NORTHWESTERN UNIVERSITY

Some Functoriality Results for Microlocal Sheaves over Legendrians and Lagrangians

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

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EVANSTON, ILLINOIS

June 2023

ABSTRACT

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In this thesis, we consider the categories of sheaves with singular support on certain Lagrangians and the categories of microlocal sheaves with support on certain Lagrangians obtained by microlocalization, and study properties of functors between these categories.

First, we study one class of the microlocal restriction functor for open inclusions, namely microlocalization on the conical Lagrangian ends. We show a duality and exact triangle arising from the microlocalization functor. Using that, we describe the adjoint functors of microlocalization, and prove that they form a spherical adjunction when the Legendrian at infinity is a full stop or swappable stop.

Using the description of the adjoint functors of microlocalization, we prove sheaf quantization theorems, constructing right inverses to the microlocalization functor

for noncompact Lagrangian submanifolds, generalizing previous works of Guillermou and Jin–Treumann. In particular, we show a sheaf quantization theorem for Lagrangian cobordisms of Arnol'd and a conditional quantization theorem for Lagrangian cobordisms in symplectic field theory.

Then, we study the microlocal specialization functor on closed subdomain embeddings of Weinstein sectors, which is right adjoint to the Viterbo restriction functor, constructed by Nadler–Shende. We show that the specialization functor is natural with respect to compositions of embeddings. Using that, we give another description of Lagrangian cobordism functor in symplectic field theory, which is compatible with the sheaf quantization functor.

Along the way, we obtain applications to symplectic and contact geometric problems, including estimations of the number of Reeb chords on Legendrians and obstructions to Lagrangian cobordisms between Legendrians.

Acknowledgements

First of all, I would like to thank my advisors Eric Zaslow and Emmy Murphy. I feel really lucky of myself to have come to Northwestern at a great time when both of you were here. I can still remember the first time I first talked to you, when you showed me the field of symplectic geometry and mirror symmetry. Both of you have spent a surprisingly large amount of time talking and discussing with me from which I learned a lot of different mathematics from you. I am surprised by your broad knowledge in mathematics. I learned from Eric how to do explicit computations on very concrete examples, while I learned from Emmy how to understand the geometric picture through an explicit and intuitive perspective. I also enjoyed talking to you about all sorts of things other than maths. Thank you for always being encouraging and supportive, and for being confident on me more than I do myself, during the past five years. It is your encouragement that allowed me to explore different directions in my research. Finally, I would like to thank Eric for suggesting the paper for my qualifying exam, which, to a large extent, guided my research in the past few years.

Next, I would like to thank my collaborator Roger Casals, who is probably the one whom I have discussed mathematics with most frequently other than my advisors. I am always surprised by your energy and passion on mathematics. I remember your email which started with saying you were quite busy and had to write a short reply and went on and on until finally you wrote that you had to go in less than five minutes. I am very grateful to your generous help on my career and I really enjoy our discussion on math from which I have benefitted a lot.

Then, I would like to thank my collaborators (and potential collaborators) Tomohiro Asano, Yuichi Ike, Christopher Kuo, David Nadler and Vivek Shende. I am very glad that we have the chance to collaborate and work on a common project, and I am largely influenced by your enthusiasm on mathematics. I would also like to thank Honghao Gao, Oleg Lazarev, Jun Zhang and Peng Zhou, with whom I have had plenty of mathematical discussions, got new mathematical understandings and formed new mathematical ideas. I would like to thank Dima Tamarkin as well, for discussion on mathematics in my last year of PhD, where I also learned a lot from him. He is very fast in learning things and is always willing to help me on the questions I have. His knowledge on microlocal analysis and homological algebra has helped me a lot.

I would like to thank Orsola Capovilla-Searle and James Hughes for your kind help in my job application. Thank you for your generosity of providing very helpful suggestions on my research statements even though we have just known each other in such a short time. Also, thank you for your hospitality when I was visiting Davis. That was a very enjoyable time. I would like to thank you and Daping Weng for being there and for talking about mathematics. I would also like to thank Johan Asplund, Côme Dattin, Honghao Gao, Xin Jin, Joseph Palmer, Yu Pan, Linhui Shen, David

Treumann and Jun Zhang for their interest in my work, their kindness of inviting me to the seminars (online or in person) and their hospitality.

In the past five years, I have benefitted from a number of people in the math department at Northwestern. I want to thank Aaron Brown, John Francis, Ezra Getzler, Bryna Kra and Boris Tsygan who have taught me a lot of mathematics. I would like to thank Bahar Acu, Dogan Karabas and Yuan Hang in the symplectic geometry group who have taught me a lot and gave me numerous advice. I would like to thank Yajit Jain, Grisha Kondyrev, Nilay Kumar, Nick McCleerey, Anthony McCormick and Piotr Pstragowski for their support and help on mathematics. I would like to thank other PhD students who were in the symplectic geometry and mathematical physics group, Zhenyi Chen, Mingyuan Hu, Yuxuan Hu, Alex Karapetyan, Juan Diego Rojas, John Snadden, Pyongwon Suh, Hanru Sun, Junxiao Wang, Randy Van Why and Ben Zhou, who have helped me much on symplectic geometry and mathematical physics. In particular, Pyongwon was the one who taught me spherical functors, which is part of the thesis, and Hang and Mingyuan also had very fruitful mathematical discussions with me in my fifth year. I would also like to thank Greg Bixler, Carlos Cortez Lemos, Yi Gu, Wanxing Liu, Fanjun Meng, Stephan Snegirov, Jiyao Tang, Kai-Hsiang Wang, Ruoyu Wang, Shiyi Weng, Mengxuan Yang and Yuxin Zhou. It was great to chat with you. I want to thank the staffs in the math department as well, without whom a lot of things would be harder for me.

I have also benefitted from a number of people from other math departments. I am very grateful to be able to stay in Princeton in my fourth year. I would like to thank Shaoyun Bai, Thomas Massoni and Mohan Swaminathan, who have helped me both in terms of math and in terms of life in Princeton. I would like to thank Julian Chaidez, Yusuf Barış Kartal and John Pardon, the people in Princeton symplectic group, for plenty of discussion in math. I would like to thank the staff there for their help. I would also like to thank other mathematicians I have met and talked to. I want to thank Youngjin Bae, Joseph Breen, Austin Christian, Yasha Eliashberg, Stéphane Guillermou, Yixuan Li, Siyang Liu, Yoon Jae Nick Nho, Hiro Lee Tanaka and Bingyu Zhang for helpful discussions. Moreover, I would like to thank Gourgeois Dimitroglou Rizell, Roman Golovko, James Pascaleff, Joshua Sabloff and Lisa Traynor for helpful discussion related to my research.

In addition, I would like to thank my friends for their emotional support during the past few years. Being a person who is kind of inactive in social activities, it was really great that you have spent some unforgettable time with me. I would like to thank my roommates Yian Yin, Binglu Wang and Wuqing Guo that have brought a lot of laughters and joy in my life, without whom my life in Evanston would be much more boring. I would like to thank Yuxin Zhou, Junwen Tang, Xinyu Li, Jiarui Wang and Zhuolin Li for the time we spent in chatting (online or in person) which was really relaxing. I would like to thank Ben Zhou for chatting about all sorts of things, from daily life to hobbies. I would like to thank Yanqing Shen and Celia Xu for the time we spent reading and chatting about literature during the pandemic. Celia was the one who invited and took me to Chicago city three years after I came here, where I had a great time listening to her passionately talking about the paintings in the Art Institute. I would like to thank Chengyang Bao, Xiang Li, Yixuan Li, Hang Qi, Lie Qian, Jingyan Shang, Yibo Sun, Yue Sun, Dun Tang, Yiyuan Wang, Sailun Xu and Ziwen Zhao for hanging out when I was visiting their places. Dun Tang has been very hospitality when I came to Berkeley and even drove me to Davis. I would like to thank my friends Sihao Chen, Xiaoxun Liu, Ke Ning, Yunhan Wen, Steven Ye, Zekun Cai, Qiuhao Chen, GuanPeng Chen, Sim U Lam and Fangzheng Shi for their support as well.

Finally, I want to thank my mom, Yi Wang, and my grandparents Xiumei Tang and Jiwu Wang, who have supported my decision. I would like to thank my dad Bingxin Li. Also, thanks to my aunt Yun Wang, my uncle Weichun Liu and my nephew Juntao Liu. My grandfather Jiwu Wang passed away this winter. This thesis is dedicated to him.

> *His soul swooned slowly as he heard the snow falling faintly through the universe and faintly falling, like the descent of their last end, upon all the living and the dead.* —- James Joyce, *The Dead.*

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CHAPTER 1

Introduction

Our goal in the thesis is to prove some general results on the functoriality of microlocal sheaves over Legendrian and Lagrangian submanifolds, which we hope would help set up the whole theory of the functoriality of mcirolocal sheaves that arises from symplectic and contact topology. More precisely, we consider exact symplectic manifolds with contact boundaries that have Lagrangian skeleta, and study functorialities arising from proper inclusions (as the generalized version of open inclusions) and subdomain embeddings (as the generalized version of closed inclusions).

In the first situation, we focus on cotangent bundles and the proper inclusion of the conical end, which induces the microlcoalization functor or the cap functor. We show that there is a duality and exact triangle, and under certain assumptions a spherical adjunction, that arise from the microlocalization. The adjunction allows us to get sheaf quantization functors, which are right inverses to the microlocalziation functor, in nice cases, for compact and noncompact submanifolds. In the second situation, we focus on the subdomain embedding of general Lagrangian skeleta, which induces the specialization functor or the right adjoint of the Viterbo restriction functor. We show that compositions of embeddings induce compositions of functors, and in particular, restricting to the setting of Lagrangian cobordisms between Legendrians we get functoriality of cobordisms.

1.1. Motivation and Background

Symplectic manifolds have the same local models given by Darboux charts. However, the ground breaking work of Gromov [**83**] showed that there are also rigidity behaviours which are not detected by the local structures, including restrictions on Lagrangian embeddings. Following Gromov's approach of pseudoholomorphic curves, Floer [**71**] realized that one can associate algebraic invariants, namely the Floer homology groups, to symplectic manifolds and their Lagrangian submanifolds, and Fukaya [**72**] upgraded the homology groups into an A_∞-category, later called Fukaya categories [**139**]. Inspired by homological mirror symmetry [**102**], Seidel considered symplectic Lefschetz fibrations and started to develop functoriality of Fukaya categories associated with Lefschetz fibrations [**139**, **141**].

On the other hand, Eliashberg-Gromov [**64**] considered (exact) symplectic manifolds with contact type boundaries, namely Liouville manifolds. Under this setup, the framework of symplectic field theory [**63**] and the relative setting [**50**] allows one to understand invariants of the symplectic manifolds/Lagrangian submanifolds as maps and functors between algebraic invariants of the contact manifolds/Legendrian submanifolds, namely the contact homology and Legendrian contact homology. Meanwhile, the geometry of symplectic manifolds with contact boundaries is also used in defining functors between Fukaya categories [**5**].

More recently, our understanding on Lefschetz fibrations and Weinstein handlebody theory [**25**, **82**] allows us to combine different viewpoints. The development of

Liouville stops [**151**] and sectors [**75**, **76**] allows people to understand the functoriality of the algebraic invariants in a uniform way. In particular, the basic functoriality properties we considered above could be interpreted as the ones arising from Liouville proper inclusions (as generalized version of open inclusions) and Liouville subsector embeddings (as generalized version of closed inclusions) [**109**].

Parallel to the development in symplectic and contact topology, microlocal theory of sheaves on manifolds was developed by Kashiwara-Schapira [**97**], after the theory of D-modules and constructible sheaves on complex manifolds. Kashiwara-Schapira also noticed the connection of the theory to symplectic geometry, namely, the invariance of the sheaf categories under contact transformations.

Nadler-Zaslow [**126**] and Nadler [**119**] and more recently Ganatra-Pardon-Shende [**74**] proved an equivalence between certain categories of constructible sheaves and suitable versions of Fukaya categories on pairs of exact symplectic manifolds with Lagrangian skeleta. Tamarkin applied microlocal theory of sheaves to some classical non-displaceability results, as instances of symplectic rigidity [**153**]. Since then, there has been a number of interesting results in symplectic and contact topology established using sheaves [**12**, **13**, **22**, **37**, **84**–**86**, **88**, **145**, **147**, **148**, **161**].

Let M be an analytic manifold and $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian at infinity. We will consider the category of sheaves on M with singular support on Λ [**97**], and the category of microsheaves, i.e. the global section of the Kashiwara-Schapira stack on Λ [**84**, **124**]. They are symplectic invariants associated to Lagrangian skeleta of exact symplectic manifolds, namely Weinstein manifolds with Legendrian stops at infinity [**88**, **124**].

More precisely, following the idea of Nadler-Zaslow [**119**, **126**], Ganatra-Pardon-Shende [**74**] showed that the subcategories of compact objects are equivalent to partially wrapped Fukaya categories

$$Sh^c_\Lambda(M) \simeq \mathrm{Perf}\,\mathcal{W}(T^*M, \Lambda)^{\mathrm{op}}, \;\; \mu Sh^c_{\mathfrak{c}_{X,\Lambda}}(\mathfrak{c}_{X,\Lambda}) \simeq \mathrm{Perf}\,\mathcal{W}(X, \Lambda)^{\mathrm{op}},$$

where $\mathfrak{c}_{X,\Lambda}$ is the Lagrangian skeleton of the Weinstein manifold with stop (X, Λ). From works on the Legendrian surgery formula [**14**, **20**, **52**, **60**], we also know an equivalence between partially wrapped Fukaya categories and Legendrian contact homology with coefficients enriched over chains on the based loop space $C_{-*}(\Omega_*\Lambda)$

$$\mathrm{Perf}\,\mathcal{W}(X, \Lambda) \simeq \mathrm{Perf}\,\mathcal{A}_{C_{-*}(\Omega_*\Lambda)}(\Lambda).$$

On the other hand, by considering an $\mathcal{A}_\infty$-category of augmentations, we have the augmentation sheaf correspondence [**127**], and generalizations into higher dimensions [**24**, **77**, **134**].

Comparing to holomorphic curve invariants, the categories of (micro)sheaves often admit simpler combinatorial descriptions. They are also more directly related to invariants in representation theory. From this perspective, our structural results will provide new understanding on the pseudo-holomorphic curve invariants.

1.2. Functorial Properties of Microlocalizations

For a subanalytic Lagrangian skeleton $\mathfrak{c}_X$ of a Weinstein manifold X and an open subset $\mathfrak{c}_{\overline{F}}$ with is the skeleton of $\overline{F}$, there is a pair of restriction and corestriction functors between microlocal sheaves. We will consider the special case of where $\mathfrak{c}_X = M \cup \Lambda \times \mathbb{R}_{>0}$ is a conical Lagrangian in T^*M and $\mathfrak{c}_{\overline{F}} = \Lambda \times \mathbb{R}_{>0}$ is the conical end.

More precisely, let M be an analytic manifold and $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian at contact boundary. Let $Sh_\Lambda(M)$ be the category of sheaves with singular support on Λ, and $\mu Sh_\Lambda(\Lambda)$ be the category of microlocal sheaves on Λ. One can define the microlocalization functor and its left adjoint

$$m_\Lambda : \; Sh_\Lambda(M) \rightleftharpoons \mu Sh_\Lambda(\Lambda) : m_\Lambda^l.$$

Under the equivalence with Fukaya categories, they are expected to correspond to the cap and cup functors $\cap : \mathcal{W}(T^*M, F) \leftrightharpoons \mathcal{W}(F) : \cup$ [**4**, **75**, **152**]. Under homological mirror symmetry, the cap and cup functors correspond to the pull-back and push-forward functors $i^* : Coh(X) \leftrightharpoons Coh(D) : i_*$ for a divisor $D \subset X$.

Spherical adjunctions are adjunctions that induce interesting autoequivalences from certain exact triangles. For example, in algebraic geometry the exact triangle coming from the adjunction $i_* \vdash i^*$ induces the autoequivalence $- \otimes \mathcal{O}_X(D)$. In Lagrangian Floer theory, Abouzaid-Ganatra's unpublished work [**4**] proved that the cap and cup functors between Fukaya categories form a spherical adjunction where the exact triangles induce the autoequivalence by wrapping once around the stop.

Sylvan proved the result for partially wrapped Fukaya categories when Λ is a swappable stop while avoiding constructing adjunctions and exact triangles geometrically [**152**]. Nadler also proved spherical adjunction for microsheaves in a specific example [**122**].

Using the language of sheaves, we are able to provide a more direct and more general result on the existence of spherical adjunctions at least on cotangent bundles, under the technical assumption that $\Lambda \subset T^{*,\infty}M$ is a full stop [**41**] or a swappable stop [**152**].

Theorem 1.2.1. *For $\Lambda \subset T^{*,\infty}M$ a compact subanalytic Legendrian full stop or swappable stop, the microlocalization functor and its left adjoint*

$$m_\Lambda : \, Sh_\Lambda(M) \rightleftharpoons \mu Sh_\Lambda(\Lambda) \, : m_\Lambda^l$$

form a spherical adjunction, where the spherical cotwist S_Λ^- (resp. dual cotwist S_Λ^+) is given by negatively (resp. positively) wrapping once around the stop $\Lambda \subset T^{*,\infty}M$.

Remark 1.2.1. *We know that the left adjoint of microlocalization m_Λ^l is isomorphic to the cup functor between partially wrapped Fukaya categories* [**74**]*, and we expect that m_Λ is the cap functor between Fukaya-Seidel categories.*

Moreover, we relate Serre duality to the spherical adjunctions. Seidel noticed that the spherical cotwist, i.e. the negative wrap-once functor on the Fukaya-Seidel category, gives the Serre functor [**140, 143**], which is mirror to the autoequivalence $-\otimes\mathcal{O}_X(-D)$ for a log Calabi-Yau pair (X, D). The Serre duality is closely related to the Calabi-Yau structure on the categories [**21**] and symplectic/Lagrangian structure on the derived moduli stack of objects [**156**].

Theorem 1.2.2 (Sabloff-Serre duality)**.** *Let M be orientable and $\Lambda \subset T^{*,\infty}M$ a compact subanalytic Legendrian full stop or swappable stop. Then the spherical cotwist S_Λ^- is the Serre functor on the subcategory of compactly supported sheaves with perfect stalks $Sh_\Lambda^b(M)_0$.*

The key ingredient of the proof is the following duality exact triangle. From a purely sheaf theory perspective, Sato noticed an exact triangle coming from microlocalization [**87, 97**]. On the other hand, in Floer theory, there is a duality exact sequence for Legendrian contact homologies [**58, 135**] and Fukaya-Seidel categories [**141**]. Let $T_t : T^{*,\infty}M \to T^{*,\infty}M$ be a Reeb flow. Define

$$Hom_+(\mathscr{F}, \mathscr{G}) = Hom(\mathscr{F}, \mathscr{G}),\ Hom_-(\mathscr{F}, \mathscr{G}) \simeq Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})).$$

We reinterpret Sato's exact triangle in a symplectic geometric way and prove the following theorem.

Theorem 1.2.3 (Sato-Sabloff duality exact triangle)**.** *For $\Lambda \subset T^{*,\infty}M$ a compact subanalytic Legendrian stop and $\mathscr{F}, \mathscr{G} \in Sh_\Lambda(M)$ such that* $\mathrm{supp}(\mathscr{F}), \mathrm{supp}(\mathscr{G})$ *are compact, we have an exact triangle*

$$Hom_-(\mathscr{F}, \mathscr{G}) \to Hom_+(\mathscr{F}, \mathscr{G}) \to \Gamma(\Lambda; \mu hom(\mathscr{F}, \mathscr{G})) \xrightarrow{+1} .$$

When M is orientable and $\mathscr{F}, \mathscr{G} \in Sh^b_\Lambda(M)$, then

$$Hom_-(\mathscr{F}, \mathscr{G}) \simeq Hom_+(\mathscr{G}, \mathscr{F})^\vee[-n-1].$$

1.3. Functorial Anti-Microlocalization as Sheaf Quantization

Consider an exact Lagrangian $L \subset T^*M$ with Legendrian lift $\widetilde{L} \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$. The microlocalization sends sheaves to microsheaves

$$m_L : Sh_{\widetilde{L}}(M \times \mathbb{R}) \to \mu Sh_L(L).$$

When L is smooth, the structure of microsheaves on L is well studied [**84**, **93**]. Classical sheaf quantization theorems of embedded Lagrangians that are compact or have conical ends have been obtained by Guillermou-Jin-Treumann [**84**, **94**], who constructed sheaves from microlocalization data which defines a right inverse to m_L whose image consists of sheaves with acyclic stalks at $M \times \{-\infty\}$.

$$\Psi_L : \mu Sh_L(L) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R})_0.$$

We explain how the sheaf quantization functor can be regarded as the left adjoint m_L^l of m_L, and hence provide a functorial understanding on the sheaf quantization theorems following [**94**]. More importantly, we generalize the results to the setting of noncompact Lagrangians where the usual construction could fail.

We will consider $Sh_\Lambda(M \times \mathbb{R})_0$ for the categories of sheaves in with singular support in $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ with acyclic stalks at $M \times \{-\infty\}$.

The first class of noncompact Lagrangians are Lagrangian cobordisms between Lagrangians defined by Arnol'd [**10**], which is an equivalence relation between Lagrangian submanifolds. An Arnol'd Lagrangian cobordism between $L, K \subset T^*M$ is a Lagrangian $V \subset T^*(M \times \mathbb{R})$ with cylindrical ends. Biran-Cornea proved that Lagrangian cobordisms give equivalence relations on the Fukaya category, and when the Lagrangian has multiple components, we get an iterated cone decomposition for objects in the Fukaya category [**17**].

Theorem 1.3.1. *Let $V \subset T^*(M \times \mathbb{R})$ be an exact Lagrangian cobordism between closed exact Lagrangians $L_1, \dots, L_r \subset T^*M$ and $K_1, \dots, K_s \subset T^*M$ with a Legendrian lift $\widetilde{V} \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times \mathbb{R})$. Then there is a fully faithful right inverse functor of m_V*

$$\Psi_V : \mu Sh_V(V) \xrightarrow{\sim} Sh_{\widetilde{V}}(M \times \mathbb{R} \times \mathbb{R})_0.$$

Despite of lack of nontrivial examples of closed exact Arnol'd Lagrangian cobordisms in T^*M, we believe that our construction will serve as the first step in understanding the relation between Arnol'd Lagrangian cobordisms and microlocal sheaves.

The second class of noncompact Lagrangians we consider are Lagrangian cobordisms between Legendrians in the sense of symplectic field theory [**27**], which is on the contrary a nonsymmetric relation between Legendrian submanifolds [**28**]. An exact Lagrangian cobordism from Λ_- to Λ_+ is an exact Lagrangian $L \subset (T^{*,\infty}_{\tau>0}(M \times \mathbb{R}) \times \mathbb{R}_{>0}, d(s\alpha_{std}))$ which agrees with the cone $\Lambda_- \times \mathbb{R}_{>0}$ (resp. $\Lambda_+ \times \mathbb{R}_{>0}$) on the negative end (resp. positive end).

Lagrangian cobordisms between Legendrians in $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ can be lifted to a Legendrian cobordism $\widetilde{L} \subset (T^{*,\infty}_{\tau>0}(M \times \mathbb{R}\mathbb{R}_{>0}), \alpha_{std})$ with conical ends [**26, 130**] following [**34**]. Therefore, one can try to study the sheaf quantization problem, i.e. constructing sheaves on $Sh_{\widetilde{L}}(N \times \mathbb{R} \times \mathbb{R}_{>0})$ from microlocal monodromy data.

However, the nonsymmetry of SFT Lagrangian cobordisms suggests that a necessary condition for constructing a sheaf on the conical Legendrian cobordism is the existence of a sheaf on the cancave end $M \times \mathbb{R} \times (0, \epsilon)$. Therefore, we can only prove a conditional sheaf quantization theorem.

Theorem 1.3.2. *Let $L \subset J^1(M) \times \mathbb{R}_{>0}$ be an exact Lagrangian cobordism from Λ_- to $\Lambda_+ \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, and $\widetilde{L} \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times \mathbb{R}_{>0})$ be the conical Legendrian lifting. Then there is a fully faithful right inverse functor of (i_-^{-1}, m_L)*

$$\Psi_L : Sh_{\Lambda_-}(M \times \mathbb{R})_0 \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0})_0$$

where $i_- : M \times \mathbb{R} \times s_- \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$ for $s_- > 0$ sufficiently small and $m_L : Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0}) \to \mu Sh_L(L)$ is the microlocalization.

