O}_X)) \simeq F(\mathscr{O}_X)[2\dim Y].$$

Since $F(\mathscr{O}_X)$ is a bounded nonzero complex, we obtain that $\dim X = \dim Y$. $\square$

5.3. Derived invariants of coherent D-modules

We showed in the previous section that $\dim X$ is determined by $\mathbf{D}^b_{coh}(\mathscr{D}_X)$. The main goal of this section is to prove that the property of being simply connected is also determined by $\mathbf{D}^b_{coh}(\mathscr{D}_X)$, and that for simply connected varieties Betti numbers are also determined by $\mathbf{D}^b_{coh}(\mathscr{D}_X)$.

Lemma 5.6. *Let X and Y be smooth projective varieties such that there is an equivalence $F : \mathbf{D}^b_{coh}(\mathscr{D}_X) \to \mathbf{D}^b_{coh}(\mathscr{D}_Y)$. Assume that Y is simply connected. Then $F(\mathscr{O}_X) = \mathscr{O}_Y[k]$ for some $k \in \mathbb{Z}$.*

Proof. From theorem 5.2 we know that $\mathscr{H}^i(F(\mathscr{O}_X)) \simeq \oplus_{j=0}^{j=a_i} \mathscr{O}_Y$ for all $i \in \mathbb{Z}$ since Y is simply connected, and there are no nontrivial vector bundles with flat connection on a simply connected variety. We want to show that there is $k \in \mathbb{Z}$ such that $a_k = 1$ and $a_i = 0$ for $i \neq k$. Take a to be the smallest integer i such that $\mathscr{H}^i(F(\mathscr{O}_X)) \neq 0$. Take b

to be the largest integer i such that $\mathscr{H}^i(F(\mathscr{O}_X)) \neq 0$. We know that

$$\operatorname{Hom}(F(\mathscr{O}_X), F(\mathscr{O}_X)[i]) = \operatorname{Hom}(\mathscr{O}_X, \mathscr{O}_X[i]) = 0$$

for $i < 0$. On the other hand it is easy to see that

$$\operatorname{Hom}(F(\mathscr{O}_X), F(\mathscr{O}_X)[a-b]) \neq 0$$

since $\mathscr{H}^i(F(\mathscr{O}_X)) = 0$ for $i > b$, $\mathscr{H}^i(F(\mathscr{O}_X)[a-b]) = 0$ for $i < b$, and

$$\mathscr{H}^b(F(\mathscr{O}_X)) = \oplus_{j=0}^{j=k_1} \mathscr{O}_Y,\ \mathscr{H}^b(F(\mathscr{O}_X)[a-b]) = \oplus_{j=0}^{j=k_2} \mathscr{O}_Y$$

for some nonzero k_1, k_2. Therefore, $a = b$ and

$$\dim \operatorname{Hom}(F(\mathscr{O}_X), F(\mathscr{O}_X)) = k_1^2.$$

Since

$$\operatorname{Hom}(F(\mathscr{O}_X), F(\mathscr{O}_X)) \simeq \mathbb{C}$$

we obtain that $k_1 = 1$ and $F(\mathscr{O}_X) \simeq \mathscr{O}_Y[-a]$. $\square$

Corollary 5.7. *Let X and Y be smooth projective varieties such that there is an equivalence $F : \mathbf{D}^b_{coh}(\mathscr{D}_X) \to \mathbf{D}^b_{coh}(\mathscr{D}_Y)$. Assume that Y is simply connected. Then X is also simply connected and we have isomorphisms $H^i(X, \mathbb{C}) \simeq H^i(Y, \mathbb{C})$ for all i.*

Proof. Consider a vector bundle with flat connection $\mathscr{E} \in \mathbf{D}^b_{coh}(\mathscr{D}_X)$. By the same argument as in the proof of the previous lemma we see that $F(\mathscr{E}) = \oplus_{i=0}^{i=a} \mathscr{O}_Y[k]$ for some $k \in \mathbb{Z}$, $a \in \mathbb{N}$ since $\operatorname{Hom}(F(\mathscr{E}), F(\mathscr{E})[-i]) \simeq \operatorname{Hom}(\mathscr{E}, \mathscr{E}[-i]) = 0$ for $i > 0$. But we also know from the same lemma that $F(\oplus_{i=0}^{i=a}\mathscr{O}_X[i]) \simeq \oplus_{i=0}^{i=a}\mathscr{O}_Y[k]$ for some $i \in \mathbb{Z}$. Which means that $\mathscr{E} \simeq \mathscr{O}_X^a$ since F is an equivalence. Therefore, there are no nontrivial vector bundles with flat connection on X, which means that X is simply connected. The second statement follows from

$$H^i(X,\mathbb{C}) \simeq \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X[i]) \simeq \mathrm{Hom}_{\mathscr{D}_Y}(F(\mathscr{O}_X), F(\mathscr{O}_X)[i])$$
$$\simeq \mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{O}_Y[k], \mathscr{O}_Y[i+k]) \simeq H^i(Y,\mathbb{C}).$$

$\square$

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