1.4. Functoriality of Embeddings and Lagrangian Cobordisms

For a Weinstein sector (Weinstein manifold with boundaries) X, with Lagrangian skeleton $\mathfrak{c}_X$ equipped with Maslov data, Nadler-Shende introduced a microlocal sheaf category on the Lagrangian skeleton $\mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X)$ [**124**]. Moreover, they constructed a fully faithful inclusion for any compact exact Lagrangian $L \subset X$ with Maslov data

$$Loc(L) \hookrightarrow \mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X).$$

Using their technique, one can produce a fully faithful functor for embeddings of Weinstein subsectors $X' \subset X$ sending sectorial boundaries to boundaries $\partial X' \subset \partial X$, where the left adjoint is the Viterbo restriction. However, it is not clear whether compositions of embeddings induce compositions of functors. We show that this is indeed the case.

Theorem 1.4.1. *Let X_0, X_1, and X_2 be Weinstein sectors with Lagrangian skeleta $\mathfrak{c}_{X_0}$, $\mathfrak{c}_{X_1}$, and $\mathfrak{c}_{X_2}$ equipped with Maslov data, such that $i_{01} : X_0 \hookrightarrow X_1$ and $i_{12} : X_1 \hookrightarrow X_2$ are Liouville embeddings sending sectorial boundaries to sectorial boundaries. Denote by $\Phi_{ij} : \mu Sh_{\mathfrak{c}_{X_i}}(\mathfrak{c}_{X_i}) \hookrightarrow \mu Sh_{\mathfrak{c}_{X_j}}(\mathfrak{c}_{X_j})$ the embeddings of microsheaf categories. Then*

$$\Phi_{12} \circ \Phi_{01} \simeq \Phi_{02} : \mu Sh_{\mathfrak{c}_{X_0}}(\mathfrak{c}_{X_0}) \hookrightarrow \mu Sh_{\mathfrak{c}_{X_2}}(\mathfrak{c}_{X_2}).$$

Subsector embeddings provide a geometric model for Lagrangian cobordisms between Legendrians in the setting of symplectic field theory [**63**]. Therefore, we deduce a Lagrangian cobordism functor using Nadler-Shende [**124**] and Theorem 1.4.1. This functor is the sheaf theory counterpart of the Lagrangian cobordism map between Legendrian contact homologies (enriched over chains on the based loop spaces)

[**50, 59, 129**]

$$\Phi_L^* : \mathcal{A}_{C_{-*}(\Omega_*\Lambda_+)}(\Lambda_+) \to \mathcal{A}_{C_{-*}(\Omega_*\Lambda_-)}(\Lambda_-) \otimes_{C_{-*}(\Omega_*\Lambda_-)} C_{-*}(\Omega_* L)$$

which will allow one to deduce more refined obstructions to Lagrangian cobordisms between Legendrian submanifolds.

Theorem 1.4.2. *Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, $\Lambda_-, \Lambda_+ \subset \partial_\infty X$ be Legendrian submanifolds, and $L \subset \partial_\infty X \times \mathbb{R}$ an exact Lagrangian cobordism from Λ_- to Λ_+. Then there is a fully faithful cobordism functor between the microsheaf categories where concatenations of cobordisms give compositions of functors*

$$\Phi_L : \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times_{Loc(\Lambda_-)} Loc(L) \hookrightarrow \mu Sh_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}).$$

In particular, when $X = T^*M$, there is a fully faithful cobordism functor

$$\Phi_L : Sh_{\Lambda_-}(M) \times_{Loc(\Lambda_-)} Loc(L) \hookrightarrow Sh_{\Lambda_+}(M).$$

Remark 1.4.1. *The left adjoint Φ_L^l of Φ_L preserves compact objects and is conjecturally isomorphic to the Lagrangian cobordism maps between Legendrian contact homologies.*

Moreover, we can show that this Lagrangian cobordism functor is compatible with the cobordism functor one can obtain using the sheaf quantziation functor in the previous section.

Theorem 1.4.3. *Let $L \subset J^1(M) \times \mathbb{R}_{>0}$ be an exact Lagrangian cobordism from Λ_- to $\Lambda_+ \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, and $\widetilde{L} \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times \mathbb{R}_{>0})$ be the conical Legendrian lifting. Then there is a commutative diagram*

$$\begin{array}{ccc} & Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0}) & \\ {\scriptstyle (i_-^{-1}, m_L)} \swarrow & & \searrow {\scriptstyle i_+^{-1}} \\ Sh_{\Lambda_-}(M \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L) & \xrightarrow{\quad \Phi_L \quad} & Sh_{\Lambda_+}(M \times \mathbb{R}) \end{array}$$

where $i_- : M \times \mathbb{R} \times s_- \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$ for $s_- > 0$ sufficiently small and $i_+ : M \times \mathbb{R} \times s_+ \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$ for $s_+ > 0$ sufficiently large.

1.5. Symplectic/Contact Consequences of Sheaf Theory

In this section, We explain some geometric results in classical symplectic/contact topology problems that we proved, and demonstrate the relation between geometric and algebraic structures coming from sheaves. We will prove symplectic/contact results using the functorial properties of sheaves.

1.5.1. Estimating the number of Reeb chords

Tamarkin's pioneering work [**153**] applying microlocal theory of sheaves to symplectic non-displaceability problems has inspired a number of non-displaceability type results [**12**, **88**]. Given the algebraic result in Theorem 1.2.3, we can estimate the number of Reeb chords of Legendrians.

We prove an estimation on the number of self Reeb chords for a Legendrian. We expect that our result is more general than the results using linear representations of Legendrian contact homologies [**45**, **56**, **58**] or generating families [**136**].

Theorem 1.5.1. *Let M be orientable, $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be a closed chord generic Legendrian submanifold and $\Bbbk$ be a field. If there exists a $\Bbbk$-coefficient sheaf of compact support and perfect stalk $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})_0$, then the number of self Reeb chords*

$$|\mathcal{Q}(\Lambda)| \geq \frac{1}{2}\sum_{i=0}^{n} b_i(\Lambda; \Bbbk).$$

Here $b_i(\Lambda; \Bbbk) = \dim_\Bbbk H^i(\Lambda; \Bbbk)$.

For Legendrian subamnifolds connected by a Hamiltonian pushoff, Asano-Ike showed a relation between persistence distance of sheaves singularly supported on the Lagrangians and the oscillation norm of the Hamiltonian [**11**]. Based on that result, we prove the following estimate on Reeb chords between the Legendrian and its Hamiltonian pushoff assuming that the norm of the Hamiltonian is small.

Recall the oscillation norm of the Hamiltonian to be

$$\|H_s\|_{\mathrm{osc}} = \int_0^1 \left(\max_{x \in T^{*,\infty}_{\tau>0}(M\times\mathbb{R})} H_s - \min_{x \in T^{*,\infty}_{\tau>0}(M\times\mathbb{R})} H_s \right) ds.$$

Denote by $l(\gamma)$ the length of a chord γ. Assume that the Maslov class $\mu(\Lambda) = 0$, and let

$$c_i(\Lambda) = \min\{l(\gamma) \,|\, \gamma \text{ is a Reeb chord, } \deg(\gamma) = i \text{ or } n - i\}.$$

Order them so that $c_{j_0}(\Lambda) \geq c_{j_1}(\Lambda) \geq ... \geq c_{j_n}(\Lambda)$.

Theorem 1.5.2. *Let M be orientable, $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be a closed Legendrian submanifold of dimension n, and $\Bbbk$ be a field. Suppose there exists a $\Bbbk$-coefficient pure sheaf of compact support and perfect stalks $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})_0$. Let H_s be any compactly supported Hamiltonian $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ such that for some $0 \leq k \leq n$*

$$\|H_s\|_{osc} < c_{j_k}(\Lambda)$$

and $\varphi^1_H(\Lambda)$ is transverse to the Reeb flow applied to Λ. Then the number of Reeb chords between Λ and $\varphi^1_H(\Lambda)$ is

$$\mathcal{Q}(\Lambda, \varphi^1_H(\Lambda)) \geq \sum_{i=0}^{k} b_{j_i}(\Lambda; \Bbbk).$$

1.5.2. Obstructions to SFT Lagrangian cobordisms

We illustrate that Theorem 1.4.2 also provides strong restrictions on the existence of SFT Lagrangian cobordisms. For example, the full faithfulness in Theorem 1.4.2 immediately implies the long exact sequences coming from the Cthulhu complex in Floer theory [**31**].

Moreover, combining Theorem 1.4.2 and the technique developed in [**26**, **157**], we show the existence and non-existence result for the following Legendrian surfaces $\Lambda_{g,k}$ considered in [**43**] and [**137**]. The obstructions are obtained using sheaves, while constructions uses the result of Eliashberg-Murphy [**66**].

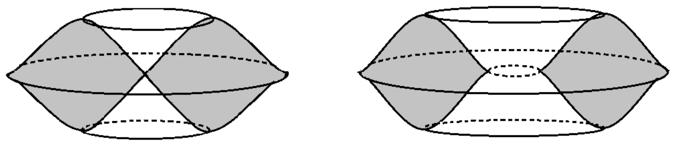

Figure 1.1. The Clifford Legendrian torus (on the left) and the unknotted Legendrian torus (on the right).

Theorem 1.5.3. *Let* $\Lambda_{Unknot}, \Lambda_{Cliff}$ *be Legendrian tori in* $T^{*,\infty}\mathbb{R}^3$ *shown in Figure 7.1. Let* $\Lambda_{g,k}$ *be the Legendrian surface with genus* g *by taking the connected sum of* k *copies of* Λ_{Cliff} *and* $g-k$ *copies of* Λ_{Unknot}*. Then*

(1) (Dimitroglou Rizell [**43**]*) for any* $k \geq 1$*, there are no Lagrangian cobordisms with vanishing Maslov class from* $\Lambda_{g,0}$ *to* $\Lambda_{g',k}$*;*

(2) for any $k \geq 1, k' \geq 0$*, there are Lagrangian cobordisms* L *from* $\Lambda_{g,k}$ *to* $\Lambda_{g,k'}$ *such that* $\dim \operatorname{coker}(H^1(L) \to H^1(\Lambda_{g,k})) \geq 2$;

(3) for any $k < k'$*, there are no Lagrangian cobordisms* L *with vanishing Maslov class from* $\Lambda_{g,k}$ *to* $\Lambda_{g,k'}$ *such that* $H^1(L) \twoheadrightarrow H^1(\Lambda_{g,k})$*; in particular there are no such Lagrangian concordances.*

Roughly speaking, the Legendrian $\Lambda_{g,k}$ is closer to being Lagrangian fillable when k is smaller (in particular, $\Lambda_{g,0}$ are the only Lagrangian fillable ones). One would expect that it is difficult to have a Lagrangian cobordism from $\Lambda_{g,k}$ to $\Lambda_{g,k'}$ if $k > k'$. Our theorem shows that, for $k > k'$, there are indeed obstructions for Lagrangian cobordisms assuming either (1) $k = 0$ or (3) $H^1(L) \to H^1(\Lambda_{g,k})$ is surjective. On the contrary, as long as we assume (2) $k \geq 1$ and $H^1(L) \to H^1(\Lambda_{g,k})$ is not surjective, then there are no obstructions.

1.6. Notations and Conventions

Geometric conventions: For a Weinstein sector X, $\partial_\infty X$ is its contact boundary, ∂X is its sectorial boundary and $\mathfrak{c}_X$ is its Lagrangian skeleton. In particular, for T^*M, $T^{*,\infty}M$ is its contact boundary, and in the paper we will identify it with the unit cotangent bundle. $T^{*,\infty}_{\tau>0}(M\times\mathbb{R})$ is the subbundle of $T^{*,\infty}(M\times\mathbb{R})$ consisting of points so that the covector coordinate in $T^*\mathbb{R}$ satisfies $\tau > 0$. For a closed submanifold $N \subset M$, $\nu^{*,\infty}_N M$ is the unit conormal bundle. For an open subset $U \subset M$ with subanalytic boundary, $\nu^{*,\infty}_{U,+/-}M$ is the outward/inward unit conormal bundle.

Let $L \subset X$ be an exact Lagrangian. $\widetilde{L} \subset X \times \mathbb{R}$ is its Legendrian lift. For Lagrangian cobordisms between Legendrian submanifolds, we say that a Lagrangian cobordism L is from Λ_- to Λ_+ if Λ_+ is at the convex end and Λ_- is at the concave end.

Categorical conventions: All categories in this paper are dg categories, and all functors will be functors in dg categories. $Sh_-, \mu Sh_-$ are the dg categories consisting of all possibly unbounded complexes of sheaves with prescribed (isotropic) singular support, $Sh^c_-, \mu Sh^c_-$ are the dg subcategories of compact objects, and $Sh^b_-, \mu Sh^b_-$ are the dg subcategories of objects with perfect stalks, and $Sh^{pp}_-, \mu Sh^{pp}_-$ are the dg subcategories of proper (i.e. pseudoperfect) modules. They are all localized along acyclic objects.

CHAPTER 2

Preliminaries in Symplectic Topology

The goal in this section is to explain concepts in symplectic and contact topology that we will use in the thesis. Since most of them are either standard or well known, we will simply refer to previous works for the proof of these results.

2.1. Contact Topology and Conical Symplectic Topology

Contact topology can be viewed as $\mathbb{R}_{>0}$-equivariant symplectic topology. Since both conventions will be useful in the discussion of microlocal sheaves, we explain the correspondence in this section. Following the philosophy, we also explain the relation between Lagrangian cobordisms in the symplectization of a 1-jet bundle and conical Legendrian cobordisms in the higher dimensional 1-jet bundle.

A contact manifold is a $(2n+1)$-dimensional manifold Y together with a maximally nonintegrable hyperplane distribution $\xi \subset TY$, and a Legendrian submanifold is an n-dimensional submanifold $\Lambda \subset Y$ such that $\xi|_\Lambda \subset T\Lambda$. Assume that $\xi \subset TY$ is defined by the kernel of a 1-form $\alpha \in \Omega^1(Y)$ called the contact form (this is equivalent to saying that the contact structure is coorientable).

2.1.1. Jet Bundles and Cotangent Bundles

In this section we explain the contactomorphism $J^1(M) \xrightarrow{\sim} T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, and the contact form and Reeb vector field we are going to work with. We also explain the contact Hamiltonians and their vector fields with respect to the specific contact form.

The 1-jet bundle $J^1(M) = T^*M \times \mathbb{R}$. Consider local coordinates $(x_0, \xi_0, t_0) \in T^*M \times \mathbb{R}$, where x_0 is the coordinate on M, ξ_0 is the coordinate on the fiber of T^*M and t_0 is the coordinate on $\mathbb{R}$. The contact structure given by $\xi_0 = \ker(dt_0 - \xi_0 dx_0)$. We choose the contact form to be $\alpha_0 = dt_0 - \xi_0 dx_0$. Now consider

$$\begin{aligned} T^*_{\tau>0}(M \times \mathbb{R}) &\rightarrow J^1(M), \\ (x, \xi, t, \tau) &\mapsto (x, \xi/\tau, t). \end{aligned}$$

After taking the quotient of $T^*_{\tau>0}(M \times \mathbb{R})$ by the dilation $(x, \xi, t, \tau) \mapsto (x, a\xi, t, a\tau)$ by $a \in \mathbb{R}_{>0}$, we get a diffeomorphism

$$T^{*,\infty}_{\tau>0}(M \times \mathbb{R}) \xrightarrow{\sim} J^1(M)$$

where $T^{*,\infty}_{\tau>0}(M \times \mathbb{R}) = \{(x, \xi, t, \tau) | |\xi|^2 + |\tau|^2 = 1, \tau > 0\} \cong T^*_{\tau>0}(M \times \mathbb{R})/\mathbb{R}_{>0}$ (If you consider the standard Liouville flow on $T^*(M \times \mathbb{R})$ and think of contact manifolds in the way that each contact form corresponds to a specific choice of a hypersurface transverse to the Liouville vector field, maybe it's better think of $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ as $\{(x, \xi, t, \tau) | \tau \equiv 1\}$). There is a natural contact structure on $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ given by restriction of the symplectic structure on $T^*(M \times \mathbb{R})$

$$\xi = \ker(\tau dt - \xi dx).$$

Then one can check that $(T^{*,\infty}_{\tau>0}(M \times \mathbb{R}), \xi)$ and $(J^1(M), \xi_0)$ are contactomorphic through that map defined above.

Under the contactomorphism, the contact form $\alpha_0 = dt_0 - \xi_0 dx_0$ is mapped to

$$\alpha = dt - (\xi/\tau)dx,$$

and the Reeb vector field $R_{\alpha_0} = \partial/\partial t$ is mapped to

$$R_\alpha = \frac{\partial}{\partial t}.$$

This contact form and Reeb vector field are the ones we will be dealing with in the paper.

Remark 2.1.1. *In the cotangent bundle $T^{*,\infty}(M \times \mathbb{R})$, the Reeb vector field that people are more familiar with may be the vector field producing the geodesic flow. The Reeb vector field we work with here is different because the contact form $\alpha = dt - (\xi/\tau)dx$ is different from the canonical one $\tau dt - \xi dx$. Indeed the contactomorphism we write down does not preserve the canonical contact forms on both sides.*

Now we consider the correspondence between contact Hamiltonians and contact vector fields determined by this contact form $\alpha = dt - (\xi/\tau)dx$. Given $H \in$

$C^\infty(T^{*,\infty}_{\tau>0}(M\times\mathbb{R}))$, the corresponding contact vector field X_H is defined by [**79**]

$$H=\alpha(X_H),\ \ \iota(X_H)d\alpha=dH(R_\alpha)\alpha-dH.$$

We claim that this contact Hamiltonian can be lifted to a homogeneous symplectic Hamiltonian on $T^*_{\tau>0}(M\times\mathbb{R})$ in the following way. Let

$$\widehat{H}(x,\xi,t,\tau)=\tau H(x,\xi/\tau,t).$$

Its corresponding symplectic Hamiltonian vector field is defined by

$$\iota(X_{\widehat{H}})\omega=-d\widehat{H},$$

where $\omega=d(\tau dt-\xi dx)=d(\tau\alpha)$. By elementary calculation, one will find that the projection $X_{\widehat{H}}$ onto the hyperplane $\tau=1$ is X_H. Therefore we will just study the homogeneous Hamiltonian $\widehat{H}$ (since in microlocal sheaf theory this will be more natural). In particular one can define the movie of a subset $\widehat{\Lambda}\subset T^*_{\tau>0}(M\times\mathbb{R})$ under the Hamiltonian isotopy $\varphi^s_{\widehat{H}}\,(s\in I)$ as

$$\widehat{\Lambda}_H=\{(x,\xi,t,\tau,s,\sigma)|(x,\xi,t,\tau)=\varphi^s_{\widehat{H}}(x_0,\xi_0,t_0,\tau_0),\sigma=-\widehat{H}\circ\varphi^s_{\widehat{H}}(x_0,\xi_0/\tau_0,t_0)\}.$$

This is an exact conical Lagrangian submanifold in $T^*_{\tau>0}(M\times\mathbb{R}\times I)$.

2.1.2. Lagrangian cobordisms and Legendrian cobordisms

In this section, we explain the relation between Lagrangian cobordisms in the symplectization of $J^1(M) \cong T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ and conical Legendrian cobordisms in $J^1(M \times \mathbb{R}_{>0}) \cong T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times \mathbb{R}_{>0})$.

Let (Y, α) be a cooriented contact manifold. The symplectization is defined as $(Y \times \mathbb{R}_{>0}, d(s\alpha))$. Following [**63**, Section 2.8], Chantraine [**27**] and Ekholm [**50**], for instance, considered the category of Lagrangian cobordisms between Legendrians in the symplectization.

Definition 2.1.1. *The category of Lagrangian cobordisms* $\mathrm{Cob}(Y)$*, has objects being Legendrian submanifolds* $\Lambda \subset Y$ *and morphisms* $Hom(\Lambda_-, \Lambda_+)$ *being exact Lagrangian submanifolds* $L \subset (Y \times \mathbb{R}_{>0}, d(s\alpha))$ *with* $s\alpha|_L = df_L$ *such that*

$$L \cap (Y \times (0, s_-]) = \Lambda_- \times (0, s_-],\ L \cap (Y \times [s_+, +\infty)) = \Lambda_+ \times [s_+, +\infty).$$

for some $s_- < s_+$*, and the primitive* f_L *is a constant on* $\Lambda_- \times (0, s_-]$ *and* $\Lambda_+ \times [s_+, +\infty)$*. We call such an* L *a Lagrangian cobordism from* Λ_- *to* Λ_+*.*

Compositions in $\mathrm{Cob}(Y)$ *are defined by concatenating Lagrangian cobordisms along their common conical ends. We will denote the concatenation of* $L_0 \in Hom(\Lambda_0, \Lambda_1)$ *and* $L_1 \in Hom(\Lambda_1, \Lambda_2)$ *by* $L_0 \cup L_1$*.*

Remark 2.1.2. *The assumption that the primitive* f_L *is a constant on* $\Lambda_- \times (-\infty, -r]$ *and* $\Lambda_+ \times [r, +\infty)$ *is made to ensure that concatenations of exact Lagrangians are still exact (see* [**29**]*).*

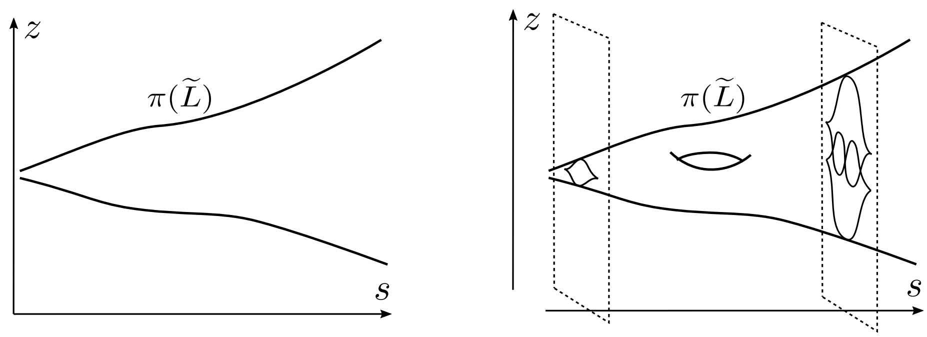

Figure 2.1. The front projection of a conical Legendrian cobordism $\widetilde{L} \subset J^1(\mathrm{pt} \times \mathbb{R}_{>0})$ (on the left) and $J^1(\mathbb{R} \times \mathbb{R}_{>0})$ (on the right).

For exact Lagrangians in the symplectization $(J^1(M) \times \mathbb{R}_{>0}, d(s\alpha_{\mathrm{std}}))$, one can consider the Legendrian lift in the contactization $((J^1(M) \times \mathbb{R}_{>0}) \times \mathbb{R}, dw + s\alpha_{\mathrm{std}})$. It is known [**34**, **130**] that there is a (strict) contactomorphism

$$\begin{aligned} \varphi: \quad ((J^1(M) \times \mathbb{R}_{>0}) \times \mathbb{R}, dw + s(dt + \xi dx)) \quad &\to \quad (J^1(M \times \mathbb{R}_{>0}), dz + \sigma ds + ydx) \\ (x, \xi, t; s; w) \quad &\mapsto \quad (x, s, s\xi, t, st + w). \end{aligned}$$

Therefore, an exact Lagrangian cobordism gives a conical Legendrian cobordism with no Reeb chords [**26**, **130**].

Definition 2.1.2. *Let $\Lambda_\pm \subset J^1(M)$ be Legendrian submanifolds. Then a conical Legendrian cobordism is a Legendrian $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ such that*

$$\widetilde{L} \cap J^1(M \times (0, s_-)) = \{(x, s, s\xi, t, st + w_{0,-}) | (x, \xi, t) \in \Lambda_-, s \in (0, s_-)\},$$

$$\widetilde{L} \cap J^1(M \times (s, +\infty)) = \{(x, s, s\xi, t, st + w_{0,+}) | (x, \xi, t) \in \Lambda_+, s \in (s_+, +\infty)\}.$$

Finally, we explain that the conical boundary condition make this type of Lagrangian cobordisms very different from the Lagrangian/Legendrian cobordisms considered by Arnol'd [**10**], defined as follows.

Definition 2.1.3. *Let $L_1, \dots, L_r$ and $K_1, \dots, K_s \subset X$ be Lagrangian submanifolds. Then an Arnol'd Lagrangian cobordism V between $L_1, \dots, L_r$ and $K_1, \dots, K_s$ is a Lagrangian submanifold $V \subset T^*(M \times \mathbb{R})$ such that*

$$V \cap T^*(M \times (-\infty, -1)) = \bigcup_{i=1}^{r} L_i \times (-\infty, -1) \times \{i\},$$
$$V \cap T^*(M \times (1, +\infty)) = \bigcup_{j=1}^{s} K_j \times (1, +\infty) \times \{j\}.$$

In particular, when $V \subset T^*(M \times \mathbb{R})$ is an exact Lagrangian, its Legendrian lift has cylindrical ends like

$$\widetilde{V} \cap J^1(M \times (-\infty, -1)) = \bigcup_{i=1}^{r} L_i \times \{(s, i, is) | s \in (-\infty, -1)\},$$
$$\widetilde{V} \cap J^1(M \times (1, +\infty)) = \bigcup_{j=1}^{s} K_j \times \{(s, j, js) | s \in (1, +\infty)\},$$

which is different from the boundary condition of conical Legendrian cobordisms. In particular, in the next section, we will see that one of the differences is whether there is a tubular neighbourhood with positive radius for a complete adapted metric on $J^1(M \times \mathbb{R})$.

2.2. Weinstein Neighbourhood for Noncompact Legendrians

For any closed Lagrangian submanifold $L \subset X$, Weinstein neighbourhood theorem asserts that there is a Weinstein tubular neighbourhood of $L \subset X$ which is symplectomorphic to a neighbourhood of the zero section $L \subset T^*L$. Similarly, for any closed Legendrian submanifold $\Lambda \subset Y$, there is a Weinstein tubular neighbourhood of $\Lambda \subset Y$ which is contactomorphic to a neighbourhood of the zero section $\Lambda \subset J^1(\Lambda)$.

However, the neighbourhood theorem for noncompact Lagrangian/Legendrian submanifolds could be nontrivial, as the radius of the tubular neighbourhood may not have a positive lower bound with respect to the given Riemannian metric. This will be essential when we discuss the sheaf quantization problem for noncompact Lagrangian/Legendrians in Section 1.3. To deal with this issue, we first introduce the notion of an adapted metric following [**68**, Section 2.2.2].

Definition 2.2.1 (Eliashberg-Gromov [**68**]). *A Riemannian metric g on a symplectic manifold X is adapted to the symplectic form ω on X if for any $H \in C^\infty(X)$*

$$\|dH\|_g = \|X_H\|_g.$$

or equivalently $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ for some g-orthonormal coframing

$$\langle dx_1, \dots, dx_n, dy_1, \dots, dy_n \rangle .$$

A Riemannian metric g on a contact manifold Y is adapted to the contact form α on Y if for any $H \in C^\infty(Y)$

$$\|dH\|_g^2 + |H|^2 = \|X_H\|_g^2$$

or equivalently $\alpha = dz - \sum_{i=1}^n dx_i \wedge dy_i$ for some g-orthonormal coframing

$$\langle dx_1, \ldots, dx_n, dy_1, \ldots, dy_n, dz \rangle .$$

Example 2.2.1. *Consider $X = T^*M$ and $\omega = d\lambda_{std}$. Then a Riemannian metric g_M on M determines an adapted Riemannian metric on T^*M by*

$$g_{T^*M} = g_M + g_M^\vee :\in T_xM \oplus T_x^*M \times T_xM \oplus T_x^*M \to \mathbb{R},$$

where $g^\vee : T^*M \otimes T^*M \to \mathbb{R}$ is the dual bilinear form to $g : TM \times TM \to \mathbb{R}$. It also determines an adapted Riemannian metric on $J^1(M)$ by

$$g_{J^1(M)} = g_{T^*M} + dz^2 : T_xM \oplus T_x^*M \oplus T_z\mathbb{R} \times T_xM \oplus T_x^*M \oplus T_z\mathbb{R} \to \mathbb{R}.$$

In particular, when g_M is complete, g_{T^*M} and $g_{J^1(M)}$ are complete as well. We call them the standard adapted metric on T^*M and $J^1(M)$.

Later we will see in Section 1.3 that the reason we discuss metrics on symplectic/contact manifolds is to understand when a noncompact Hamiltonian vector field can be integrated. Adapted metrics allow us to estimate the norm of the Hamiltonian vector fields in terms of their C^1-norm. On the other hand, complete metrics allow us to deduce existence of the integration flow from the estimation of the norm of vector fields.

It is proved that any symplectic manifold admits a complete adapted metric [**68**]. It seems unclear whether a contact manifold always has a complete adapted metric, but in this thesis we will only need the case of cotangent bundles and 1-jet bundles.

Definition 2.2.2. *Let $L \subset X$ be a submanifold. A (tubular) neighbourhood U of L of positive radius $r > 0$ with respect to a metric g on X is a (tubular) neighbourhood U such that for any $x \in X$ with $d_g(x, L) \leq r$, we have $x \in U$.*

Lemma 2.2.1. *Let $L \subset (X, d\lambda_X)$ be an exact Lagrangian submanifold. Suppose L has a tubular neighbourhood of positive radius $r > 0$ with respect to a complete adapted metric g_X, then the Legendrian lift $\widetilde{L} \subset (X \times \mathbb{R}, dt - \lambda_X)$ also has a tubular neighbourhood of positive radius $r > 0$ with respect to the complete adapted metric $g_X \oplus g_{\mathbb{R},std}$, where $g_{\mathbb{R},std}$ is the Euclidean metric.*

For Arnol'd Lagrangian cobordisms between closed Lagrangians, the following lemma is immediate, by noticing that the cylindrical end $L_i \times (-\infty, -1) \times \{i\}$ or $K_j \times (1, +\infty) \times \{j\}$ has a tubular neighbourhood of positive radius.

Lemma 2.2.2. *Let $V \subset X \times T^*\mathbb{R}$ be an Arnol'd Lagrangian cobordism between closed embedded Lagrangians $L_1, \ldots, L_r \subset X$ and $K_1, \ldots, K_s \subset X$. Then V has a tubular neighbourhood of positive radius with respect to some complete adapted metric on $X \times T^*\mathbb{R}$.*

For exact Lagrangians in T^*M with closed Legendrian boundary in $T^{*,\infty}M$, by considering the standard adapted metric on T^*M, the following lemma is also almost immediate:

Lemma 2.2.3. *Let $L \subset T^*M$ be a Lagrangian filling with closed Legendrian boundary $\Lambda \subset T^{*,\infty}M$. Then L has a tubular neighbourhood of positive radius $r > 0$ with respect to the adapted metric on T^*M.*

Proof. We only need to find a tubular neighbourhood of positive radius outside a compact set $T^*_{|\xi|\leq s_0}M$ of the zero section, where $L \cap T^*_{|\xi|>s_0}M = \Lambda \times (s_0, +\infty)$. Observe that the standard adapted metric g_{T^*M} can be written as $s^2 g_{T^{*,\infty}M} + ds^2$. When s_0 is large, it is bounded from below by the product metric $g_{T^{*,\infty}M} + ds^2$. Since Λ has a tubular neighbourhood of positive radius $r > 0$, we can conclude that so does L. □

However, for Lagrangian cobordisms between closed Legendrians, such a tubular neighbourhood does not exist, for the simple reason that the symplectic area near the concave end of the symplectization has an upper bound, while a tubular neighbourhood of positive radius for the conical/cylindrical submanifold cannot have a bounded symplectic area.

For simplicity, we only deal with the particular case of Lagrangian cobordisms in $J^1(M) \subset T^{*,\infty}(M \times \mathbb{R})$. In this case the symplectization is symplectomophic to a cotangent bundle

$$J^1(M) \times \mathbb{R}_{>0} \xrightarrow{\sim} T^*(M \times \mathbb{R}_{>0}),\ (x, \xi, t; s) \mapsto (x, s, s\xi, t).$$

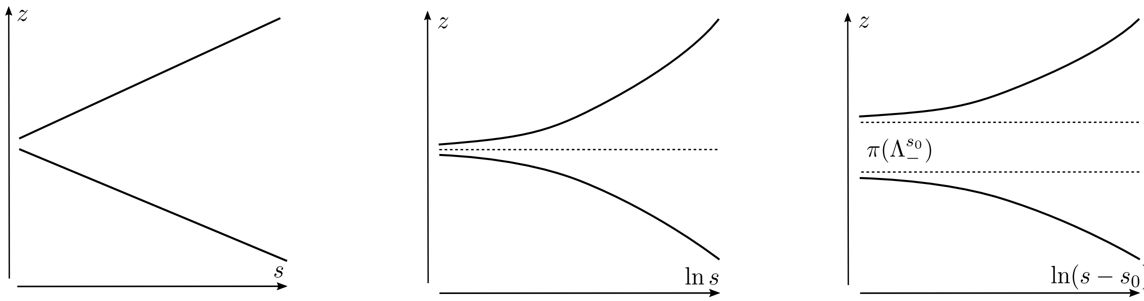

Figure 2.2. The figure on the left is the front projection of a conical Legendrian cobordism $\widetilde{L} \subset J^1(\mathrm{pt} \times \mathbb{R}_{>0})$. The figure in the middle is the front projection after applying the diffeomorphism $s \mapsto \ln s$, where the complete adapted metric induced by $g_M + s^{-2}ds^2$ becomes the Euclidean metric. The figure on the right is the front projection after applying the diffeomorphism $s \mapsto \ln(s - s_0)$, where the complete adapted metric induced by $g_M + (s - s_0)^{-2}ds^2$ becomes the Euclidean metric.

We will consider a different standard complete adapted metric on $T^*(M \times \mathbb{R}_{>0})$ induced by the complete metric $g_M + s^{-2}ds^2$ on $M \times \mathbb{R}_{>0}$, which is $g_M + g_M^\vee + s^{-2}ds^2 + s^2dt^2$; see Figure 2.2 middle (note that the metric $s^{-2}ds^2$ is identical to the Euclidean metric under the diffeomorphism $s \mapsto \ln s$).

Lemma 2.2.4. *Let $L \subset T^*(M \times \mathbb{R}_{>0}) \cong J^1(M) \times \mathbb{R}_{>0}$ be a Lagrangian cobordism between closed Legendrians from Λ_- to $\Lambda_+ \subset J^1(M)$. Then for any sufficiently small $s_0 > 0$, $L \cap T^*(M \times (s_0, +\infty))$ has a tubular neighbourhood of positive radius $r > 0$ with respect to the adapted metric on $T^*(M \times \mathbb{R}_{>0})$.*

Proof. First consider $L \cap T^{*,\infty}M \times (s_0, s_1')$ where s_0 is small, s_1 is sufficiently large and $s_1' > s_1$. Since the intersection is a bounded subset, there exists a tubular neighbourhood of positive radius $r > 0$. Then consider $L \cap T^{*,\infty}M \times (s_1, +\infty)$. Since

s_1 is sufficiently large, we may assume that

$$L \cap T^{*,\infty}M \times (s_1, +\infty) = \{(x, s, s\xi) | (x, \xi) \in \Lambda_+, s \in (s_1, +\infty)\}.$$

Then since Λ_+ is a closed Legendrian, it has a tubular neighbourhood of positive radius $r > 0$ with respect to the any complete metric. The adapted metric on $T^*(M \times (s_1, +\infty))$ is given by

$$g_{T^*(M \times \mathbb{R}_{>0})} = g_M + g_M^\vee + s^{-2}ds^2 + s^2dt^2.$$

Under the identification $J^1(M) \times \mathbb{R}_{>0} \xrightarrow{\sim} T^*(M \times \mathbb{R}_{>0}),\ (x, \xi, t; s) \mapsto (x, s, s\xi, t)$, the cone on the right hand side is identified with the product cylinder on the left hand side $\Lambda_+ \times (s_1, +\infty)$. The metric is identified with

$$g_M + s^2 g_M^\vee + s^2dt^2 + s^{-2}ds^2$$

which is bounded from below by the product metric $g_{J^1(M)} + s^{-2}ds^2$ on $J^1(M) \times (s_1, +\infty)$. Therefore, by considering the product neighbourhood of $\Lambda_+ \times (s_1, +\infty)$, we get a tubular neighbourhood of positive radius. $\square$

Now we restrict to the case $J^1(M) \subset T^{*,\infty}(M \times \mathbb{R})$. We can restrict to the open submanifold $J^1(M) \times (s_0, +\infty)$ which is symplectomorphic to $T^*(M \times (s_0, +\infty))$. Consider the complete adapted metric on the submanifold induced by $g_M + (s - s_0)^{-2}ds^2$ on $M \times (s_0, +\infty)$; see Figure 2.2 right (note that the metric $(s - s_0)^{-2}ds^2$ is identical to the Euclidean metric under the diffeomorphism $s \mapsto \ln(s - s_0)$). The advantage of this new metric is that it is complete on $T^*(M \times (s_0, +\infty))$, so that we can deal with the subset independently

$$L \cap T^*(M \times (s_0, +\infty)) \subset T^*(M \times (s_0, +\infty))$$

when studying Hamiltonian vector fields and their integration flows in later sections.

However, under this metric, $L \cap T^*(M \times (s_0, +\infty))$ is no longer a bounded subset in an ambient manifold, so it is no longer true that $L \cap T^*(M \times (s_0, +\infty))$ has a tubular neighbourhood of positive radius $r > 0$ with respect to this new complete adapted metric. We only have a weaker result (weaker in the sense of Lemma 2.2.1).

Lemma 2.2.5. *Let $L \subset J^1(M) \times \mathbb{R}_{>0} \cong T^*(M \times \mathbb{R}_{>0})$ be a Lagrangian cobordism between closed Legendrians from Λ_- to $\Lambda_+ \subset J^1(M)$. Then for any sufficiently small $s_0 > 0$, the Legendrian lift $\widetilde{L} \cap J^1(M \times (s_0, +\infty))$ has a tubular neighbourhood of positive radius $r > 0$ with respect to the complete adapted metric on $J^1(M \times (s_0, +\infty))$.*

Proof. We notice that the same argument in Lemma 2.2.4 shows that for any $s_0' > s_0$, $L \cap J^1(M \times (s_0', +\infty))$ admits a tubular neighbourhood of positive radius $r > 0$. Therefore, by Lemma 2.2.1, it suffices to show that the Legendrian lift $\widetilde{L} \cap J^1(M \times (s_0, s_0'))$ admits a tubular neighbourhood of positive radius.

On $J^1(M \times (s_0, s_0'))$, we know that the Legendrian lift of the cobordism is

$$\begin{aligned} \widetilde{L} \cap J^1(M \times (s_0, s_0')) &= \{(x, s, s\xi, t, st) | (x, \xi, t) \in \Lambda_-, s \in (s_0, s_0')\} \\ &= \{(x, s, \xi, s^{-1}z, z) | (x, \xi, z) \in \Lambda_-^s, s \in (s_0, s_0')\}. \end{aligned}$$

where $\Lambda_-^s = \{(x, s\xi, st) | (x, \xi, t) \in \Lambda_-\}$ is a closed Legendrian. Λ_-^s has a tubular neighbourhood of positive radius $r > 0$ with respect to the complete adapted metric on $J^1(M)$

$$g_{J^1(M)} = g_M + g_M^\vee + dz^2.$$

On the other hand, the complete adapted metric on $J^1(M \times (s_0, +\infty))$ is given by

$$g_{J^1(M \times (s_0, +\infty))} = g_M + g_M^\vee + (s - s_0)^{-2} ds^2 + (s - s_0)^2 dt^2 + dz^2.$$

When $s > s_0 > 0$, we know that $g_{J^1(M \times (s_0, +\infty))}$ is bounded from below by the product metric $g_{J^1(M)} + (s - s_0)^{-2} ds^2 + (s - s_0)^2 dt^2$. By considering the product neighbourhood of $\Lambda_-^{s_0} \times (s_0, s_0')$, we get a tubular neighbourhood of positive radius for the cylinder $\Lambda_-^{s_0} \times (s_0, s_0')$. Finally, we estimate the distance between the cylinder and the cone

$$\{(x, s, \xi, 0, z) | (x, \xi, z) \in \Lambda_-^{s_0}, s \in (s_0, s_0')\}, \ \{(x, s, \xi, s^{-1}z, z) | (x, \xi, z) \in \Lambda_-^s, s \in (s_0, s_0')\}.$$

Consider pairs of the form $(x, s, \xi, 0, z)$ and $(x, s, ss_0^{-1}\xi, s^{-1}z, ss_0^{-1}z)$, and set $r = \max_{(x,\xi,z)\in\Lambda_-^{s_0}}(|\xi|^2 + z^2)^{1/2}$, we know that the distance is bounded by

$$\sup_{(x,\xi,z)\in\Lambda_-^{s_0}, s\in(s_0,s_0')} d_{J^1(M\times(s_0,+\infty))}\big((x, s, \xi, 0, z), (x, s, ss_0^{-1}\xi, s^{-1}z, ss_0^{-1}z)\big)$$

$$\leq \sup_{(x,\xi,z)\in\Lambda_-^{s_0}} d_{(s_0,+\infty)}((s_0')^{-1}z, 0) + d_{J^1(M)}\big((x, \xi, z), (x, s_0's_0^{-1}\xi, s_0's_0^{-1}z)\big)$$

$$\leq \sup_{(x,\xi,z)\in\Lambda_-^{s_0}} (s - s_0)((s_0')^{-1}z - 0) + \big((\xi - s_0's_0^{-1}\xi)^2 + (z - s_0's_0^{-1}z)^2\big)^{1/2}$$

$$\leq (s_0' - s_0)(s_0')^{-1}r + (s_0's_0^{-1} - 1)r = (s_0's_0^{-1} - s_0(s_0')^{-1})r.$$

We know that the distance can be arbitrarily small when s_0' is sufficiently close to s_0. Therefore, a tubular neighbourhood of positive radius for $\Lambda_-^{s_0} \times (s_0, s_0')$ gives a (possibly smaller) tubular neighbourhood of positive radius for the cone. □

2.3. Genericity Assumption and Gradings on Legendrians

When proving results on estimations of Reeb chords, we need some assumptions on genericity and then would be able to study the Maslov grading on Reeb chords on the Legendrian. They are explained as follows.

2.3.1. Genericity Assumptions of Legendrians

In this section we introduce the notions of chord generic Legendrian submanifolds and admissible Legendrian isotopies. They are generic under C^1-topology in the space of embeddings/isotopies.

Definition 2.3.1. *Let $\Lambda \subset J^1(M)$ be a Legendrian submanifold. Λ is called chord generic if the Lagrangian projection*

$$\pi_{Lag} : \Lambda \to T^*M$$

is a Lagrangian immersion with only transverse double points.

Lemma 2.3.1 (Ekholm-Etnyre-Sullivan, [**54**, Lemma 3.5])**.** *Let Λ be a Legendrian submanifold. Then for any $\epsilon > 0$ there is a chord generic Legendrian submanifold Λ_ϵ that is ϵ-close to Λ in the C^1-topology.*

Remark 2.3.1. *In fact being ϵ-close in the C^1-topology implies that Λ is Hamiltonian isotopic to Λ_ϵ by the Legendrian neighbourhood theorem. In addition the C^0-norm of the Hamiltonian isotopy can also be smaller than ϵ.*

By Legendrian isotopy extension theorem, any Legendrian isotopy can be realized as an ambient Hamiltonian isotopy. Therefore to discuss Hamiltonian isotopies it suffices to discuss Legendrian isotopies.

Definition 2.3.2. *Let $n \geq 2$, $\Lambda \subset J^1(M)$ be a Legendrian submanifold and $H \in C^\infty(J^1(M))$ a contact Hamiltonian. Then the Legendrian isotopy $\Lambda_s = \varphi_H^s(\Lambda)$ $(s \in I)$ is admissible if there are $s_1, ..., s_k \in I$ such that*

(1). for $s \neq s_1, ..., s_k$, Λ_s is a chord generic Legendrian;

(2). for $s \in (s_i - \epsilon, s_i + \epsilon)$ where $\epsilon > 0$ is sufficiently small, Λ_s is still chord generic away from some contact ball $U \in J^1(M)$, and in the contact ball $U \simeq \mathbb{R}^{2n+1}$,

$$\Lambda_t \cap U \simeq (\{(x, 0, 0) | x \in \mathbb{R}\} \times L_1) \cup \big(\{(x, 3x^2 + s, x^3 + sx) | x \in \mathbb{R}\} \times L_2\big)$$

such that $L_1 \pitchfork L_2$ are Lagrangian subspaces in $\mathbb{R}^{2n-2}$.

Lemma 2.3.2 (Ekholm-Etnyre-Sullivan, [**54**, Lemma 3.6])**.** *Let $\Lambda_s\,(s \in I)$ be a Legendrian isotopy consisting of chord generic Legendrians connecting Λ_1 and Λ_1. Then for any $\epsilon > 0$ there exists an admissible Legendrian isotopy connecting Λ_0 and Λ_1 that is ϵ-close to $\Lambda_s\,(s \in I)$ in the C^1-topology.*

Remark 2.3.2. *Ekholm-Etnyre-Sullivan's definition for admissible Legendrian isotopies requires more conditions, but for our purpose the definition above is already enough.*

2.3.2. Grading of Reeb chords on Legendrians

In this section we discuss the grading of Reeb chords and Maslov potential.

Recall that the symplectic structure on T^*M will give a contractible choice of almost complex structures on the tangent bundle $T(T^*M)$, which canonically turns $T(T^*M)$ into a complex vector bundle. On T^*M there is a canonical Lagrangian fibration given by the cotangent fibers. A framing on this Lagrangian fibration together with the almost complex structure J determines a canonical trivialization of the complex vector bundle $T(T^*M)$.

Definition 2.3.3. *Let $\Lambda \to J^1(M)$ be a Legendrian immersion, and consider the Lagrangian projection onto T^*M. For any $\gamma : S^1 \hookrightarrow \Lambda \to T^*M$, consider the canonically trivialized complex vector bundle $\gamma^* T(T^*M)$ and the Lagrangian subbundle $\gamma^* T\Lambda$. Then the Maslov index of γ is*

$$m(\gamma) : \mathbb{Z} \xrightarrow{\sim} \pi_1(S^1) \to \pi_1(U(n)/O(n)) \xrightarrow{\sim} \mathbb{Z}.$$

The Maslov class of Λ is the homomorphism

$$\mu(\Lambda) : \pi_1(\Lambda) \to \mathbb{Z}, \gamma \mapsto m(\gamma).$$

In fact $\mu(\Lambda) \in H^1(\Lambda)$.

Now we define the Maslov potential for a Legendrian submanifold Λ with $\mu(\Lambda) = 0$. Currently Maslov potential is only defined combinatorially for Legendrian knots, since in higher dimensions it is hard (in fact, impossible) to classify the singularities of the front projection. Therefore here we only define the Maslov potential on a strand.

Definition 2.3.4. *Let $\Lambda \subset J^1(M)$ be a Legendrian submanifold such that the front projection $\pi_{front} : \Lambda \to M \times \mathbb{R}$ is a smooth hypersurface on an open dense subset. For a curve $\gamma : I \to \Lambda$, a Maslov potential is a step function*

$$d : \gamma(I) \to \mathbb{Z}$$

such that for any $a, b \in \gamma(I)$*,* $d(a) - d(b)$ *is equal to the number of down cusps minus the number of up cusps, and the value at a cusp is equal to points in* $\gamma(I)$ *in a small neighbourhood with greater* t *coordinates. Here a cusp is going up (down) if* $\gamma^* dt > 0$ *(*$\gamma^* dt < 0$*).*

Remark 2.3.3. *It is not clear at all that the Maslov potential can be globally well-defined. However, when* $\mu(\Lambda) = 0$ *there is indeed a well-defined Maslov potential*

$$d : \Lambda \to \mathbb{Z}$$

such that its restriction to any curve will be a Maslov potential on that strand. For a possible choice of the Maslov potential, see [**84**]*.*

The following definition is coming from the formula obtained by Ekholm-Etnyre-Sullivan [**55**, Section 3.5]. It may not be a good definition from a geometric viewpoint. However it is the most convenient one for us.

Definition 2.3.5. *Let* $\Lambda \subset J^1(M)$ *be a chord generic Legendrian submanifold,* γ *be a Reeb chord on* Λ *starting from* a *and ending at* b*, and* d *be a Maslov potential on any strand on* Λ *connecting* a *and* b*. Let* h_a, h_b *the functions* $\mathbb{R}^n \to \mathbb{R}$ *be functions such that in small contact balls* U_a, U_b *around* a *and* b*,*

$$\Lambda \cap U_j = \{(x, dh_j(x), h_j(x)) | x \in \mathbb{R}\}.$$

Let $h_{ab}(x) = h_b(x) - h_a(x)$*. Then the degree of* γ *is*

$$n - \deg(\gamma) = d(a) - d(b) + \mathrm{ind}(D^2 h_{ab}) - 1.$$

Lemma 2.3.3 (Ekholm-Etnyre-Sullivan, [**55**, Lemma 3.4])**.** *Let* $\Lambda \subset J^1(M)$ *be a chord generic Legendrian submanifold with* $\mu(\Lambda) = 0$*,* γ *be a Reeb chord on* Λ *starting from* a *and ending at* b*. Then* $\deg(\gamma)$ *is independent of the strand on* Λ *and the Maslov potential* d *we choose.*

Basically, the degree $\deg(\gamma)$ is well-defined because it is equal to a shifted Conley-Zehnder index of γ. We won't discuss Conley-Zehnder indices here. Interested readers may refer to [**55**, Section 2.3] or [**54**, Section 2.2].

2.4. Weinstein Manifolds and Weinstein Sectors

Finally, we explain the basic concepts of Weinstein manifolds and Weinstein sectors, which are developed since the work of Weinstein [**160**]. For the details see [**40**, **61**, **76**]. For the details of ideal contact boundaries of symplectic manifolds, see [**81**].

Definition 2.4.1. *Let* $(X, d\lambda)$ *be an exact symplectic manifold with ideal contact boundary* $\partial_\infty X$*. Let the Liouville vector field* Z_λ *be defined by* $\iota(Z_\lambda)d\lambda = \lambda$*, which we assume to be outward pointing along the ideal contact boundary.* X *is a (finite type) Weinstein manifold if there is a proper Morse(-Bott) function* f *on* X *such that* Z_λ

is a gradient-like vector field. Write $X_c = f^{-1}((-\infty, c])$*. Then the skeleton of* X *is*

$$\mathfrak{c}_X = \bigcup_{c\in\mathbb{R}} \bigcap_{z>0} \varphi_{Z_\lambda}^{-z}(X_c).$$

Example 2.4.1. *Let* $f_0 : M \to \mathbb{R}$ *be a Morse(-Bott) function on a closed manifold* M *with Riemannian metric* g*. Let* $f(x, \xi) = f_0(x) + |\xi|_g^2$ *and* $\lambda = \sum_{i=1}^n \xi_i dx_i + df_0$*. This pair defines a Weinstein structure on* T^*M *with the standard symplectic structure. In particular, when* $f_0 \equiv 0$*, we have the standard Liouville structure on* T^*M*.*

It follows that the stable submanifolds of critical points of the Morse function f are isotropic submanifolds [**40**, Lemma 11.13]. Therefore, the skeleton $\mathfrak{c}_X$, which is the union of stable submanifolds, is a stratified space stratified by isotropic submanifolds.

Definition 2.4.2. *Let* $(X, d\lambda)$ *be an exact symplectic manifold with contact boundary* $\partial_\infty^- X \sqcup \partial_\infty^+ X$*. Let the Liouville vector field* Z_λ *be defined by* $\iota(Z_\lambda)d\lambda = \lambda$*, which we assume to be transverse to the contact boundary, inward pointing along* $\partial_\infty^- X$ *and outward pointing along* $\partial_\infty^+ X$*.* X *is a Weinstein cobordism from* $\partial_\infty^- X$ *to* $\partial_\infty^+ X$ *if there is a proper Morse function* f *on* X *such that* $f^{-1}(0) = \partial_\infty^- X$*,* $f^{-1}(1) = \partial_\infty^+ X$ *and* Z_λ *is a gradient-like vector field.*

More generally, one can define Weinstein sectors, which are Weinstein manifolds with boundaries, following [**76**].

Definition 2.4.3. *Let $(X, d\lambda)$ be an exact symplectic manifold with boundary ∂X, whose ideal contact boundary $\partial_\infty X$ is a contact manifold with boundary. Let the Liouville vector field Z_λ be defined by $\iota(Z_\lambda)d\lambda = \lambda$, outward pointing along the ideal contact boundary $\partial_\infty X$ and tangent to ∂X. X is a Liouville sector if there is a function $I : \partial X \to \mathbb{R}$ such that*

(1) $Z_\lambda I = I$ near the ideal contact boundary $\partial\partial_\infty X$;

(2) dI is pointing positively along the characteristic foliation $\ker(\omega|_{\partial X})$ on ∂X.

X is a Weinstein sector if there is a function f on X such that Z_λ is a gradient-like vector field. Write $X_c = f^{-1}((-\infty, c])$. Then the skeleton of X is

$$\mathfrak{c}_X = \bigcup_{c\in\mathbb{R}} \bigcap_{z>0} \varphi_{Z_\lambda}^{-z}(X_c).$$

Example 2.4.2. *Let X be a Weinstein manifold with Morse function f_X and $F \subset \partial_\infty X$ be a Weinstein hypersurface with Morse function f_F. Then one can define a Weinstein sector by removing a Weinstein tubular neighbourhood of F* [**61**, **76**].

We define the notion of proper sectorial inclusions and Liouville subsector embeddings. We will not use these notions except in Section 7.1, but it will be helpful to keep in mind this viewpoint, which will appear throughout the thesis.

Definition 2.4.4. *A proper sectorial inclusion is a proper exact symplectic embedding of Weinstein sectors. In particular, it sends ideal contact boundaries to ideal contact boundaries.*

In terms of the Lagrangian skeleta, it should be viewed as open inclusions of skeleta (up to possible Liouville deformations). For example, following [**75**, Section 8.2], for X a Weinstein manifold and $F \subset \partial_\infty X$ a Weinstein hypersurface, there is a proper sectorial inclusion $F \times T^*[0,1] \hookrightarrow X$. This will be the main example we discuss in Chapter 4 and 6.

Definition 2.4.5. *A Liouville subsector embedding is an exact symplectic embedding of Weinstein sectors that sends sectorial boundaries to sectorial boundaries. A Liouville subsector embedding is a Liouville sectorial embedding such that the compliment is a Weinstein cobordism with sectorial boundary.*

In terms of the Lagrangian skeleta, Liouville subsector embedding should be viewed as closed embeddings of skeleta (up to possible Liouville deformations). Note that a Liouville subdomain embedding between Weinstein domains is not always a Weinstein subdomain embedding. In fact, there are embeddings of Weinstein domains whose compliment does not have the homotopy type of a half-dimensional CW-complex [**66**].

CHAPTER 3

Preliminaries in Microlocal Sheaves

Sheaves have played a central role in many branches of mathematics. Microlocal theory of sheaves on manifolds, introduced by Kashiwara-Schapira, strongly inspired by studies in differential equations, is a theory that tries to understand sheaves through their first order approximation, characterized by the stalks of certain local cohomologies. In this chapter, we review the basic theory of microlocal sheaves which will be needed for our results.

3.1. Microlocal Theory of Sheaves

Kashiwara and Schapira developed the microlocal theory of sheaves on manifolds in their celebrated book [**97**]. We briefly review the results in microlocal sheaf theory that we are going to use in this paper.

Definition 3.1.1. *Let $\underline{Sh}(M)$ be the dg category of sheaves on M, i.e. the dg category of (unbounded) chain complexes of sheaves on M over a field $\Bbbk$, and $Sh(M)$ the dg derived category of sheaves on M, i.e. the dg localization of $\underline{Sh}(M)$ along all acyclic objects.*

Remark 3.1.1. *We can consider the dg-categories of chain complexes of sheaves or sheaves of chain complexes. There is a natural functor from the former to the* *latter, by associating to the sheafification of the corresponding presheaf of chain complexes to each complex of sheaves. This is an equivalence for a smooth manifold of finite Lebesgue covering dimension; see* [**112**, Appendix C].

Gronthendieck six-functor formalism is well developed for sheaves on manifolds. One can define the internal $\mathscr{H}om(-,-)$ and tensor product $-\otimes-$ of sheaves. Given a continuous map $f : M \to N$, we have an adjunction between pull back and push forward

$$f_* : Sh(M) \rightleftharpoons Sh(N) : f^*,$$

and we also have an adjunction between proper push forward and proper pull back

$$f^! : Sh(N) \rightleftharpoons Sh(M) : f_!.$$

The readers may refer to Kashiwara-Schapira [**97**, Section 2 & 3.1] for important properties of the six functors on bounded complexes of sheaves, and see [**150**] for the generalization of Grothendieck six-functors to the setting of unbounded complexes of sheaves.

Example 3.1.2. *We denote by $\Bbbk_M$ the constant sheaf on M. For a locally closed subset $i_V : V \hookrightarrow M$, abusing notations, we will write*

$$\Bbbk_V = i_{V!}\Bbbk_V \in Sh(M).$$

In particular, $\Bbbk_V \in Sh(M)$ will have stalk $\Bbbk$ for $x \in V$ and stalk 0 for $x \notin V$. Note that when $V \hookrightarrow M$ is a closed subset, we can also write $\Bbbk_V = i_{V*}\Bbbk_V$.

We can define the linear dual and Verdier dual of a sheaf. Recall that for $p : M \to \{*\}$, the dualizing sheaf of M is $\omega_M = p^! \Bbbk$. When M is orientable with dimension n, $\omega_M = \Bbbk_M[n]$. For the detailed discussion, see Kashiwara-Schapira [**97**, Section 3.3].

Definition 3.1.2. *Let $\mathscr{F} \in Sh(M)$. The linear dual of $\mathscr{F}$ is*

$$\mathrm{D}'_M \mathscr{F} = \mathscr{H}om(\mathscr{F}, \Bbbk_M).$$

The Verdier dual of $\mathscr{F}$ is

$$\mathrm{D}_M \mathscr{F} = \mathscr{H}om(\mathscr{F}, \omega_M).$$

Then we are ready to introduce the notion of singular support, which was introduced by Kashiwara-Schapira [**97**, Section 5] as the key concept of microlocal theory of sheaves on manifolds.

Definition 3.1.3. *Let $\mathscr{F} \in Sh(M)$. Then its singular support $SS(\mathscr{F})$ is the closure of the set of points $(x, \xi) \in T^*M$ such that there exists a smooth function $\varphi \in C^1(M)$, $\varphi(x) = 0, d\varphi(x) = \xi$ and*

$$\Gamma_{\varphi^{-1}([0,+\infty))}(\mathscr{F})_x \neq 0.$$

The singular support at infinity is $SS^\infty(\mathscr{F}) = SS(\mathscr{F}) \cap T^{*,\infty}M$.

For $\widehat{\Lambda} \subset T^*M$ a conical subset (resp. $\Lambda \subset T^{*,\infty}M$ any subset), let $Sh_{\widehat{\Lambda}}(M) \subset Sh(M)$ (resp. $Sh_\Lambda \subset Sh(M)$) be the full subcategory consisting of sheaves such that $SS(\mathscr{F}) \subset \widehat{\Lambda}$ (resp. $SS^\infty(\mathscr{F}) \subset \Lambda$).

Example 3.1.3. *Let* $\mathscr{F} = \Bbbk_{\mathbb{R}^n \times [0,+\infty)}$. *Then* $SS(\mathscr{F}) = \mathbb{R}^n \times \{(x,\xi)|x \geq 0, \xi = 0$ *or* $x = 0, \xi \geq 0\}$, $SS^\infty(\mathscr{F}) = \nu^{*,\infty}_{\mathbb{R}^n \times \mathbb{R}_{>0},-}\mathbb{R}^{n+1} = \{(x_1, ..., x_n, 0, 0, ..., 0, 1)\}$, *which is the inward conormal bundle of* $\mathbb{R}^n \times \mathbb{R}_{>0}$.

Let $\mathscr{F} = \Bbbk_{\mathbb{R}^n \times (0,+\infty)}$. *Then* $SS(\mathscr{F}) = \mathbb{R}^n \times \{(x,\xi)|x \geq 0, \xi = 0$ *or* $x = 0, \xi \leq 0\}$, $SS^\infty(\mathscr{F}) = \nu^{*,\infty}_{\mathbb{R}^n \times \mathbb{R}_{>0},+}\mathbb{R}^{n+1} = \{(x_1, ..., x_n, 0, 0, ..., 0, -1)\}$, *which is the outward conormal bundle of* $\mathbb{R}^n \times \mathbb{R}_{>0}$.

The singular support of a sheaf detect when derived sections of the sheaf fail to propagate. To make it precise, we explain several important lemmas on propagations of sections of complexes of sheaves [**97**, Section 5]. See Robalo-Schapira [**132**] for the generalization of non-characteristic propagations to the setting of unbounded complexes and Jin-Treumann [**94**, Section 2] for generalizations to the setting of modules over E_2-ring spectra.

First, on a vector space, we introduce the notion of a convolution and state the microlocal cut-off lemma. This is a special case of non-characteristic propagations.

Definition 3.1.4. *Let* V *be an* $\mathbb{R}$*-vector space. Let*

$$\pi_1 : V \times V \to V, (v_1, v_2) \mapsto v_1, \;\; \pi_2 : V \times V \to V, (v_1, v_2) \mapsto v_2,$$
$$s : V \times V \to V, (v_1, v_2) \mapsto v_1 + v_2.$$

For $\mathscr{F}, \mathscr{G} \in Sh(V)$, *define the convolution as*

$$\mathscr{F} \star \mathscr{G} = s_*(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\mathscr{G}),$$

$$\mathscr{F} \star' \mathscr{G} = s_!(\pi_1^{-1}\mathscr{F} \otimes \pi_2^{-1}\mathscr{G}).$$

Let V be an $\mathbb{R}$-vector space and $\gamma \subset V$ be a closed cone, meaning that γ is invariant under $\mathbb{R}_{>0}$-dilation. Then the polar set of γ is

$$\gamma^\vee = \{u \in V^\vee |\, \langle u, v\rangle \geq 0, \forall\, v \in \gamma\}.$$

For a subset $A \subset M$, the interior of A is denoted by A°.

Lemma 3.1.1 (Microlocal cut-off lemma, [**97**, Proposition 5.2.3], [**84**, Proposition 2.9])**.** *Let V be an $\mathbb{R}$-vector space, $\gamma \subset V$ be a closed cone and $\mathscr{F} \in Sh(V)$. Then $SS(\mathscr{F}) \subset V \times (\gamma^\vee)^\circ$ iff*

$$\Bbbk_\gamma \star \mathscr{F} \xrightarrow{\sim} \Bbbk_0 \star \mathscr{F}.$$

Remark 3.1.4. *In Kashiwara-Schapira they use γ° as the polar set and* $\mathrm{Int}(\gamma^\circ)$ *for its interior but here we use different notions.*

Then we explain the machinery of microlocal Morse theory or non-characteristic propagation theory for sheaves on general manifolds that will be frequently used in this paper. We state the results here.

Proposition 3.1.2 (non-characteristic deformation lemma [**97**, Proposition 2.7.2])**.** *Let $\mathscr{F} \in Sh(M)$ and $\{U_t\}_{t\in\mathbb{R}}$ be a family of open subsets and $Z_t = \bigcap_{t>s} \overline{U_t \backslash U_s}$. Suppose that*

(1) $U_t = \bigcup_{s<t} U_s$, for $-\infty < t < +\infty$;

(2) $\overline{U_t \backslash U_s} \cap \mathrm{supp}(\mathscr{F})$ is compact, for $-\infty < s < t < +\infty$;

(3) $\Gamma_{M\backslash U_t}(\mathscr{F})_x = 0$, for $x \in Z_s \backslash U_t$, $-\infty < s \leq t < +\infty$.

Then for any $t \in \mathbb{R}$ *we have*

$$\Gamma\Big(\bigcup_{s\in\mathbb{R}} U_s, \mathscr{F}\Big) \xrightarrow{\sim} \Gamma(U_t, \mathscr{F}).$$

By considering the special case when $U_s \subset M$ are sublevel sets of a smooth function $f : M \to \mathbb{R}$, we have the microlocal Morse lemma.

Proposition 3.1.3 (microlocal Morse lemma [**97**, Corollary 5.4.19])**.** *Let* $\mathscr{F} \in Sh(M)$ *and* $f : M \to \mathbb{R}$ *be a smooth function that is proper on* $\mathrm{supp}(\mathscr{F})$*. Suppose for any* $x \in f^{-1}([a,b))$, $df(x) \notin SS(\mathscr{F})$*. Then*

$$\Gamma(f^{-1}((-\infty,b)), \mathscr{F}) \xrightarrow{\sim} \Gamma(f^{-1}((-\infty,a)), \mathscr{F}).$$

Example 3.1.5 ([**148**, Section 3.3])**.** *Suppose* $\Lambda = \nu^{*,\infty}_{\mathbb{R}^n\times\mathbb{R}_{>0},-}\mathbb{R}^{n+1} \subset T^{*,\infty}\mathbb{R}^{n+1}$ *is the inward conormal bundle of* $\mathbb{R}^n \times \mathbb{R}_{>0}$ *at infinity, and* $\mathscr{F} \in Sh^b_\Lambda(\mathbb{R}^{n+1})$*. Then by microlocal Morse lemma,* $\mathscr{F}|_{\mathbb{R}^n\times\{0\}}$, $\mathscr{F}|_{\mathbb{R}^n\times(0,+\infty)}$ *and* $\mathscr{F}|_{\mathbb{R}^n\times(-\infty,0)}$ *are locally constant sheaves, and*

$$\Gamma(\mathbb{R}^n\times\{0\}, \mathscr{F}) \simeq \Gamma(\mathbb{R}^{n+1}, \mathscr{F}) \simeq \Gamma(\mathbb{R}^n\times[0,+\infty), \mathscr{F}).$$

Suppose that the locally constant sheaves are

$$\mathscr{F}|_{\mathbb{R}^n\times[0,+\infty)} = F_+|_{\mathbb{R}^n\times[0,+\infty)},\;\; \mathscr{F}|_{\mathbb{R}^n\times(-\infty,0)} = F_-|_{\mathbb{R}^n\times(-\infty,0)}.$$

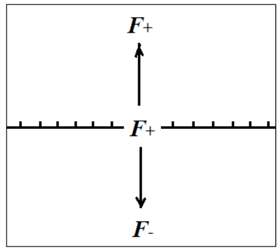

Figure 3.1. The singular support of a sheaf and the combinatoric description.

Then $\mathscr{F}$ is determined by the diagram (Figure 3.1)

$$F_- \longleftarrow F_+ \xrightarrow{\sim} F_+$$

Microlocal Morse theory not only shows when derived sections of sheaves on sublevel sets propagate, but also detects how derived sections of sheaves on sublevel sets fail to propagate. Quantitatively, we have the following microlocal Morse inequality.

Proposition 3.1.4 (microlocal Morse inequality [**97**, Proposition 5.4.20])**.** *Let $\mathscr{F} \in Sh(M)$ and $f : M \to \mathbb{R}$ be a smooth function that is proper on* $\mathrm{supp}(\mathscr{F})$*. Let $\Lambda_\varphi = \{(x, d\varphi(x)) | x \in M\}$, and suppose that*

$$SS(\mathscr{F}) \cap \Lambda_\varphi = \{(x_1, \xi_1), ..., (x_n, \xi_n)\}$$

and $V_i = \Gamma_{\varphi \geq \varphi(x_i)}(\mathscr{F})_{x_i}$ is finite dimensional. Then $\Gamma(M, \mathscr{F})$ is also finite dimensional and for any $l \in \mathbb{Z}$

$$\sum_{1 \leq i \leq n} \sum_{j \leq l} (-1)^{l-j} \dim H^j(V_i) \geq \sum_{j \leq l} (-1)^{l-j} \dim H^j(M, \mathscr{F}).$$

In particular for any $j \in \mathbb{Z}$, $\sum_{1\le i\le n} \dim H^j(V_i) \ge \dim H^j(M, \mathscr{F})$.

From the above discussion, we have seen that the core of investigating the behaviour of sheaves on manifolds is to estimate the singular support of the sheaf. Here are some singular support estimates we are going to use. Let $f : M \to N$ be a smooth map. Then we have the following maps between vector bundles

$$T^*M \xleftarrow{f_d} M \times_N T^*N \xrightarrow{f_\pi} T^*N,$$

where f_π is the natural map determined by fiber product, and f_d is the pullback map of covectors or differential forms. More explicitly, for $(x, \eta) \in M \times_N T^*N$ where $\eta \in T^*_{f(x)}N$,

$$f_\pi(x, \eta) = (f(x), \eta), \quad f_d(x, \eta) = (x, f^*\eta).$$

Proposition 3.1.5 ([**97**, Proposition 5.4.5])**.** *Let* $\mathscr{F} \in Sh(N)$ *and* $f : M \to N$ *be a submersion. Then*

$$SS(f^{-1}\mathscr{F}) = f_d f_\pi^{-1}(SS(\mathscr{F})).$$

Proposition 3.1.6 ([**97**, Proposition 5.4.4])**.** *Let* $\mathscr{F} \in Sh(M)$ *and* $f : M \to N$ *be a proper smooth map. Then*

$$SS(f_*\mathscr{F}) \subset f_\pi f_d^{-1}(SS(\mathscr{F})).$$

Remark 3.1.6. *In Kashiwara-Schapira, they call a smooth/continuous map as a morphism between manifolds, and call a submersion as a smooth morphism beween manifolds. Here we instead use the terminologies that may be more familiar to geometric topologists.*

Proposition 3.1.7 ([**97**, Proposition 5.4.14])**.** *Let* $\mathscr{F}, \mathscr{G} \in Sh(M)$*. Suppose* $(-SS(\mathscr{F})) \cap SS(\mathscr{G}) \subset M \subset T^*M$*. Then*

$$SS(\mathscr{F} \otimes \mathscr{G}) \subset SS(\mathscr{F}) + SS(\mathscr{G}).$$

Suppose $SS(\mathscr{F}) \cap SS(\mathscr{G}) \subset M \subset T^*M$*. Then*

$$SS(\mathscr{H}om(\mathscr{F}, \mathscr{G})) \subset (-SS(\mathscr{F})) + SS(\mathscr{G}).$$

Under the assumption, when $\mathscr{F}$ *is constructible, then* $\mathscr{H}om(\mathscr{F}, \mathscr{G}) \simeq \mathrm{D}'_M\mathscr{F} \otimes \mathscr{G}$*.*

The singular support estimation for pullback functors usually requires more assumptions. Let $f : M \to N$ be a smooth map. Then a subset $\Lambda \subset T^*N$ is called non-characteristic with respect to f if

$$\Lambda \cap \nu^*_{f(M)}N \subset M \subset M \times_N T^*N.$$

where $\nu^*_{f(M)}N$ is the kernel of $f_\pi : M \times_N T^*N \to T^*N$.

Proposition 3.1.8 ([**97**, Proposition 5.4.13]). *Let $\mathscr{G} \in Sh(M)$ and $f : M \to N$ be a smooth map such that $SS(\mathscr{G})$ is non-characteristic with respect to f. Then*

$$SS(f^{-1}\mathscr{G}) \subset f_d(f_\pi^{-1}(SS(\mathscr{G}))),$$

and there is a natural isomorphism $f^{-1}\mathscr{G} \otimes \omega_{M/N} \xrightarrow{\sim} f^!\mathscr{G}$.

Here are some singular support estimates that we are going to use. Let $A, B \subset T^*M$. Then define $(x, \xi) \in A \widehat{+} B$ iff there exists $(a_n, \alpha_n) \in A, (b_n, \beta_n) \in B$ such that

$$a_n, b_n \to x,\ \alpha_n + \beta_n \to \xi,\ |a_n - b_n||\alpha_n| \to 0.$$

Let $i : M \to N$ be a closed embedding. Then for $A \subset T^*N$, define $(x, \xi) \in i^\#(A)$ iff there exists $(y_n, \eta_n, x_n, \xi_n) \in A \times T^*M$ such that

$$x_n, y_n \to x,\ \eta_n - \xi_n \to \xi,\ |x_n - y_n||\eta_n| \to 0.$$

Proposition 3.1.9 ([**97**, Theorem 6.3.1]). *Let $j : U \hookrightarrow N$ be an open embedding, $\mathscr{F} \in Sh(U)$. Then*

$$SS(j_*\mathscr{F}) \subset SS(\mathscr{F}) \widehat{+} \nu^*_{U,-}N,$$

where $\nu^*_{U,-}N$ is the inward conormal bundle of $U \subset N$.

Proposition 3.1.10 ([**97**, Corollary 6.4.4]). *Let $i : M \to N$ be a closed embedding, $\mathscr{F} \in Sh(N)$. Then*

$$SS(i^{-1}\mathscr{F}) \subset i^\# SS(\mathscr{F}).$$

The first hint on the relation between sheaves and symplectic geometry is the result of Kashiwara-Schapira, which shows that singular supports are coisotropic subsets [**97**, Theorem 6.5.4]. In our paper, we will mostly study the simple case when the singular support is Lagrangian.

Definition 3.1.5. *A subset $Z \subset M$ is subanalytic at $x \in M$ if there exists an open neighbourhood U of x, and compact manifolds $Y_j^i\,(i = 1, 2, 1 \leq j \leq N)$ and $f_j^i : Y_j^i \to M$ analytic functions such that*

$$Z \cap U = U \cap \left(\bigcup_{1 \leq j \leq N} f_j^1(Y_j^1) \backslash f_j^2(Y_j^2) \right).$$

Z is a subanalytic set if it is subanalytic at any point.

Definition 3.1.6 ([**97**, Definition 8.4.3])**.** *For a sheaf $\mathscr{F} \in Sh(M)$, when $SS(\mathscr{F})$ is a subanalytic Lagrangian subset (resp. when $SS^\infty(\mathscr{F})$ is a subanalytic Legendrian), then $\mathscr{F}$ is called a weakly constructible sheaf.*

Remark 3.1.7. *In some modern literatures* [**104**, **124**, **146**]*, people call such sheaves constructible sheaves, since they work with unbounded complexes of sheaves and it is unnatural to assume perfect stalks in the corresponding large categories. Here, we follow the original convention in* [**97**] *since we will use both notations.*

In particular, for a weakly constructible sheaf $\mathscr{F}$, by Sard theorem, when $\epsilon > 0$ is sufficiently small, the outward conormal bundle $\nu^{*,\infty}_{\partial B_\epsilon(x),+}M$ will be disjoint from the subanalytic Legendrian $SS^\infty(\mathscr{F})$, and thus by microlocal Morse lemma we can identify the stalk of a weakly constructible sheaf with the local sections.

Lemma 3.1.11 ([**97**, Lemma 8.4.7]). *When $\mathscr{F} \in Sh(M)$ is a weakly constructible sheaf, for any $x \in M$ and $\epsilon > 0$ sufficiently small we have*

$$\mathscr{F}_x \simeq \Gamma(\overline{B_\epsilon(x)}, \mathscr{F}) \simeq \Gamma(B_\epsilon(x), \mathscr{F}).$$

There is a stronger notion of a constructible sheaf [**97**, Section 8.4], which we introduce now.

Definition 3.1.7 ([**97**, Definition 8.4.3]). *For a sheaf $\mathscr{F} \in Sh(M)$, when $\mathscr{F}$ is a weakly constructible sheaf and $\mathscr{F}_x$ is a perfect complex for any $x \in M$, then $\mathscr{F}$ is called a constructible sheaf.*

Proposition 3.1.12 ([**97**, Proposition 3.4.4]). *Let $\mathscr{F} \in Sh(M)$ be a constructible sheaf. Then*

$$\pi_1^{-1}\mathrm{D}_M\mathscr{F} \otimes \pi_2^{-1}\mathscr{G} \simeq \mathscr{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^{!}\mathscr{G}),$$

$$\pi_1^{-1}\mathrm{D}'_M\mathscr{F} \otimes \pi_2^{-1}\mathscr{G} \simeq \mathscr{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^{-1}\mathscr{G}).$$

Proposition 3.1.13 ([**97**, Proposition 3.4.6]). *Let $\mathscr{F}, \mathscr{G} \in Sh(M)$ be constructible sheaves. Then*

$$R\mathscr{H}om(\mathscr{F}, \mathscr{G}) \simeq R\mathscr{H}om(\mathrm{D}_M\mathscr{F}, D\mathscr{G}) \simeq \mathrm{D}_M\left(\mathrm{D}_M\mathscr{F} \otimes \mathscr{G}\right).$$

Remark 3.1.8. *In fact, the above propositions hold as long as $\mathscr{F}, \mathscr{G} \in Sh(M)$ are so called cohomologically constructible sheaves* [**97**, Definition 3.4.1]. *We can* *easily show that cohomologically construtible sheaves are constructible using Lemma 3.1.11. However, we do not know whether the converse is true.*

3.2. Microsheaves or Kashiwara-Schapira Stack

Whn studying microlocal theory of sheaves, one of the most important results of Kashiwara-Schapira is their theory of microlocalization, which enhances sheaves on M to microsheaves on T^*M to obtain more precise description on the microlocal behaviour of sheaves using the algebraic theory of microsheaves instead of the geometric theory of singular supports.

We review the definition and properties of microlocalization and microsheaves or Kashiwara-Schapira stacks, which has been introduced and studied in [**97**, Section 6], [**84**, Section 6] or [**121**, Section 3.4]. Here we follow the definition in [**124**, Section 5].

Definition 3.2.1. *Let $\widehat{\Lambda} \subset T^*M$ be a conical subset. Then define a presheaf of dg categories on T^*M supported on $\widehat{\Lambda}$ to be*

$$\mu Sh_{\widehat{\Lambda}}^{pre}: \widehat{\Omega} \mapsto Sh_{\widehat{\Lambda} \cup T^*M \backslash \widehat{\Omega}}(M)/Sh_{T^*M \backslash \widehat{\Omega}}(M),$$

The sheafification of $\mu Sh_{\widehat{\Lambda}}^{pre}$ is $\mu Sh_{\widehat{\Lambda}}$. In particular, we write $\mu Sh = \mu Sh_{T^*M}$ for the sheaf of categories on T^*M.

Let $Sh_{(\widehat{\Lambda})}(M)$ be the subcategory of sheaves $\mathscr{F}$ such that there exists some neighbourhood $\widehat{\Omega}$ of $\widehat{\Lambda}$ satisfying $SS(\mathscr{F}) \cap \widehat{\Omega} \subset \widehat{\Lambda}$. For $\mathscr{F}, \mathscr{G} \in Sh_{(\widehat{\Lambda})}(M)$, let the sheaf of *homomorphisms in the sheaf of categories* $\mu Sh_{\widehat{\Lambda}}$ *be*

$$\mu hom(\mathscr{F},\mathscr{G})|_{\widehat{\Lambda}} : \widehat{\Omega} \mapsto Hom_{\mu Sh_{\widehat{\Lambda}}(\widehat{\Omega})}(\mathscr{F},\mathscr{G}).$$

Write $\mu hom(\mathscr{F},\mathscr{G})$ *to be the sheaf of homomorphisms in* μSh.

Let $\Lambda \subset T^{*,\infty}M$ *be a subset where* $T^{*,\infty}M$ *is identified with the unit cotangent bundle. Then* μSh_Λ *is defined by* $\mu Sh_\Lambda = \mu Sh_{\Lambda\times\mathbb{R}_{>0}}|_\Lambda$.

Remark 3.2.1. *We define the sheafification in the (large) category of dg categories whose morphisms are exact functors. When* $\widehat{\Lambda}$ *is a conical subanalytic Lagrangian, the sheafification takes value in the (large) category of presentable dg categories whose morphisms are colimit preserving functors* [**124**, Remark 6.1]. *When* $\widehat{\Lambda}$ *is not conical subanalytic Lagrangian, it is then not necessarily true that the sheafification in exact dg categories agree with the one in presentable dg categories.*

Remark 3.2.2. *Kashiwara-Schapira defined* μhom *using the mcirolocalization fucntor of Sato* [**97**, Definition 4.1.1] *rather than the internal Hom on* μSh. *Let* $\mathscr{F}$, $\mathscr{G} \in \mathrm{Sh}(M)$, *they set*

$$\mu hom(\mathscr{F},\mathscr{G}) \coloneqq \mu_{\Delta_M}\mathscr{H}om(\pi_1^*\mathscr{F},\pi_2^!\mathscr{G})$$

where μ_{Δ_M} *is the microlocalization along the diagonal* [**97**, Section 4.3]. *It follows from* [**97**, Theorem 6.1.2] *and* [**84**, Corollary 5.5.] *that there is an canonical isomorphism*

$$\mu hom(\mathscr{F},\mathscr{G})|_\Omega \to \mathscr{H}om_{\mu Sh(\Omega)}(\mathscr{F},\mathscr{G}).$$

Thus, we abuse the notation and simply use μhom to denote the internal Hom of μSh valued in sheaves on conic open sets of T^*M.

Denote by m_Λ the natural quotient functor on the sheaf of categories, which, on the level of global sections, induces

$$m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda).$$

We call m_Λ the microlocalization functor. One may notice that the microlocalization functor factors through the restriction functors on the cotangent bundle

$$m_\Lambda : Sh_\Lambda(M) \to \mu Sh_{M\cup\widehat{\Lambda}}(M \cup \widehat{\Lambda}) \to \mu Sh_\Lambda(\Lambda).$$

The next lemma follows from the identity $\Gamma(T^*M, \mu hom(\mathscr{F}, \mathscr{G})) = Hom(\mathscr{F}, \mathscr{G})$ [**97**, Equation (4.3.1)] and the fact that $\mathrm{supp}(\mu hom(\mathscr{F}, \mathscr{G})) \subset SS(\mathscr{F}) \cap SS(\mathscr{G})$ [**97**, Corollary 5.4.10].

Lemma 3.2.1 ([**121**, Remark 3.18])**.** *Let $\widehat{\Lambda} \subset T^*M$ be a conical subanalytic Lagrangian. Then there is an isomorphism*

$$Sh_{M\cup\widehat{\Lambda}}(M) \xrightarrow{\sim} \mu Sh_{M\cup\widehat{\Lambda}}(M \cup \widehat{\Lambda}).$$

Remark 3.2.3. *Using the invariance of μSh under contact transformations* [**97**, Section 7.2] *and* [**124**, Lemma 6.3]*, which will be discussed in the next section, the right hand side only depends on the germ of $M \cup \widehat{\Lambda}$, and can be viewed as a sheaf of* *categories either in* $M \cup \widehat{\Lambda} \subset T^*M$ *or in some* $T^{*,\infty}N$ *through a Legendrian embedding* $M \cup \widehat{\Lambda} \hookrightarrow T^{*,\infty}N$ *(see also* [**124**, Remark 8.25]*).*

Proposition 3.2.2 ([**84**, Equation 6.4], [**87**, Equation 1.4.6])**.** *For* $p = (x, \xi) \in \Lambda \subset T^{*,\infty}M$ *a smooth point on a Legendrian* $\Lambda \subset T^{*,\infty}M$*, the stalk* μSh_p *satisfies the following: for* $\mathscr{F}, \mathscr{G} \in Sh_{(\Lambda)}(M)$*,* $\varphi \in C^1(M)$ *such that* $\varphi(x) = 0, d\varphi(x) = \xi$*,*

$$Hom_{\mu Sh_p}(\mathscr{F}, \mathscr{G}) = \mu hom(\mathscr{F}, \mathscr{G})_p = Hom(\Gamma_{\varphi \geq 0}(\mathscr{F})_x, \Gamma_{\varphi \geq 0}(\mathscr{G})_x).$$

Theorem 3.2.3 ([**84**, Proposition 6.6 & Lemma 6.7], [**124**, Corollary 5.4])**.** *For* $p = (x, \xi) \in \Lambda \subset T^{*,\infty}M$ *a smooth point on a Legendrian* $\Lambda \subset T^{*,\infty}M$*, the stalk* $\mu Sh_{\Lambda,p} \simeq \mathrm{Mod}(\Bbbk)$*.*

Theorem 3.2.4 (Guillermou, [**84**, Theorem 11.5])**.** *Let* $\Lambda \subset T^{*,\infty}M$ *be a smooth Legendrian submanifold. Suppose the Maslov class* $\mu(\Lambda) = 0$ *and* Λ *is relative spin, then as sheaves of categories*

$$\mu Sh_\Lambda \xrightarrow{\sim} Loc_\Lambda.$$

Proposition 3.2.5 (Guillermou, [**84**, Theorem 7.6 (iv), 7.9, 8.10 & Lemma 11.4])**.** *Let* $\Lambda \subset T^{*,\infty}M$ *be a Legendrian submanifold. Suppose the Maslov class* $\mu(\Lambda) = 0$ *and* Λ *is relative spin. When the front projection of* Λ *is a smooth hypersurface near* p *and* $\varphi \in C^1(M)$ *is a local defining function for* Λ*, then*

$$m_{\Lambda,p}(\mathscr{F}) = \Gamma_{\varphi \geq 0}(\mathscr{F})_x[-d(p)].$$

For two different points p and $p' \in \Lambda$, $d(p) - d(p')$ is equal to the difference of any Maslov potential at p and p'.

Example 3.2.4. *Suppose $\Lambda = \nu^{*,\infty}_{\mathbb{R}^n \times \mathbb{R}_{>0},-}\mathbb{R}^{n+1} \subset T^{*,\infty}\mathbb{R}^{n+1}$ is the inward conormal of $\mathbb{R}^n \times \mathbb{R}_{>0}$ and $\mathscr{F} \in Sh^b_\Lambda(\mathbb{R}^{n+1})$. Then $\mathscr{F}$ is determined by*

$$F_- \longleftarrow F_+ \xrightarrow{\sim} F_+$$

For $p = (0, ..., 0, 0; 0, ..., 0, 1) \in \Lambda$ we can pick $\varphi(x) = x_{n+1}$ and get

$$\Gamma_{\varphi \geq 0}(\mathscr{F})_{(0,...,0)} = \mathrm{Cone}(F_+ \to F_-)[-1] \simeq \mathrm{Tot}(F_+ \to F_-).$$

Therefore one can see that the definition of the microstalk coincides with the definition of the microlocal monodromy defined by Shende-Treumann-Zaslow [**148**, Section 5.1]*, and indeed*

$$m_{\Lambda,p}(\mathscr{F}) \simeq \mu mon(\mathscr{F})_p[-1].$$

Now we are able to define the notion of microstalks, which defines the equivalence in Theorem 3.2.3. Using that we are able to define simple sheaves and pure sheaves, or microlocal rank r sheaves.

Definition 3.2.2. *Let $\Lambda \subset T^{*,\infty}M$ be a Legendrian submanifold. Suppose $\mu(\Lambda) = 0$ and Λ is relative spin. For $p = (x, \xi) \in \Lambda$, the microstalk of $\mathscr{F} \in Sh^b(M)$ at p is*

$$m_{\Lambda,p}(\mathscr{F}) = m_\Lambda(\mathscr{F})_p.$$

$\mathscr{F} \in Sh_\Lambda(M)$ is called microlocal rank r if $m_{\Lambda,p}(\mathscr{F})$ is concentrated at a single degree with rank r. In this case $\mathscr{F}$ is called pure, and when $r = 1$ it is also called simple.

Proposition 3.2.6 ([**87**, Equation 1.4.4])**.** *Let $\Lambda \subset T^{*,\infty}M$ be a Legendrian submanifold. $\mathscr{F} \in Sh_\Lambda(M)$ is microlocal rank r at $p \in \Lambda$ iff*

$$\mu hom(\mathscr{F}, \mathscr{F})_p \simeq \Bbbk^{r^2}.$$

One can estimate the singular support of the sheaf $\mu hom(\mathscr{F}, \mathscr{G})$ in T^*M. Recall that for $A, B \subset X$, we define the normal cone $C(A, B)$ such that $(x, \xi) \in TX$ iff there exists $a_n \in A, b_n \in B, c_n \in \mathbb{R}$ such that

$$a_n, b_n \to x, \;\; c_n(a_n - b_n) \to \xi, \;\; n \to \infty.$$

Proposition 3.2.7 ([**97**, Corollary 5.4.10 & Corollary 6.4.3])**.** *Let $\mathscr{F}, \mathscr{G} \in Sh(M)$. Then*

$$SS(\mu hom(\mathscr{F}, \mathscr{G})) \subset C(SS(\mathscr{F}), SS(\mathscr{G})).$$

In particular, $\mathrm{supp}(\mu hom(\mathscr{F}, \mathscr{G})) \subset SS(\mathscr{F}) \cap SS(\mathscr{G})$.

Remark 3.2.5. *By Proposition 3.1.9* [**97**, Proposition 5.4.4]*, we can show that* [**97**, Corollary 6.4.4 & 6.4.5] *for $\pi : T^*M \to M$ and $\dot\pi : \dot T^*M \to M$ we have*

$$SS(\pi_* \mu hom(\mathscr{F}, \mathscr{G})) \subset \pi_\pi (d\pi^*)^{-1} C(SS(\mathscr{F}), SS(\mathscr{G})) = -SS(\mathscr{F}) \mathbin{\widehat{+}} SS(\mathscr{G}),$$

$$SS(\dot\pi_* \mu hom(\mathscr{F}, \mathscr{G})) \subset \dot\pi_\pi (d\dot\pi^*)^{-1} C(SS(\mathscr{F}), SS(\mathscr{G})) = -SS(\mathscr{F}) \mathbin{\widehat{+}}_\infty SS(\mathscr{G}).$$

Finally, we know that the microlocalization induces morphisms

$$\mu hom(\mathscr{F},\mathscr{G}) \to \mu hom(\mathscr{F},\mathscr{G})|_{S^*M}, \quad \mathscr{H}om(F,G) \to \dot{\pi}_*(\mu hom(\mathscr{F},\mathscr{G})|_{S^*M}).$$

By [**97**, Equation (4.3.1)], we immediately know that the second morphism fits into the following Sato's fiber sequence, which characterizes algebraically the effect of singular support on Homs between sheaves.

Theorem 3.2.8 (Sato's exact triangle, [**84**, Equation 2.17], [**87**, Equation 1.3.5])**.** *Let $\mathscr{F},\mathscr{G} \in Sh(M)$. Then there is an exact triangle*

$$\Delta^*\mathscr{H}om(\pi_1^{-1}\mathscr{F},\pi_2^{-1}\mathscr{G}) \to \mathscr{H}om(\mathscr{F},\mathscr{G}) \to \dot{\pi}_*(\mu hom(\mathscr{F},\mathscr{G})|_{T^{*,\infty}M}) \xrightarrow{+1}.$$

In particular, when $\mathscr{F}$ is constructible, by Proposition 3.1.12

$$D'\mathscr{F} \otimes \mathscr{G} \to \mathscr{H}om(\mathscr{F},\mathscr{G}) \to \dot{\pi}_*(\mu hom(\mathscr{F},\mathscr{G})|_{T^{*,\infty}M}) \xrightarrow{+1}.$$

Remark 3.2.6. *Since $\mathscr{H}om(\mathscr{F},\mathscr{G}) = \Delta^!\mathscr{H}om(\pi_1^{-1}\mathscr{F},\pi_2^!\mathscr{G})$, we know that the first map is induced by the natural map $\Delta^*\mathscr{H} \otimes \omega_{\Delta/M\times M} \to \Delta^!\mathscr{H}$. The failure of the map being an isomorphism is due to the characteristicity of $SS(\mathscr{H})$ with respect to Δ by Proposition 3.1.8.*

3.3. Functors and Quantizations of Hamiltonian Isotopies

The very first step to understand the relation between symplectic geometry and microlocal sheaves is to understand how sheaves change with respect to Hamiltonian isotopies. This requires one to define functors from geometric Hamiltonian isotopies, which is called sheaf quantization. In this section we review the equivalence functor from a Hamiltonian isotopy defined by Guillermou-Kashiwara-Schapira [**88**].

Definition 3.3.1. *Let $\widehat{H}_s : T^*M \times I \to T^*M$ be a homogeneous Hamiltonian on T^*M. Then the Lagrangian graph of the Hamiltonian isotopy $\varphi^s_{\widehat{H}}$ $(s \in I)$ is*

$$\mathrm{Graph}_{\widehat{H}} = \{(x, x', \xi, \xi', s, \sigma) | (x', \xi') = \varphi^s_{\widehat{H}}(x, \xi), \sigma = -\widehat{H}_s \circ \varphi^s_{\widehat{H}}(x, \xi)\} \subset T^*(M \times M \times I).$$

For a conical Lagrangian $\widehat{\Lambda}$, the Lagrangian movie of $\widehat{\Lambda}$ under the Hamiltonian isotopy $\varphi^s_{\widehat{H}}$ $(s \in I)$ is by definition

$$\widehat{\Lambda}_{\widehat{H}} = \{(x, \xi, s, \sigma) | (x, \xi) = \varphi^s_{\widehat{H}}(x_0, \xi_0), \sigma = -\widehat{H}_s \circ \varphi^s_{\widehat{H}}(x_0, \xi_0), (x_0, \xi_0) \in \widehat{\Lambda}\} \subset T^*(M \times I).$$

Theorem 3.3.1 (Guillermou-Kashiwara-Schapira, [**88**, Proposition 3.12])**.** *Let $\widehat{H}_s : T^*M \times I \to T^*M$ be a homogeneous Hamiltonian on T^*M and $\widehat{\Lambda}$ a conical Lagrangian in T^*M. Then there are functors that give equivalences*

$$Sh_{\widehat{\Lambda}}(M) \xleftarrow{\sim} Sh_{\widehat{\Lambda}_{\widehat{H}}}(M \times I) \xrightarrow{\sim} Sh_{\varphi^1_{\widehat{H}}(\widehat{\Lambda})}(M)$$

given by restriction functors i_0^{-1} and i_1^{-1} where $i_s : M \times \{s\} \hookrightarrow M \times I$ is the inclusion.

Remark 3.3.1. *One can show that the theorem also works for a U-parametric family of Hamiltonian isotopies for a contractible manifold U.*

For the category of microlocal sheaves $\mu Sh_\Lambda(\Lambda)$, Kashiwara-Schapira [**97**, Theorem 7.2.1] showed that it is invariant under contact transformations, which are just (local) contactomorphisms. Nadler-Shende explained how this will imply the invariance of $\mu Sh_\Lambda(\Lambda)$ under (global) Hamiltonian isotopies.

Theorem 3.3.2 (Nadler-Shende [**124**, Lemma 6.6])**.** *Let $H_s : T^{*,\infty}M \times I \to \mathbb{R}$ be a contact Hamiltonian on $T^{*,\infty}M$ and Λ a Legendrian in $T^{*,\infty}M$. Then there are equivalences*

$$\mu Sh_\Lambda(\Lambda) \xleftarrow{\sim} \mu Sh_{\Lambda_H}(\Lambda_H) \xrightarrow{\sim} \mu Sh_{\varphi_H^1(\Lambda)}(\varphi_H^1(\Lambda))$$

given by restriction functors i_0^{-1} and i_1^{-1} where $i_s : T^{*,\infty}M \times \{s\} \hookrightarrow T^{*,\infty}(M \times I)$ is the inclusion. We denote their inverses by Ψ_H^0 and Ψ_H^1, and $\Psi_H = i_1^{-1} \circ \Psi_H^0$.

Proof. For any open subset $\Omega \subset T^{*,\infty}M$, consider the contact movie $\Omega_{H,s,\epsilon} \subset T^{*,\infty}(M \times I)$ in the time interval $I_{s,\epsilon} = (s - \epsilon, s + \epsilon)$. Then i_s^{-1} induces equivalences of categories

$$Sh_{\Lambda_H \cup \Omega_{H,s,\epsilon}^c}(M \times I_{s,\epsilon}) \xrightarrow{\sim} Sh_{\varphi_H^s(\Lambda \cup \Omega^c)}(M), \;\; Sh_{\Omega_{H,s,\epsilon}^c}(M \times I_{s,\epsilon}) \xrightarrow{\sim} Sh_{\varphi_H^s(\Omega^c)}(M).$$

Since $Sh(M \times I_{s,\epsilon}) = Sh(M \times I)/Sh_{T^*(M \times I \setminus I_{s,\epsilon})}(M \times I)$, we get an equivalence of presheaves

$$i_s^{-1} : \varinjlim_{\epsilon \to 0} \mu Sh_{\Lambda_H}^{\mathrm{pre}}(\Omega_{H,s,\epsilon}) \xrightarrow{\sim} \mu Sh_{\varphi_H^s(\Lambda)}^{\mathrm{pre}}(\varphi_H^s(\Omega)),$$

where the left hand side is the pull back of a presheaf, since $\Omega_{H,s,\epsilon}\,(\epsilon > 0)$ form a neighbourhood basis of $\varphi_H^s(\Omega)$. Therefore, after sheafification, we can get an equivalence given by the pull back

$$i_s^{-1} : \mu Sh_{\Lambda_H}(\varphi_H^s(\Lambda)) \xrightarrow{\sim} \mu Sh_{\varphi_H^s(\Lambda)}(\varphi_H^s(\Lambda)).$$

Then, since $\mu Sh^{\mathrm{pre}}_{\Lambda_H}(\Omega_{H,s,\epsilon}) \simeq \mu Sh^{\mathrm{pre}}_{\Lambda_H}(\Omega_{H,s',\epsilon})$, we also know that $\mu Sh^{\mathrm{pre}}_{\Lambda_H}$ forms a presheaf that is locally constant in the I direction (along contact movies of points). Since I is contractible, we can conclude that there is an equivalence given by the restriction

$$\mu Sh_{\Lambda_H}(\Lambda_H) \xrightarrow{\sim} \mu Sh_{\Lambda_H}(\varphi^s_H(\Lambda)).$$

This completes the proof of the theorem. □

Remark 3.3.2. *One can show that the theorem also works for a U-parametric family of Hamiltonian isotopies for a contractible manifold U, following Remark 3.3.1.*

Remark 3.3.3. *From our proof, one may notice that there is a commutative diagram*

$$\begin{array}{ccc} Sh_{\Lambda_H}(M \times I) & \xrightarrow{i_s^{-1}} & Sh_{\varphi^s_H(\Lambda)}(M) \\ \downarrow & & \downarrow \\ \mu Sh_{\Lambda_H}(\Lambda_H) & \xrightarrow{i_s^{-1}} & \mu Sh_{\varphi^s_H(\Lambda)}(\varphi^s_H(\Lambda)). \end{array}$$

3.4. Various Versions of Microlocal Sheaf Categories

We have defined the sheaf of stable categories Sh_Λ and μSh_Λ consisting of sheaves and respectively microsheaves. However, in general we may want to work with either the subcategories of compact objects or proper objects. We explain how to restrict to these categories. Most of the discussions can be found in [**121**, Section 3.6 & 3.8] and [**74**, Section 4.5].

Throughout the discussion, we will be considering the microlocal sheaf category μSh_Λ on a subanalytic Legendrian (or conical Lagrangian) subset.

Following [**74**, Lemma 4.11] or [**153**, Section 2.2], we know that the category of sheaves $Sh_\Lambda(M) \subset Sh(M)$ is a right orthogonal complement category and is well generated. When $\Lambda \subset T^{*,\infty}M$ is closed, $Sh_\Lambda(M)$ is complete and cocomplete. Moreover, we also know that $Sh_\Lambda(M)$ is presentable [**74**, Lemma 4.12].

Definition 3.4.1. *For $\mathscr{F} \in \mu Sh_\Lambda(\Omega)$, we call it a compact object if the Yoneda module $Hom_{\mu Sh_\Lambda(\Omega)}(\mathscr{F}, -)$ commutes with filtered colimits. Let $\mu Sh_\Lambda^c(\Omega) \subset \mu Sh_\Lambda(\Omega)$ be the full subcategory of compact objects.*

In particular, when we consider for a subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$ the category of compact objects

$$Sh_\Lambda^c(M) = \mu Sh_{M \cup \widehat{\Lambda}}^c(T^*M),$$

we can prove that it is a smooth category in the sense of [**103**, Definition 8.1.2] (see also [**111**, Definition 4.6.4.13]), namely that (for the small category $\mathscr{A}$ under consideration) the diagonal bimodule

$$\mathscr{A}_\Delta(X,Y) = Hom_{\mathscr{A}}(X,Y)$$

is a perfect $\mathscr{A}^{op} \times \mathscr{A}$-bimodule.

Proposition 3.4.1 ([**74**, Corollary 4.25])**.** *Let M be compact and $\Lambda \subset T^{*,\infty}M$ be a subanalytic isotropic subset. Then $Sh_\Lambda^c(M)$ is a smooth category.*

Since $Sh_\Lambda(M)$ is well generated and presentable, we know that μSh_Λ is both a sheaf and a cosheaf of categories with respect to restrictions and corestrictions. In fact, for $V \subseteq U$, the restriction functor

$$r_{UV}^* : \mu Sh_\Lambda(U) \to \mu Sh_\Lambda(V)$$

preserves limits and colimits and thus admits left and right adjoints [**74**, Lemma 4.12]. Since r_{UV}^* preserves colimits, its left adjoint, which is called the corestriction functor

$$r_{UV,!} : \mu Sh_\Lambda(V) \to \mu Sh_\Lambda(U).$$

preserves compact objects. Hence the corestriction functor restricts to the subsheaf of category of compact objects

$$r_{UV,!} : \mu Sh_\Lambda^c(V) \to \mu Sh_\Lambda^c(U).$$

Note that $\mu Sh_{\Lambda\cap U}(U) = \mu Sh_\Lambda(U)$, so this is indeed a functor on global sections of categories $\mu Sh_{\Lambda\cap V}^c(V) \to \mu Sh_{\Lambda\cap U}^c(U)$.

Remark 3.4.1. *For closed subanalytic isotropic subsets $\Lambda \subset T^{*,\infty}M$, the microlocalization and its left adjoint*

$$m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda), \;\; m_\Lambda^l : \mu Sh_\Lambda(\Lambda) \to Sh_\Lambda(M)$$

are special cases of restriction functors and corestriction functors. In particular, the left adjoint of microlocalization m_Λ^l preserves compact objects

$$m_\Lambda^l : \mu Sh_\Lambda^c(\Lambda) \to Sh_\Lambda^c(M).$$

Given sheaves of categories μSh_X and μSh_Y, where $X \subseteq Y$ is a closed subset, there is an inclusion functor between sheaves of categories

$$\iota_{XY*} : \mu Sh_X \to \mu Sh_Y$$

which also preserves limits and colimits. Since it preserves limits and is accessible, there is a left adjoint called the pullback functor

$$\iota_{XY}^* : \mu Sh_Y \to \mu Sh_X.$$

Since ι_{XY*} preserves colimits, ι_{XY}^* preserves compact objects. Hence the corestriction functor preserves the sub-cosheaf of categories of compact objects. By considering global sections, we get a pullback functor $\iota_{XY}^* : \mu Sh_Y^c(Y) \to \mu Sh_X^c(X)$.

Remark 3.4.2. *For closed subanalytic isotropic subsets $\Lambda \subset \Lambda' \subset T^{*,\infty}M$, the inclusion functor and its left adjoint*

$$\iota_{\Lambda\Lambda'*} : Sh_\Lambda(M) \hookrightarrow Sh_{\Lambda'}(M), \;\; \iota_{\Lambda\Lambda'}^* : Sh_{\Lambda'}(M) \to Sh_\Lambda(M)$$

are special cases of the inclusion and pullback functors above. In particular, the pullback functor preserves compact objects

$$\iota^*_{\Lambda\Lambda'} : Sh^c_{\Lambda'}(M) \to Sh^c_{\Lambda}(M).$$

This is also called the stop removal functor [**74**, Corollary 4.22] *(one can compare it to the stop removal functors in partially wrapped Fukaya categories* [**75**, Theorem 1.16]*).*

Let Λ be a singular isotropic and $(x, \xi) \in \Lambda$ be a smooth point. Up to a shift, there is a microstalk functor $\mu_{(x,\xi)} : Sh_{\Lambda}(M) \to \mathrm{Mod}(\Bbbk)$ [**97**, Proposition 7.5.3], which admits descriptions by sub-level sets of functions whose differential is transverse to Λ [**74**, Theorem 4.10]. By applying its left adjoint to the generator $\Bbbk$, we see that it is tautologically corepresented by the compact object $\mu^l_{(x,\xi)}(\Bbbk) \in Sh^c_{\Lambda}(M)$. Furthermore, when there is an inclusion $\Lambda \subseteq \Lambda'$ and $(x, \xi) \in \Lambda'$, the corepresentative $\mu^l_{(x,\xi)}(\Bbbk) \in \mathrm{Sh}^c_{\Lambda'}(M)$ is sent under $Sh^c_{\Lambda'}(M) \twoheadrightarrow Sh^c_{\Lambda}(M)$ to a similar corepresentative in $Sh^c_{\Lambda}(M)$ and, they are tautologically sent to the zero object when (x, ξ) is a smooth point in $\Lambda' \setminus \Lambda$. The converse is also true:

Proposition 3.4.2 ([**74**, Theorem 4.13])**.** *Let $\Lambda \subseteq \Lambda'$ be subanalytic isotropics and let $\mathscr{D}^{\mu}_{\Lambda',\Lambda}(T^*M)$ denote the fiber of the canonical functor $Sh^c_{\Lambda'}(M) \twoheadrightarrow Sh^c_{\Lambda}(M)$. Then $\mathscr{D}^{\mu}_{\Lambda',\Lambda}(T^*M)$ is generated by the corepresentatives of the microstalk functors $\mu_{(x,\xi)}$ for smooth Legendrian points $(x, \xi) \in \Lambda' \setminus \Lambda$.*

On the other hand, we can consider the subcategory with perfect stalks, which turns out to be the subcategory of proper modules (equivalently, pseudoperfect modules) in the category of (micro)sheaves.

Definition 3.4.2. *Let* $\mu Sh^b_\Lambda(U) \subset \mu Sh_\Lambda(U)$ *be the full subcategory of objects with perfect stalks, and* $\mu Sh^{pp}_\Lambda(U) = \mathrm{Fun}^{ex}(\mu Sh^c_\Lambda(U)^{op}, \mathrm{Perf}(\Bbbk))$ *be the category of proper modules in* $\mu Sh^c_\Lambda(U)$*, where* $\mathrm{Fun}^{ex}(-,-)$ *is the stable category of exact functors.*

Since restriction functors in μSh_Λ preserves (micro)stalks, the sheaf of categories μSh_Λ can be restricted to a subsheaf of categories μSh^b_Λ. Meanwhile, since $\mu Sh^c_\Lambda(U)$ forms a cosheaf of categories under corestriction functors, we know that the full subcategories of proper submodules μSh^{pp}_Λ also forms a sheaf of categories under restriction functors.

The following theorem shows that $\mu Sh^b_\Lambda(U)$ is the equivalent to the subcategories of proper modules μSh^{pp}_Λ in $\mu Sh^c_\Lambda(U)$.

Theorem 3.4.3 (Nadler [**121**, Theorem 3.21], [**74**, Corollary 4.23])**.** *Let* $\Lambda \subseteq T^{*,\infty}M$ *be a subanalytic isotropic subset. Then the natural pairing* $\mu hom(-,-)$ *defines an equivalence*

$$\mu Sh^b_\Lambda(U) \simeq \mu Sh^{pp}_\Lambda(U) = \mathrm{Fun}^{ex}(\mu Sh^c_\Lambda(U)^{op}, \mathrm{Perf}(\Bbbk)).$$

In particular, $Sh^b_\Lambda(M) \simeq Sh^{pp}_\Lambda(M)$*.*

Using the above theorem, for a subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$ the category of proper modules

$$Sh^{pp}_{\Lambda}(M) = \mu Sh^{pp}_{M \cup \widehat{\Lambda}}(T^*M),$$

is a proper category (see [**103**, Definition 8.2.1] or [**111**, Definition 4.6.4.2]), namely that (for the small category $\mathscr{A}$ under consideration) the diagonal bimodule $\mathscr{A}_\Delta$ is a proper bimodule, i.e. for any $X, Y \in \mathscr{A}$,

$$Hom_{\mathscr{A}}(X, Y) \in \mathrm{Perf}(\Bbbk).$$

Proposition 3.4.4 ([**74**, Corollary 4.25])**.** *Let M be compact and $\Lambda \subseteq T^{*,\infty}M$ be a subanalytic isotropic subset. Then $Sh^{pp}_{\Lambda}(M)$ is a proper category.*

Since $Sh^c_{\Lambda}(M)$ is a smooth category, we know by [**74**, Lemma A.8] that $Sh^{pp}_{\Lambda}(M) \subseteq Sh^c_{\Lambda}(M)$. Therefore we have the following corollary.

Corollary 3.4.5. *Let M be compact and $\Lambda \subset T^{*,\infty}M$ be a subanalytic isotropic subset. Then $Sh^b_{\Lambda}(M) \subseteq Sh^c_{\Lambda}(M)$.*

Remark 3.4.3. *From the discussion above, we can show that for closed subanalytic isotropic subsets $\Lambda \subset T^{*,\infty}M$, the microlocalization functor preserves proper objects*

$$m_\Lambda : Sh^b_{\Lambda}(M) \to \mu Sh^b_{\Lambda}(\Lambda),$$

and so does the inclusion functor

$$\iota_{\Lambda\Lambda' *} : Sh^b_{\Lambda}(M) \hookrightarrow Sh^b_{\Lambda'}(M).$$

CHAPTER 4

Microlocalization and Doubling along Legendrians

Our goal in this section is to prove the duality and long exact sequence regarding microlocalization and understand the microlocalization functor

$$m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda).$$

by the doubling construction in sheaf theory (which is also known as the antimicrolocalization functor [**124**] or the Guillermou convolution functor [**94**]).

First, we will consider an arbitrary Reeb flow T_t, $t \in \mathbb{R}$, on $T^{*,\infty}M$ and prove the Sabloff-Serre duality and Sato-Sabloff long exact sequence.

Theorem 4.0.6 (Theorem 1.2.3)**.** *Let $\Lambda \subset T^{*,\infty}M$ be a closed subanalytic Legendrian and $c(\Lambda)$ be the length of the shortest Reeb chord on Λ with respect to the Reeb flow T_t. Let $\mathscr{F} \in Sh_\Lambda(M)$ and $\mathscr{G} \in Sh_{\Lambda'}(M)$. Then for $0 < \epsilon < c(\Lambda)/2$, there is an exact triangle*

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})) \to Hom(\mathscr{F}, T_\epsilon(\mathscr{G})) \to \Gamma(\Lambda, \mu hom(\mathscr{F}, \mathscr{G})) \xrightarrow{+1} .$$

Let ω_M be the dualizing sheaf of M. There is a duality

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G}) \otimes \omega_M)^\vee = Hom(\mathscr{G}, T_\epsilon(\mathscr{F})).$$

It turns out that the duality and long exact sequence show that there exists a fully faithful functor which is the right inverse of microlocalization. This is the doubling functor.

Theorem 4.0.7. *Let $\Lambda \subset T^{*,\infty}M$ be a closed subanalytic Legendrian and $c(\Lambda)$ be the length of the shortest Reeb chord on Λ with respect to the Reeb flow T_t. Then for $0 < \epsilon < c(\Lambda)/2$, there is a fully faithful functor*

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_\epsilon(\Lambda) \cup T_{-\epsilon}(\Lambda)}(M).$$

The doubling functor in sheaf theory goes back to Guillermou [**84**, Section 13-15], and is also formulated in a different way in Nadler-Shende [**124**, Section 6]. Here we will generalize that functor to arbitrary Reeb flows on $T^{*,\infty}M$. In Lagrangian Floer theory, the stop doubling construction has been discussed in the setting of Fukaya-Seidel categories [**4**] (see also [**2**, **6**]) as the cup functor

$$\cup_F : \mathcal{F}(F) \to \mathcal{FS}(X, \pi),$$

and also in the setting of partially wrapped Fukaya categories as the doubling trick [**75**, Example 8.7], cup functor or Orlov functor [**152**]

$$\cup_F : \mathcal{W}(F) \to \mathcal{W}(X, F).$$

Recently the doubling trick has been used in the theory of (twisted) generating families [**3**, Theorem C].

Our key ingredient to deduce the doubling construction is sheaf theoretic wrappings. In Section 4.1, we prove the Sato-Sabloff exact sequence. When discussing the Sato-Sabloff exact sequence, we also show a Sabloff duality using the Verdier duality on sheaves, which has appeared in a number of works in symplectic geometry [**58**, **135**, **141**].

Then in Section 4.2, using the Sato-Sabloff fiber sequence, we define the doubling construction, which allows us to prove sheaf quantization results for a large family of exact Lagrangian submanifolds in Section 4.4, which in particular includes the sheaf quantization result for Lagrangian cobordisms between embedded Lagrangian submanifolds, and the conditional sheaf quantization result for Lagrangian cobordisms between Legendrian submanifolds.

Sheaf quantziation, namely constructing sheaves from known symplectic/contact geometric data has been to core problem in the field, studied in a number of celebrated works [**12**, **84**, **88**, **94**, **153**]. Given exact Lagrangians with Legendrian lifts $\widetilde{L} \subset J^1(M)$ that are either closed or with Legendrian boundaries at the ideal contact boundary $T^{*,\infty}M$, Guillermou and Jin-Treumann constructed fully faithful functors

$$\Psi_L : Loc(L) \hookrightarrow Sh_{\widetilde{L}}(M \times \mathbb{R})$$

that are inverses of taking microlocalization.

We are able to generalize the sheaf quantization results in more general settings. Recall that $Sh_\Lambda(M \times \mathbb{R})_0$ consists of sheaves with acyclic stalks at $M \times \{-\infty\}$.

One class of noncompact exact Lagrangians that are of particular interest are Lagrangian cobordisms in the sense of Arnol'd [**9**]. Their relation with Lagrangian Floer theory and Fukaya categories have been studied in a number of recent works, namely Lagrangian cobordisms induce equivalences on the Fukaya category, and Lagrangian cobordisms with extra ends induce morphisms in the Fukaya category which give iterated mapping cone decompositions [**16**, **17**, **125**, **155**]. Our result is the first step to understand the relationship between Lagrangian cobordisms and microlocal sheaves. This is based on joint work with T. Asano and Y. Ike.

Theorem 4.0.8 (Theorem 1.3.1)**.** *Let* $V \subset T^*(M \times \mathbb{R})$ *be an exact Arnol'd Lagrangian cobordism between* $L_1, \dots, L_p$ *and* $K_1, \dots, K_q$ *with a Legendrian lift* $\widetilde{V} \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times \mathbb{R})$*. Then there is a fully faithful right inverse functor of* m_V

$$\Psi_V : \mu Sh_V(V) \xrightarrow{\sim} Sh_{\widetilde{V}}(M \times \mathbb{R} \times \mathbb{R})_0.$$

However, for Lagrangian cobordisms between Legendrian submanifolds in the sense of symplectic field theory, we know for sure that there does not always exist a sheaf quantization which produces sheaves in $Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0})$ (for example the trivial endocobordism of a stabilized or loose Legendrian). The best to hope is a conditional sheaf quantization result.

We will prove such a conditional sheaf quantization result, which explains that given a local system in $Loc(L)$, the necessary condition of existence of a sheaf quantization at the negative end $Sh_{\Lambda_-}(M \times \mathbb{R})$ is in fact also the sufficient condition.

Theorem 4.0.9 (Theorem 1.3.2)**.** *Let $L \subset J^1(M) \times \mathbb{R}_{>0}$ be an exact Lagrangian cobordism between Legendrians from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$, with Legendrian lift $\widetilde{L} \subset J^1(M) \times \mathbb{R}_{>0}$. Then there is a fully faithful right inverse functor of (i_-^{-1}, m_L)*

$$\Psi_L : Sh_{\Lambda_-}(M \times \mathbb{R})_0 \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0})_0$$

where $i_- : M \times \mathbb{R} \times s_- \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$ for $s_- > 0$ sufficiently small and $m_L : Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0}) \to \mu Sh_L(L)$ is the microlocalization.

In $J^1(\mathbb{R})$ and $J^1(S^1)$, Pan-Rutherford showed that the dg algebra map can be viewed as a bimodule [**130**]. By enhancing with loop space coefficients, we expect that the enhanced dg algebra map can be viewed as correspondences parametrized by chains on the based loop space of L

$$\mathcal{A}(\Lambda_-) \otimes_{C_{-*}(\Omega_*\Lambda_-)} C_{-*}(\Omega_* L) \to \mathcal{A}(\widetilde{L}) \leftarrow \mathcal{A}(\Lambda_+)$$

where the first map is an equivalence when L is embedded. Forgetting the data of $C_{-*}(\Omega_* L)$, we in particular have a diagram (of dg algebras with no loop space coefficients)

$$\mathcal{A}(\Lambda_-) \to \mathcal{A}(\widetilde{L}) \leftarrow \mathcal{A}(\Lambda_+).$$

Similarly, in microlocal sheaf theory, one may also consider the restriction functors to both ends which define correspondences between the sheaf categories with singular supports on $\Lambda_\pm$ parametrized by local systems on L. Our result realizes the Lagrangian cobordism functor as correspondences,

4.1. Sabloff Duality and Sato-Sabloff Exact Triangle

The goal in this section is to prove Theorem 4.0.6.We will explain how Reeb flows induce continuation maps and how the cones of the continuation maps are determined by the Hom in $\mu Sh_\Lambda(\Lambda)$.

4.1.1. Continuation maps for positive Hamiltonians

In Lagrangian Floer theory people consider wrappings by positive Hamiltonian at infinity. Here we consider the effect of a positive Hamiltonian flow and how they define continuation maps. By a positive Hamiltonian, we mean the following.

Definition 4.1.1 (Eliashberg-Polterovich [**67**])**.** *Let* $(Y, \ker\alpha)$ *be a cooriented contact manifold. Then a time-dependent Hamiltonian* $H \in C^\infty([0,1]\times Y)$ *is called positive if* $H(u,x) \geq 0$, $\forall\,(u,x) \in [0,1]\times Y$.

Remark 4.1.1. *One can show that any time-independent positive Hamiltonian vector flow is the Reeb flow for some contact form* $e^f\alpha$ *on the contact manifold* $(Y, \ker\alpha)$.

Remark 4.1.2. *Denote by* $G\circ H$ *the composition of* G *and* H *such that* $\varphi^u_{G\circ H} = \varphi^u_G \circ \varphi^u_H$. *Then the time-1 Hamiltonian flow defined by* $G\circ H$ *is homotopic to the concatenation of the concatenation of time-1 flows of* G *and* H *through positive Hamiltonian isotopies.*

Write $q : M\times\mathbb{R} \to M$ and $u : M\times\mathbb{R}\to\mathbb{R}$ be the projection maps. For a subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$, consider the Legendrian movie of Λ under the identity flow

$$\Lambda_q = \{(x, \xi, u, 0) | (x, \xi) \in \Lambda\}.$$

Let $\varphi_H^u : T^{*,\infty}M \to T^{*,\infty}M$ be any positive Hamiltonian flow defined by the $H : T^{*,\infty}M \to \mathbb{R}$ and consider the Legendrian movie of Λ under the Reeb flow

$$\Lambda_H = \{(x, \xi, u, \nu) | (x, \xi) \in \varphi_H^u(x_0, \xi_0), \nu = -H(x_0, \xi_0), (x_0, \xi_0) \in \Lambda\}.$$

Abusing notations, we will write $\Psi_H^0 : Sh_\Lambda(M) \xrightarrow{\sim} Sh_{\Lambda_H}(M \times \mathbb{R})$ and $\Psi_{H,u} : Sh_\Lambda(M) \xrightarrow{\sim} Sh_{\varphi_H^u(\Lambda)}(M \times \mathbb{R})$ to be the equivalence functor induced by the Hamiltonian flow.

Lemma 4.1.1. *Let H be a positive Hamiltonian on $T^{*,\infty}M$, and $\mathscr{F} \in Sh(M)$ such that* $\operatorname{supp}(\mathscr{F})$ *is compact. Denote by $\Psi_H^0 : Sh_\Lambda(M) \xrightarrow{\sim} Sh_{\Lambda_H}(M \times \mathbb{R})$ and $\Psi_{H,u} : Sh_\Lambda(M) \xrightarrow{\sim} Sh_{\varphi_H^u(\Lambda)}(M)$. Then*

$$\Psi_{H,u}\mathscr{F} = \Bbbk_{(-\infty,u)}[-1] \circ \Psi_H^0\mathscr{F} = q_!(u^{-1}\Bbbk_{(-\infty,u)}[-1] \otimes \Psi_H^0\mathscr{F}).$$

Proof. First of all, since $H > 0$ we know that $SS^\infty(\Psi_H^0(\mathscr{F})) \subset T^{*,\infty}_{\nu<0}(M \times \mathbb{R})$. Therefore by microlocal Morse lemma Proposition 3.1.3 we know that

$$\Psi_{H,u}\mathscr{F} = \varinjlim_{\epsilon>0} q_! i_{(u-\epsilon,u],!} i^!_{(u-\epsilon,u]} \Psi_H^0\mathscr{F} = q_! i_{(-\infty,u],!} i^!_{(-\infty,u]} \Psi_H^0\mathscr{F}.$$

Then the lemma follows since $i_{(-\infty,u],!} i^!_{(-\infty,u]} \Psi_H^0\mathscr{F} = u^{-1}\Bbbk_{(-\infty,u)}[-1] \otimes \Psi_H^0\mathscr{F}$. $\square$

4.1.2. Sato-Sabloff fiber sequence

For compact subanalytic Legendrians $\Lambda_0, \Lambda_1 \subset T^{*,\infty}M$, we let $c(\Lambda_0, \Lambda_1)$ be the minimal absolute value of lengths of Reeb chords between Λ_0 and Λ_1 with respect to the Reeb flow T_t. Abusing notations, we also use T_t to denote the associated functor of its time-t flow which acts on sheaves on M. The key proposition of this section is that the Hom in $\mu Sh_\Lambda(\Lambda)$ can be computed as a difference between the positive and negative wrappings.

Similar considerations have also appeared in previous works of for example Guillermou [**84**, Section 11–13] and Tamarkin [**154**, Equation (1)].

Proposition 4.1.2. *Let Λ_0, $\Lambda_1 \subset T^{*,\infty}M$ be compact subanalytic Legendrians, $\mathscr{F} \in Sh_{\Lambda_0}(M)$, $\mathscr{G} \in Sh_{\Lambda_1}(M)$ and* $\mathrm{supp}(F) \cup \mathrm{supp}(G)$ *is compact. When $0 < \epsilon < c(\Lambda_0, \Lambda_1)$, there is a commutative diagram*

$$\begin{array}{ccc} Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})) & \xrightarrow{\ c\ } & Hom(\mathscr{F}, T_{\epsilon}(\mathscr{G})) \\ \wr\downarrow & & \downarrow\wr \\ \Gamma(M,\, \Delta^* \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^* \mathscr{G})) & \longrightarrow & Hom(\mathscr{F}, \mathscr{G}) \end{array}$$

where the top arrow c is the continuation map associated to the Reeb flow and the bottom arrow is the canonical map in Theorem 3.2.8.

Theorem 4.1.3 (Sato-Sabloff exact triangle)**.** *For $\mathscr{F}, \mathscr{G} \in Sh_\Lambda(M)$, there is a exact triangle*

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})) \xrightarrow{c} Hom(\mathscr{F}, T_\epsilon(\mathscr{G})) \to \Gamma(\Lambda, \mu hom(\mathscr{F}, \mathscr{G})) \xrightarrow{+1} .$$

where the first map is induced by the continuation map $T_{-\epsilon}(\mathscr{G}) \to T_\epsilon(\mathscr{G})$ and the second map is given by the canonical microlocalization map.

Remark 4.1.3. *We remark that the above computation also works in the case when we take microlocalization along a single connected component $\Lambda_i \subset \Lambda \subset T^{*,\infty}M$. Let $\widetilde{T}_t : T^{*,\infty}M \to T^{*,\infty}M$ be a Hamiltonian flow such that $\widetilde{T}_\epsilon|_{T_\epsilon(\Lambda_i)} = T_\epsilon$ is the Reeb flow while $\widetilde{T}_t|_{\Lambda\backslash\Lambda_i} = \mathrm{id}$. Then there is a fiber sequence*

$$Hom(\mathscr{F}, \widetilde{T}_{-\epsilon}(\mathscr{G})) \to Hom(\mathscr{F}, \widetilde{T}_{-\epsilon}(\mathscr{G})) \to \Gamma(\Lambda_i, \mu hom(\mathscr{F}, \mathscr{G}))$$

Before entering the proof of Proposition 4.1.2, we recall the notations in the previous section. Consider the Legendrian movie of Λ under the identity flow

$$\Lambda_q = \{(x, \xi, u, 0) | (x, \xi) \in \Lambda, u \in \mathbb{R}\}.$$

Let $T_t : T^{*,\infty}M \to T^{*,\infty}M$ be any Reeb flow. Consider the Legendrian movie of Λ under the Reeb flow

$$\Lambda_T = \{(x, \xi, u, \nu) | (x, \xi) \in T_u(x_0, \xi_0), \nu = -H \circ T_u(x_0, \xi_0), (x_0, \xi_0) \in \Lambda\}.$$

A standard trick is to consider the total sheaf Hom, $\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})$. The following singular support estimate is essential and will be used throughout the thesis.

Let $\mathcal{Q}_\pm(\Lambda_0, \Lambda_1)$ be the set of unoriented Reeb chords from Λ_0 to Λ_1, namely

$$\mathcal{Q}_\pm(\Lambda_0, \Lambda_1) = \{(x_0, \xi_0, x_1, \xi_1) \in \Lambda_0 \times \Lambda_1 | \, \exists\, u \in \mathbb{R}, T_u(x_0, \xi_0) = (x_1, \xi_1)\}.$$

For a Reeb chord such that $T_u(x_0, \xi_0) = (x_1, \xi_1)$, we call $u \in \mathbb{R}$ the length of the Reeb chord.

Lemma 4.1.4. *Let $\Lambda_{0,1} \subset T^{*,\infty}M$ be subanalytic Legendrians, $T_u : T^{*,\infty}M \to T^{*,\infty}M$ be any Reeb flow and $\mathscr{F} \in Sh_{\Lambda_0}(M), \mathscr{G} \in Sh_{\Lambda_1}(M)$. Then*

$$SS^\infty(\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})) \cap \{(x, 0, u, \nu) \in T^{*,\infty}(M \times \mathbb{R}) | \nu > 0\} = \varnothing,$$

$$SS^\infty(\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})) \cap \{(x, 0, u, \nu) \in T^{*,\infty}(M \times \mathbb{R}) | \nu < 0\} \hookrightarrow \mathcal{Q}_\pm(\Lambda_0, \Lambda_1),$$

where the u coordinates in the intersection correspond to lengths of Reeb chords in $\mathcal{Q}_\pm(\Lambda_0, \Lambda_1)$. In particular, $\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})$ is $\mathbb{R}_u$-noncharacteristic away from the length spectrum of Reeb chords.

Proof. Since $SS^\infty(q^*\mathscr{F}) \cap SS^\infty(\Psi_T^0\mathscr{G}) = \Lambda_{0,q} \cap \Lambda_{1,T} = \varnothing$, we can apply the singular support estimate Proposition 3.1.7

$$SS^\infty(\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})) \subset (-SS^\infty(q^*\mathscr{F})) + SS^\infty(\Psi_T^0\mathscr{G}) = (-\Lambda_{0,q}) + \Lambda_{1,T}.$$

Hence $(x, 0, u, \nu) \in (-\Lambda_{0,q}) + \Lambda_{1,T}$ if and only if there exists $(x_0, \xi_0) \in \Lambda_0, (x_1, \xi_1) \in \Lambda_1$ such that $(x_1, \xi_1) = T_u(x_0, \xi_0)$, or in other words there is a Reeb chord from Λ_0 to Λ_1

of length u. In particular, we know that $\nu = -H(x_0, \xi_0) < 0$ is determined by such a pair. Hence when $\nu > 0$, there will never be $(x, 0, u, \nu) \in (-\Lambda_{0,q}) + \Lambda_{1,T}$. Therefore

$$SS^\infty(\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})) \cap \mathrm{Graph}(du) = \varnothing,$$

$$SS^\infty(\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})) \cap \mathrm{Graph}(-du) \hookrightarrow \mathcal{Q}_\pm(\Lambda_0, \Lambda_1),$$

where our injection maps $(x, 0, u, \nu)$ to the Reeb chord of length u connecting $(x_0, \xi_0) \in \Lambda_0$ and $(x_1, \xi_1) = T_u(x_0, \xi_0) \in \Lambda_1$. □

Proof of Proposition 4.1.2. Denote by i_u the inclusion of the slice of $M \times M \times u \to M \times M \times \mathbb{R}_u$. We first prove the more straightforward statement of $Hom(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} Hom(\mathscr{F}, T_\epsilon(\mathscr{G}))$. The above Lemma 4.1.4 implies that, by Proposition 3.1.10, the ϵ-slice of the total internal Hom sheaf $\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})$ is the same as

$$i_\epsilon^*\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G}) = i_\epsilon^!\mathscr{H}om(q^*\mathscr{F}, \Psi_T^0\mathscr{G})[-1] = \mathscr{H}om(F, T_\epsilon(G)).$$

Then we may use Lemma 4.1.1, and the isomorphism $\Bbbk_{(-\infty,0)} \xrightarrow{\sim} \lim_{u\to 0^+} \Bbbk_{(-\infty,u)}$ will induce

$$\mathscr{H}om(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \lim_{u\to 0^+} \mathscr{H}om(\mathscr{F}, T_u(\mathscr{G})).$$

Applying $\Gamma(M, -)$, we obtain that

$$Hom(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \lim_{t\to 0^+} Hom(\mathscr{F}, T_t(\mathscr{G})).$$

However, the latter limit when restricting to $0 < \epsilon < c(\Lambda)$ is a constant diagram induced by a Legendrian isotopy and the projection

$$\lim_{u \to 0^+} Hom(\mathscr{F}, T_u(\mathscr{G})) \xrightarrow{\sim} Hom(\mathscr{F}, T_\epsilon(\mathscr{G}))$$

is an isomorphism for $0 < \epsilon < c(\Lambda)$.

To prove the statement for $Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})) \to \Gamma(M, \Delta^* \mathscr{H}om(\mathscr{F}, \mathscr{G}))$, let $\pi_{i,\mathbb{R}} : M \times M \times \mathbb{R} \to M \times \mathbb{R}$ denote the $\mathbb{R}$-parameter version of the projection to the i-th component. Theorem 3.2.8 implies that there is a canonical map

$$\Delta^* \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^* \mathscr{G}) \to \Delta^! \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^! \mathscr{G}) = \mathscr{H}om(\mathscr{F}, \mathscr{G}).$$

Here, we use the fact that $\Delta^! 1_{M \times M} = \omega_M^{-1}$ is an invertible sheaf so we can multiply the morphism with its inverse ω_M. Similarly, there is an canonical map

$$(\Delta \times \mathrm{id}_{\mathbb{R}})^* \mathscr{H}om(\pi_{1,\mathbb{R}}^* q^* \mathscr{F}, \pi_{2,\mathbb{R}}^* \Psi_T^0 \mathscr{G}) \to \mathscr{H}om(q^* \mathscr{F}, \Psi_T^0 \mathscr{G})$$

which is an isomorphism over $(-c(\Lambda), 0)$ by the similar singular support estimation as the above Lemma 4.1.4. Thus, by consider the ϵ-slice for $-c(\Lambda) < \epsilon < 0$ and the 0-slice, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\Delta^* \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^* T_{-\epsilon}(\mathscr{G})) & \longrightarrow & \Delta^* \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^* \mathscr{G}) \\
\wr \downarrow & & \downarrow \\
\mathscr{H}om(\mathscr{F}, T_{-\epsilon}(\mathscr{G})) & \xrightarrow{c} & \mathscr{H}om(\mathscr{F}, \mathscr{G})
\end{array}$$

Applying Proposition 3.1.8 to $(\Delta \times \mathrm{id}_{\mathbb{R}})^* \mathcal{H}om(\pi_{1,\mathbb{R}}^* \mathscr{F}, \pi_{2,\mathbb{R}}^* \Psi_T^0 \mathscr{G})$, we obtain that

$$\operatorname*{colim}_{-u \to 0^-} \mathscr{H}om(\mathscr{F}, T_{-u}(\mathscr{G})) \xleftarrow{\sim} \operatorname*{colim}_{-u \to 0^-} \Delta^* \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^* T_{-u}(\mathscr{G})) \xrightarrow{\sim} \Delta^* \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^* \mathscr{G}).$$

Since $\mathrm{supp}(\mathscr{F})$ and $\mathrm{supp}(\mathscr{G})$ are compact, $\Gamma(M, -) = \Gamma_c(M, -)$ is colimit preserving, and thus we conclude the isomorphism that

$$\mathrm{colim}_{-u \to 0^-} Hom(\mathscr{F}, T_{-u}(\mathscr{G})) \xrightarrow{\sim} \Gamma(M, \Delta^* \mathcal{H}om(\pi_1^* \mathscr{F}, \pi_2^* \mathscr{G})).$$

The same argument as in the positive case then implies that the colimit diagram is constant and thus the inclusion

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})) \to \operatorname*{colim}_{-u \to 0^-} Hom(\mathscr{F}, T_{-t}(\mathscr{G}))$$

is an isomorphism for $-c(\Lambda) < -\epsilon < 0$.

Finally, we notice that the diagram commute in the statement commute because it is a composition of the following two commutative diagram:

$$\begin{array}{ccc} \Gamma(M,\, \Delta^* \mathscr{H}om(\pi_1^* \mathscr{F}, \pi_2^* \mathscr{G})) & \longrightarrow & Hom(\mathscr{F}, \mathscr{G}) \\ \downarrow \wr & \circlearrowleft \nearrow \circlearrowleft & \downarrow \wr \\ Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})) & \longrightarrow & Hom(\mathscr{F}, T_{\epsilon}(\mathscr{G})) \end{array}$$

□

Remark 4.1.4. *The identity $Hom(\mathscr{F}, \mathscr{G}) \simeq Hom(\mathscr{F}, T_\epsilon(\mathscr{G}))$ is often referred to as the perturbation trick, and has in fact appeared in previous works of Guillermou*

[**84**, Corollary 16.6] *for the special case of vertical translation on* $M \times \mathbb{R}$*, and Zhou for arbitrary Reeb flows* [**163**]*. The proof here follows* [**104**, Proposition 3.18]*.*

Thus, we have finished the proof of the exact triangle statement in Theorem 4.0.6.

4.1.3. Sabloff-Serre duality

In this section, we illustrate an additional property that arises from the Sato-Sabloff fiber sequence and prove a Sabloff-Serre duality that

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G}) \otimes \omega_M) = Hom(\mathscr{F}, T_\epsilon(\mathscr{G}))^\vee.$$

Such duality between a positive Reeb pushoff and a negative Reeb pushoff has been understood in symplectic geometry in a number of works. In Legendrian contact homology, this is known as the Sabloff duality [**58, 135**], and in Fukaya-Seidel categories, this is known as the Poincaré-Lefschetz duality proved by Seidel [**141**].

To the study of Serre functor, we need the following elementary lemma. Let $\pi_M : M \times N \to M$, $\pi_N : M \times N \to N$ be projection maps.

Lemma 4.1.5. *Let* M *and* N *be manifolds. Then*

(1) $\pi_N^! 1_N = \pi_M^* \omega_M$,

(2) $\omega_{M \times N} = \omega_M \boxtimes \omega_N$.

As a corollary, we see the inverse of ω_M *is isomorphic to* $\Delta^! 1_{M \times M}$.

Proof. Consider the pullback diagram:

$$\begin{array}{ccc} M \times N & \xrightarrow{\pi_N} & N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow{p_M} & \{*\} \end{array}$$

For (1), the base change $p_M^* p_{N!} = \pi_{N!}\pi_M^*$ implies that there exists a canonical map $\pi_M^* p_N^! \to \pi_N^! p_M^*$. This map is in general not an isomorphism but in our case, we may assume M and N are Euclidean spaces by checking the map locally. Then the isomorphism follows from the isomorphism $\Bbbk = \Gamma_c(\mathbb{R}^k, \Bbbk)[k]$ and $\omega_{\mathbb{R}^k} = \Bbbk_{\mathbb{R}^k}[k]$. For (2), we can use (1) of this lemma and Proposition 3.1.8 and compute that

$$\omega_M \boxtimes \omega_N = \pi_M^*\omega_M \otimes \pi_N^*\omega_N = \pi_M^*\omega_M \otimes \pi_M^!\Bbbk_M = \pi_M^!\omega_M = \omega_{M\times N}.$$

To obtain the corollary, we again apply Proposition 3.1.8 again and compute that

$$\Delta^!(\Bbbk_{M\times M}) \otimes \omega_M = \Delta^!(\Bbbk_{M\times M}) \otimes \Delta^*(\pi_1^*\omega_M) = \Delta^!(\pi_1^*\omega_M) = \Delta^!\pi_2^!(\Bbbk_M) = \Bbbk_M. \quad \square$$

The following proposition is the main result in this section, i.e. the Sabloff-Serre duality.

Proposition 4.1.6 (Sabloff-Serre duality)**.** *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian, $\mathscr{F}, \mathscr{G} \in Sh^b_\Lambda(M)$ such that* $\mathrm{supp}(\mathscr{F}), \mathrm{supp}(\mathscr{G})$ *are compact. Then*

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G}) \otimes \omega_M) \simeq Hom(\mathscr{G}, \mathscr{F})^\vee.$$

In particular, when M is oriented, $Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G}))[-n] \simeq Hom(\mathscr{G}, \mathscr{F})^\vee$.

Proof. By Proposition 4.1.2 and Lemma 3.1.11,

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G}) \otimes \omega_M) = p_* \left(\Delta^* \mathcal{H}om(\pi_1^* \mathscr{F}, \pi_2^*(\mathscr{G} \otimes \omega_M))\right)$$
$$= p_*(\mathrm{D}'_M(\mathscr{F}) \otimes \mathscr{G} \otimes \omega_M) = p_*(\mathrm{D}_M(\mathscr{F}) \otimes \mathscr{G}).$$

The compact support assumption then implies that

$$Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G}) \otimes \omega_M)^\vee = Hom(p_!(\mathrm{D}_M(\mathscr{F}) \otimes G), \Bbbk) = Hom(\mathrm{D}_M(\mathscr{F}) \otimes \mathscr{G}, \omega_M)$$
$$= Hom\left(\mathscr{G}, \mathrm{D}_M \circ \mathrm{D}_M(\mathscr{F})\right) = Hom(\mathscr{G}, \mathscr{F}).$$ □

Thus, we have finished the proof of the duality statement, and hence completed the proof of Theorem 4.0.6.

4.2. Doubling Construction or the Guillermou Functor

Let us construct the doubling functor in this section. Our strategy is to define the doubling w_Λ locally and then glue together the local pieces. Therefore, first we will construct the local model of w_Λ.

Consider $\mathscr{F} \in \mu Sh_\Lambda(\Lambda)$. Then we claim that there exists an open covering $\{U_\alpha\}_{\alpha \in I}$ of M and $\mathscr{F}_\alpha \in Sh_{\Lambda \cap \Omega_\alpha}(U_\alpha)$ such that

$$m_{\Lambda \cap \Omega_\alpha}(\mathscr{F}_\alpha) = \mathscr{F}|_{\Lambda \cap \Omega_\alpha} \in \mu Sh_\Lambda(\Lambda \cap T^{*,\infty} U_\alpha).$$

Consider a representative $\mathscr{F}^* \in Sh_{(\Lambda)}(U)$, i.e. for some neighbourhood Ω of $\Lambda \cap T^{*,\infty}U$, $SS(\mathscr{F}) \cap \Omega \subset \Lambda \cap T^{*,\infty}U$. We need to find $\mathscr{F} \in Sh_\Lambda(U)$ with required properties. This is a corollary of the refined microlocal cut-off lemma [**97**, Proposition 6.1.4].

Lemma 4.2.1 (Guillermou [**84**, Lemma 6.7] or [**87**, Lemma 10.2.5])**.** *Let $\Lambda \subseteq T^{*,\infty}M$ be a locally closed subanalytic Legendrian such that $\pi|_\Lambda : \Lambda \to M$ is finite. Then for $(x, \xi) \in \Lambda$, there is a neighbourhood U of $x \in M$ and $\mathscr{F}_U \in Sh_{\Lambda \cap T^{*,\infty}U}(U)$ such that $m_{\Lambda \cap T^{*,\infty}U}(\mathscr{F}_U) = \mathscr{F} \in \mu Sh_\Lambda(S^*U)$.*

In this section we will write down the doubling functor explicitly on each local chart, and use the results in Section 4.1.2 to show it defines a fully-faithful functor. To be more precise, we would like to construct a sheaf $w_\Lambda(\mathscr{F})$ which locally on an open subset U_α will be of the form

$$w_\Lambda(\mathscr{F})_{U_\alpha} = \mathrm{Cofib}(T_{-\epsilon}(\mathscr{F}_\alpha) \to T_\epsilon(\mathscr{F}_\alpha)).$$

However there is some technical issue that, under the Reeb flow T_t on $T^{*,\infty}M$, it is not even true that $T_{\pm\epsilon}(\Lambda \cap T^{*,\infty}U_\alpha) \subset T^{*,\infty}U_\alpha$, and hence the above formula does not seem to be meaningful even at the first place.

Our solution to the above problem is as follows. We will need to push forward the sheaves on U to sheaves on M so as to apply the Reeb flow on the ambient manifold $T^{*,\infty}M$. The singular support of the resulting sheaves consists of both the

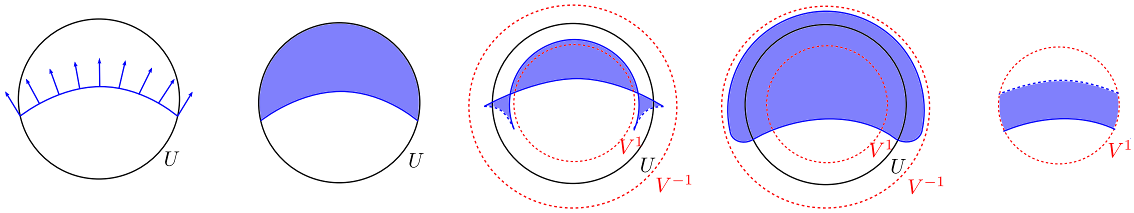

Figure 4.1. We consider the open subset U (in black), the Legendrian Λ (in blue), and the Reeb flow being the geodesic flow, where $T_{\pm\epsilon}(\Lambda \cap T^{*,\infty}U_\alpha) \nsubseteq S^*U_\alpha$. Let $\mathscr{F}_U$ be the sheaf as in the 2nd figure. Then $T_{\pm\epsilon}(j_{U*}\mathscr{F}_U)$ are illustrated in the 3rd and 4th figure. The supports of the sheaves are in V^{-1}, while the singular support coming from $T_{\pm\epsilon}(\nu^*_{U,\pm}M)$ are outside V^1. Finally, $w_\Lambda(\mathscr{F})_V$ is shown in the 5th figure.

Reeb pushoff of the Legendrian $T_{\pm\epsilon}(\Lambda \cap T^{*,\infty}U_\alpha)$ and the Reeb pushoff of the unit conormal bundle of the boundary $T_{\pm\epsilon}(\nu^*_{U_\alpha,\pm}M)$.

To block off the effect coming from the second part (which may come into $T^{*,\infty}U_\alpha$ under the Reeb flow), we will need to restrict the sheaf to a smaller neighbourhood $V_\alpha \subset U_\alpha$. This accounts for the following complicated definition.

From now on, when considering an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$, we will always write $U_{\alpha_1\cdots\alpha_k} = \bigcap_{i=1}^k U_{\alpha_i}$ for simplicity.

Definition 4.2.1. *Let $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$ be an open covering of M, $\Lambda \subseteq T^{*,\infty}M$ a closed subanalytic Legendrian and Λ' a generic Hamiltonian perturbation of Λ. Then $\mathscr{U}$ is a good covering with respect to $\Lambda \subset T^{*,\infty}M$ if*

(1) $U_{\alpha_1\cdots\alpha_k}\,(\alpha_1,\cdots,\alpha_k \in I)$ are contractible;

(2) $\partial U_\alpha\,(\alpha \in I)$ are piecewise smooth with transverse intersections;

(3) $\nu^*_{U_{\alpha_1\cdots\alpha_k},+}M \cap \Lambda' = \varnothing\,(\alpha_1,\cdots,\alpha_k \in I)$.

Given a good covering $\mathscr{U}$ with respect to Λ, a good family of refinement with respect to Λ is $\mathscr{V}^t = \{V_\alpha^t\}_{\alpha \in I}$ where $t \in [-1, 1]$ is a family of open covering with $\mathscr{V}^0 = \mathscr{U}$ such that

(1) $V_\alpha^{t'} \subseteq V_\alpha^t \,(\alpha \in I)$ *for any* $-1 \leq t \leq t' \leq 1$;

(2) $V_{\alpha_1 \cdots \alpha_k}^t \,(\alpha_1, \cdots, \alpha_k \in I)$ *are contractible for any* $-1 \leq t \leq 1$;

(3) $\partial V_\alpha^t \,(\alpha \in I)$ *are piecewise smooth with transverse intersections for any* $-1 \leq t \leq 1$;

(4) *there exists some Riemannian metric* g *on* M *and some* $\epsilon > 0$ *so that*

$$\mathrm{dist}_g(\partial U_\alpha, \partial V_\alpha^{\pm 1}) \geq \epsilon.$$

(5) $\nu^*_{V^t_{\alpha_1 \cdots \alpha_k}, +} M \cap \Lambda' = \varnothing \,(\alpha_1, \cdots, \alpha_k \in I)$ *for any* $-1 \leq t \leq 1$;

For simplicity, we will call $\mathscr{V} = \mathscr{V}^1$ a good refinement of $\mathscr{U}$.

Remark 4.2.1. *This definition is in the same spirit as* [**87**, Definition 11.4.1]*. The reason that we also need to choose a family of good refinement instead of only a good covering is that here we need to consider an arbitrary Reeb flow, while Guillermou considered only the vertical translation in $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ and chose only open subsets of the form $U_i \times I_i \subset M \times \mathbb{R}$. Here we are adding the contractibility assumption simply the discussion when constructing good refinements.*

Lemma 4.2.2. *For any open covering $\mathscr{U}_0$ on M and a closed subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$, there exists a refinement $\mathscr{U}$ with respect to Λ such that $\mathscr{U}$ admits a good family of refinements $\mathscr{V}^t$.*

Proof. The existence of a refinement $\mathscr{U}$ of $\mathscr{U}_0$ satisfying (1) & (2) follows from convex neighbourhood theorem in Riemannian geometry. The reason that

$$\nu^*_{U_{\alpha_1\cdots\alpha_k},+}M \cap \Lambda' = \varnothing,\ \alpha_1,\cdots,\alpha_k \in I$$

for a generic perturbation Λ' of Λ is because $\bigcup_{\alpha_1,\cdots,\alpha_k\in I}\nu^*_{U_{\alpha_1\cdots\alpha_k},+}M$ is also a subanalytic Legendrian and hence the sum of dimensions is less than $2\dim M - 1$.

The existence of a family of refinement $\mathscr{V}^t$ of $\mathscr{U}$ satisfying (1)–(4) is again convex neighbourhood theorem. The reason that

$$\nu^*_{V^t_{\alpha_1\cdots\alpha_k},+}M \cap \Lambda' = \varnothing,\ \alpha_1,\cdots,\alpha_k \in I,\ -1 \leq t \leq 1$$

for a generic perturbation Λ' is that we can choose $\mathscr{V}^t$ so that $\bigcup_{\alpha_1,\cdots,\alpha_k\in I}\nu^*_{V^t_{\alpha_1\cdots\alpha_k},+}M$ are small perturbations of $\bigcup_{\alpha_1,\cdots,\alpha_k\in I}\nu^*_{U_{\alpha_1\cdots\alpha_k},+}M$. This completes the proof. $\square$

Remark 4.2.2. *From now on, without loss of generality (by Theorem 3.3.1 and Theorem 3.3.2) we will always assume that $\Lambda = \Lambda'$. In other words, we assume that $\nu^*_{U_{\alpha_1\cdots\alpha_k},+}M \cap \Lambda = \varnothing$ and $\nu^*_{V^t_{\alpha_1\cdots\alpha_k},+}M \cap \Lambda = \varnothing$ for any $-1 \leq t \leq 1$.*

The main microlocal properties of families of good refinements of coverings that we are going to use are given as follows.

Lemma 4.2.3. *Let $\mathscr{U}$ be a good covering of M with a good family of refinements $\mathscr{V}^t$ with respect to Λ. Write $j_\alpha : U_\alpha \hookrightarrow M$. Then given $\mathscr{F} \in Sh(U_\alpha)$, for $\epsilon > 0$*

sufficiently small, we have

$$T_{\pm\epsilon}(j_{\alpha!}\mathscr{F})|_{V_\alpha} \xrightarrow{\sim} T_{\pm\epsilon}(j_{\alpha*}\mathscr{F})|_{V_\alpha}.$$

Proof. Consider the mapping cone

$$\mathrm{Cone}(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}) \to T_{\pm\epsilon}(j_{\alpha*}\mathscr{F})) \simeq T_{\pm\epsilon}\mathrm{Cone}(j_{\alpha!}\mathscr{F} \to j_{\alpha*}\mathscr{F}).$$

Then by Proposition 3.1.9, $SS^\infty(\mathrm{Cone}(j_{\alpha!}\mathscr{F} \to j_{\alpha*}\mathscr{F})) \subseteq T^{*,\infty}M|_{\partial U_\alpha}$, since $j_{\alpha!}\mathscr{F} \simeq j_{\alpha*}\mathscr{F}$ in the interior of T^*U_α. By Theorem 3.3.1 we know that

$$SS^\infty(T_{\pm\epsilon}\mathrm{Cone}(j_{\alpha!}\mathscr{F} \to j_{\alpha*}\mathscr{F})) \subseteq T_{\pm\epsilon}(T^{*,\infty}M|_{\partial U_\alpha}).$$

Since $T_{\pm\epsilon}(T^{*,\infty}M|_{\partial U_\alpha}) \cap S^*V_\alpha = \varnothing$, we know that in particular for $\epsilon > 0$ sufficiently small,

$$SS^\infty(T_{\pm\epsilon}\mathrm{Cone}(j_{\alpha!}\mathscr{F} \to j_{\alpha*}\mathscr{F})) \cap S^*V_\alpha = \varnothing.$$

Moreover, since $j_{\alpha!}\mathscr{F} \simeq j_{\alpha*}\mathscr{F}$ in T^*V_α, by the above singular support estimate, we can conclude that $T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}) \simeq T_{\pm\epsilon}(j_{\alpha*}\mathscr{F})$ in T^*V_α, which proves the isomorphism as claimed. □

Lemma 4.2.4. *Let $\mathscr{U}$ be a good covering of M with a good family of refinements $\mathscr{V}^t$ with respect to Λ. Write $j_\alpha : U_\alpha \hookrightarrow M$. Then given $\mathscr{F}, \mathscr{G} \in Sh_{\Lambda \cap T^{*,\infty}U_\alpha}(U_\alpha)$, for $\epsilon > 0$ sufficiently small, we have*

$$Hom(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}), T_{\pm\epsilon}(j_{\alpha*}\mathscr{G})) \xrightarrow{\sim} Hom(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F})|_{V_\alpha}, T_{\pm\epsilon}(j_{\alpha*}\mathscr{G})|_{V_\alpha}).$$

Proof. Consider the sheaf $\mathscr{H}om(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}), T_{\pm\epsilon}(j_{\alpha*}\mathscr{G}))$. We know by Proposition 3.1.7 that

$$SS(\mathscr{H}om(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}), T_{\pm\epsilon}(j_{\alpha*}\mathscr{G}))) \subseteq T_{\pm\epsilon}(-SS(j_{\alpha!}\mathscr{F})) \widehat{+} T_{\pm\epsilon}(SS(j_{\alpha*}\mathscr{G})).$$

Consider the family of good refinements V_α^t, $t \in [-1, 1]$, where $V_\alpha^0 = U_\alpha$ and $V_\alpha^1 = V_\alpha$. We know by Proposition 3.1.9 that

$$SS^\infty(j_{\alpha!}\mathscr{F}) \subseteq \Lambda \widehat{+} \nu^*_{U_\alpha,+}M, \;\; SS^\infty(j_{\alpha*}F) \subseteq \Lambda \widehat{+} \nu^*_{U_\alpha,-}M.$$

Therefore by the condition on the family of good refinements V_α^t

$$\nu^*_{V_\alpha^t,+}M \cap (-SS^\infty(j_{\alpha!}\mathscr{F})) = \nu^*_{V_\alpha^t,+}M \cap SS^\infty(j_{\alpha*}\mathscr{F}) = \varnothing.$$

Indeed, for $(x, \xi) \in \nu^*_{V_\alpha^t,+}M$, $(x, \xi) \in \nu^*_{U_\alpha,-}M \widehat{+} (\pm\Lambda)$ only if there exists $(x_n, -\xi_n) \in \nu^*_{U_\alpha,-}M$, $(y_n, \eta_n) \in \pm\Lambda$ such that $x_n, y_n \to x$, $-\xi_n + \eta_n \to \xi$, $|x_n - y_n||\xi_n| \to 0$. However, the fact that $\Lambda \cap \nu^*_{V_\alpha^t}M = \varnothing$ immediately implies that $\eta_n \to 0$. This forces $-\xi_n \to \xi$, which implies $\xi = 0$, so in the unit cotangent bundle the intersections are empty. Hence for $\epsilon > 0$ sufficiently small, we have $\mathrm{supp}(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}))$, $\mathrm{supp}(T_{\pm\epsilon}(j_{\alpha*}\mathscr{G})) \subset V_\alpha^{-1}$, and

$$\nu^*_{V_\alpha^t,+}M \cap T_{\pm\epsilon}(-SS^\infty(j_{\alpha!}F)) = \nu^*_{V_\alpha^t,+}M \cap T_{\pm\epsilon}(SS^\infty(j_{\alpha*}F)) = \varnothing.$$

Therefore, by microlocal Morse lemma Proposition 3.1.3, restricting from V_α^{-1} to V_α^1 we have

$$\Gamma(M, \mathscr{H}om(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}), T_{\pm\epsilon}(j_{\alpha*}\mathscr{G}))) \simeq \Gamma(V_\alpha, \mathscr{H}om(T_{\pm\epsilon}(j_{\alpha!}\mathscr{F}), T_{\pm\epsilon}(j_{\alpha*}\mathscr{G}))),$$

which shows the isomorphism. □

Lemma 4.2.5. *Let $\mathscr{U}$ be a good covering of M with a good family of refinements $\mathscr{V}^t$ with respect to Λ. Write $j_\alpha : U_\alpha \hookrightarrow M$. Then given $\mathscr{F}, \mathscr{G} \in Sh_{\Lambda \cap T^{*,\infty}U_\alpha}(U_\alpha)$, for $\epsilon > 0$ sufficiently small, we have*

$$\Gamma(T^{*,\infty}V_\alpha^{-1}, \mu hom(j_{\alpha!}\mathscr{F}, j_{\alpha*}\mathscr{G})) \xrightarrow{\sim} \Gamma(T^{*,\infty}V_\alpha^1, \mu hom(\mathscr{F}, \mathscr{G})).$$

Proof. Consider the good family of refinements V_α^t where $V_\alpha^0 = U_\alpha$ and $V_\alpha^1 = V_\alpha$. Note that by Proposition 3.1.9 we have

$$SS^\infty(j_{\alpha!}\mathscr{F}) \subset \Lambda \,\widehat{+}\, \nu^*_{U_\alpha,+}M, \;\; SS^\infty(j_{\alpha*}\mathscr{F}) \subset \Lambda \,\widehat{+}\, \nu^*_{U_\alpha,-}M,$$

and then by Proposition 3.2.7 [**97**, Corollary 6.4.3] one can deduce that

$$SS^\infty(\mu hom(j_{\alpha!}\mathscr{F}, j_{\alpha*}\mathscr{G})) \subset C(\Lambda \,\widehat{+}\, \nu^*_{U_\alpha,+}M, \Lambda \,\widehat{+}\, \nu^*_{U_\alpha,-}M) \subseteq T^{*,infty}(T^{*,\infty}M).$$

Then by Proposition 3.1.9 and Remark 3.2.5 we know that

$$SS^\infty(\dot{\pi}_*\mu hom(j_{\alpha!}\mathscr{F}, j_{\alpha*}\mathscr{G})) \subset -(\Lambda \,\widehat{+}\, \nu^*_{U_\alpha,+}M) \,\widehat{+}_\infty\, (\Lambda \,\widehat{+}\, \nu^*_{U_\alpha,-}M).$$

We know $(x,\xi) \in (-\Lambda \widehat{+} \nu^*_{U_\alpha,-}M) \widehat{+} (\Lambda \widehat{+} \nu^*_{U_\alpha,-}M))$ if and only if there are $(x_n, -\xi_n) \in -\Lambda \widehat{+} \nu^*_{U_\alpha,-}M$ and $(y_n, \eta_n) \in \Lambda \widehat{+} \nu^*_{U_\alpha,-}M$ such that $x_n, y_n \to x,\ -\xi_n + \eta_n \to \xi,\ |x_n - y_n||\xi_n| \to 0$. When we consider $x \in \partial U_\alpha$, we know that $(x,\xi) \in \pm\Lambda \widehat{+} \nu^*_{U_\alpha,-}M$ and hence $(x,\xi) \notin \nu^*_{V_\alpha^t,+}M$. When we consider $x \in U_\alpha$, we know that $(x,\xi) \in \Lambda$ and hence $(x,\xi) \notin \nu^*_{V_\alpha^t,+}M$ because $\pm\Lambda \cap \nu^*_{V_\alpha^t,+}M = \varnothing$. These two facts imply that in the unit cotangent bundle the following intersection is empty

$$\nu^{*,\infty}_{V_\alpha^t,+}M \cap SS^\infty(\dot{\pi}_* \mu hom(j_{\alpha !}\mathscr{F}, j_{\alpha *}\mathscr{G})) = \varnothing.$$

Therefore, by microlocal Morse lemma Proposition 3.1.3, restricting from V_α^{-1} to V_α^1, we can show that the isomorphism holds. □

Given $\mathscr{F} \in \mu Sh_\Lambda(\Lambda)$, by Lemma 4.2.1 there exists an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ and a collection of sheaves $\{\mathscr{F}_\alpha\}_{\alpha \in I}$ where $\mathscr{F}_\alpha \in Sh_\Lambda(U_\alpha)$ such that $\mathscr{F}|_{\Lambda \cap T^{*,\infty}U_\alpha} = m_\Lambda(\mathscr{F}_\alpha)$. Write $j_\alpha : U_\alpha \hookrightarrow M$. Now choose a good family of refinements $\mathscr{V}^t\,(t \in [-1,1])$ of $\mathscr{U}$, and define (by Lemma 4.2.3)

$$\begin{aligned} w_\Lambda(\mathscr{F})_{V_\alpha} &= \mathrm{Cone}(T_{-\epsilon}(j_{\alpha *}\mathscr{F}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha *}\mathscr{F}_\alpha)|_{V_\alpha}) \\ &= \mathrm{Cone}(T_{-\epsilon}(j_{\alpha !}\mathscr{F}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha !}\mathscr{F}_\alpha)|_{V_\alpha}). \end{aligned}$$

Proposition 4.2.6. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian, $\mathscr{U}$ a good open covering and $\mathscr{V}^t$ a good family of refinements with respect to Λ. Then for $\epsilon > 0$ sufficiently small, there is a natural isomorphism*

$$Hom(w_\Lambda(\mathscr{F})_{V_\alpha}, w_\Lambda(\mathscr{G})_{V_\alpha}) \xrightarrow{\sim} \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F}_{V_\alpha}, \mathscr{G}_{V_\alpha})).$$

Proof. Writing down the definition of $w_\Lambda(\mathscr{F})_{V_\alpha}$ and $w_\Lambda(\mathscr{G})_{V_\alpha}$, we have

$$\begin{aligned} &Hom(w_\Lambda(\mathscr{F})_{V_\alpha}, w_\Lambda(\mathscr{G})_{V_\alpha}) \\ &\quad\simeq \mathrm{Cone}\big(Hom(\mathrm{Cone}(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}), T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}) \\ &\qquad\qquad \to Hom(\mathrm{Cone}(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}), T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha})\big). \end{aligned}$$

For the first term, we claim that

$$\begin{aligned} &Hom(\mathrm{Cone}(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}), T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}) \\ &\quad\simeq \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F}_{V_\alpha}, \mathscr{G}_{V_\alpha})). \end{aligned}$$

To prove this, we apply the Sato-Sabloff exact sequence Theorem 4.1.3 and get

$$\begin{aligned} &Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha), T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)) \to Hom(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha), T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)) \\ &\qquad \to \Gamma(T^{*,\infty}V_\alpha^{-1}, \mu hom(j_{\alpha!}\mathscr{F}_\alpha, j_{\alpha*}\mathscr{G}_\alpha)). \end{aligned}$$

Given the non-characteristic assumption on $\mathscr{V}^t$ with respect to Λ, we can apply Lemma 4.2.4 to restrict the corresponding $\mathscr{H}om(-,-)$ sheaves from V_α^{-1} to V_α^1, and get quasi-isomorphisms

$$Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha), T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)) \simeq Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}, T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}),$$

$$Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha), T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)) \simeq Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}, T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}).$$

On the other hand, we can also apply Lemma 4.2.5 to restrict the corresponding $\mu hom(-,-)$ sheaves from $T^{*,\infty}V_\alpha^{-1}$ to $T^{*,\infty}V_\alpha^1$, and get

$$\Gamma(T^{*,\infty}V_\alpha^{-1}, \mu hom(j_{\alpha!}\mathscr{F}_\alpha, j_{\alpha*}\mathscr{G}_\alpha)) \simeq \Gamma(T^{*,\infty}V_\alpha^1, \mu hom(\mathscr{F}_\alpha, \mathscr{G}_\alpha)).$$

Since the restriction maps commute with all the maps in the Sato-Sabloff fiber sequence, this proves our first claim. For the second term, we claim that there is a quasi-isomorphism

$$Hom(\mathrm{Cone}(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}), T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}) \simeq 0.$$

Indeed by Proposition 4.1.2 we know that

$$Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha), T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)) \simeq Hom(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha), T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)) \simeq Hom(j_{\alpha!}\mathscr{F}_\alpha, j_{\alpha*}\mathscr{G}_\alpha).$$

and the isomorphism is witnessed by the precomposition with the canonical continuation map $T_{-\epsilon}(j_{\alpha*}\mathscr{F}_\alpha) \to T_\epsilon(j_{\alpha*}\mathscr{F}_\alpha)$. Again by the non-characteristic assumption on $\mathscr{U}, \mathscr{V}$ with respect to Λ, we can apply Lemma 4.2.4 to restrict the corresponding $\mathscr{H}om(-,-)$ sheaves from V_α^{-1} to V_α^1, and the quasi-isomorphisms still hold. This proves our second claim. □

Corollary 4.2.7. *Let $\Lambda \subset T^{*,\infty}M$ be a closed subanalytic Legendrian, $\mathscr{U}$ a good open covering and $\mathscr{V}^t$ a good family of refinements with respect to Λ. Then for $\epsilon > 0$*

sufficiently small,

$$Hom(w_\Lambda(\mathscr{F})_{V_{\alpha_i}}|_{V_{\alpha_1\cdots\alpha_k}}, w_\Lambda(\mathscr{G})_{V_{\alpha_j}}|_{V_{\alpha_1\cdots\alpha_k}}) \simeq \Gamma(T^{*,\infty}V_{\alpha_1\cdots\alpha_k}, \mu hom(\mathscr{F}_{\alpha_i}|_{V_{\alpha_1\cdots\alpha_k}}, \mathscr{G}_{\alpha_j}|_{V_{\alpha_1\cdots\alpha_k}})).$$

Proof. Write $j_{\alpha_1\cdots\alpha_k} : U_{\alpha_1\cdots\alpha_k} \hookrightarrow M$. Note that

$$\begin{aligned} w_\Lambda(\mathscr{F})_{\alpha_i}|_{V_{\alpha_1\cdots\alpha_k}} &= \mathrm{Cone}(T_{-\epsilon}(j_{\alpha_i *}\mathscr{F}_{\alpha_i})|_{V_{\alpha_i}} \to T_\epsilon(j_{\alpha_i *}\mathscr{F}_{\alpha_i})|_{V_{\alpha_i}})|_{V_{\alpha_1\cdots\alpha_k}} \\ &\simeq \mathrm{Cone}(T_{-\epsilon}(j_{\alpha_1\cdots\alpha_k *}\mathscr{F}_{\alpha_i})|_{V_{\alpha_1\cdots\alpha_k}} \to T_\epsilon(j_{\alpha_1\cdots\alpha_k *}\mathscr{F}_{\alpha_i})|_{V_{\alpha_1\cdots\alpha_k}}). \end{aligned}$$

Then the corollary immediately follows from Proposition 4.2.6. □

By Proposition 4.2.6 and Corollary 4.2.7, we can conclude that indeed the family of sheaves $(w_\Lambda(\mathscr{F})_\alpha)_{\alpha\in I}$ on the refined open cover of $\mathscr{V}$ can be glued to a global object.

Proof of Theorem 4.0.7. Consider the doubling functor

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \to Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M).$$

We apply Proposition 4.2.6 and Corollary 4.2.7 to show that w_Λ is fully faithful, i.e.

$$Hom(w_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} \Gamma(T^{*,\infty}M, \mu hom(\mathscr{F}, \mathscr{G})).$$

First of all, since $\Lambda \subset T^{*,\infty}M$ is compact, we may choose assume that there are only finite open subsets $U_\alpha \in \mathscr{U}$ such that $\pi(\Lambda) \cap U_\alpha \neq \varnothing$. Hence there exists a uniform $\epsilon > 0$ sufficiently small such that Proposition 4.2.6 and Corollary 4.2.7 hold for all $U_\alpha \in \mathscr{U}$. By Proposition 4.2.6 and Corollary 4.2.7, and the fact that these quasi-isomorphisms commute with restriction maps, we have the following diagram

$$\begin{array}{ccccc} \bigoplus_{\alpha\in I} Hom(w_\Lambda(\mathscr{F})_{V_\alpha}, w_\Lambda(\mathscr{G})_{V_\alpha}) & \longrightarrow & \bigoplus_{\alpha,\beta\in I} Hom(w_\Lambda(\mathscr{F})_{V_{\alpha\beta}}, w_\Lambda(\mathscr{G})_{V_{\alpha\beta}}) & \rightrightarrows & \cdots \\ \downarrow \wr & & \downarrow \wr & & \\ \bigoplus_{\alpha\in I} \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F},\mathscr{G})) & \longrightarrow & \bigoplus_{\alpha,\beta\in I} \Gamma(T^{*,\infty}V_{\alpha\beta}, \mu hom(\mathscr{F},\mathscr{G})) & \rightrightarrows & \cdots \end{array}$$

Note that $\mathscr{V}$ is a good covering such that any finite intersection is contractible. Therefore, by taking the homotopy colimit of the above diagram, we get the quasi-isomorphism of global sections

$$\begin{aligned} Hom(w_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) &\xrightarrow{\sim} \lim_{\alpha\in I} Hom(w_\Lambda(\mathscr{F})_{V_\alpha}, w_\Lambda(\mathscr{G})_{V_\alpha}) \\ &\xrightarrow{\sim} \lim_{\alpha\in I} \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F},\mathscr{G})) \xrightarrow{\sim} \Gamma(T^{*,\infty}M, \mu hom(\mathscr{F}, G)). \end{aligned}$$

Finally, we need to check that the doubling functor can be defined for any $0 < \epsilon < c(\Lambda)/2$ where $c(\Lambda)$ is the length of the shortest Reeb chord on Λ. This is because when $0 < \epsilon < c(\Lambda)/2$, $T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)$ are related by Hamiltonian isotopies supported away from Λ. We can choose $H : T^{*,\infty}M \to \mathbb{R}$ with compact support (since Λ is compact) such that

$$H|_\Lambda = 0, \;\; H|_{\bigcup_{\epsilon\in[c,c']} T_\epsilon(\Lambda)} = 1, \;\; H|_{\bigcup_{\epsilon\in[c,c']} T_{-\epsilon}(\Lambda)} = -1.$$

Then the contact Hamiltonian flow is the integration of the corresponding compactly supported Hamiltonian vector field. $\square$

Remark 4.2.3. *The condition that Λ is compact plays an important role in the proof. First, it ensures that there exists a uniform $\epsilon > 0$ such that the doubling functor is locally defined among all $V_\alpha \in \mathscr{V}$. Secondly, it ensures that there exists a compactly supported Hamiltonian isotopy relating $T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)$ for $0 < \epsilon < c(\Lambda)/2$ (otherwise, though we can similarly define the Hamiltonian function, it is unclear whether the Hamiltonian vector field is complete).*

Finally, we remark that the doubling construction immediately implies the following fiber sequence.

Corollary 4.2.8. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian. Then there is a fiber sequence of functors*

$$T_{-\epsilon} \to T_\epsilon \to w_\Lambda \circ m_\Lambda.$$

4.2.1. Remark on relative doubling construction

We expect that there is also a more geometric approach to the construction of a relative doubling functor. Following [**124**, Section 6.3], we consider a locally closed subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$ with contact collar $\partial\Lambda \times (0, 1) \hookrightarrow \Lambda$. Then we try to define

$$w_{(\Lambda,\partial\Lambda)} : \mu Sh_\Lambda(\Lambda) \to Sh_{(\Lambda,\partial\Lambda)_\epsilon^{\prec}}(M),$$

where $(\Lambda, \partial\Lambda)^{\prec}_{\epsilon} = \widetilde{T}_{-\epsilon}(\Lambda) \cup \widetilde{T}_{\epsilon}(\Lambda)$ where $\widetilde{T}_t$ is defined by a non-negative contact Hamiltonian $\widetilde{H}$ such that

$$\widetilde{H}|_{\bigcup_{t\in[-\epsilon,\epsilon]} T_t(\Lambda\backslash\partial\Lambda\times(0,1))} = 1, \;\; \widetilde{H}|_{\partial\Lambda\times(0,1/2)} = 0.$$

Suppose $\Lambda = \mathfrak{c}_F$ is the Lagrangian skeleton of a Weinstein hypersurface $F \subset T^{*,\infty}M$. For any exact Lagrangian $L \subset F$ with Legendrian boundary $\partial_\infty L$, consider the limit set of L under the Liouville flow Z_F on F when time goes to $-\infty$, which defines a relative Lagrangian skeleton $\mathfrak{c}_F \cup (\partial_\infty L \times \mathbb{R})$. Following Nadler-Shende [**124**, Section 6.3], we may also expect an explicit construction for the relative doubling functor satisfying properties in this section

$$w_{\mathfrak{c}_F\cup(\partial_\infty L\times\mathbb{R})} : \mu Sh_{\mathfrak{c}_F\cup(\partial_\infty L\times\mathbb{R})}(\mathfrak{c}_F \cup (\partial_\infty L \times \mathbb{R})) \to Sh_{(\mathfrak{c}_F\cup(\partial_\infty L\times\mathbb{R}))^{\prec}_{\epsilon}}(M).$$

Then one may remove the relative part of the stop and send the right hand side into $Sh_{\mathfrak{c}_F}(M)$.

In this setup, one can probably explicitly see that a Lagrangian cocore of $\mathfrak{c}_F$ is sent to the corresponding sheaf theoretic linking disk. Then, following the first author's work [**104**], we are able to define a functor

$$\mu Sh^c_{\mathfrak{c}_F}(\mathfrak{c}_F) \to \mathfrak{w}\mathrm{sh}_{\mathfrak{c}_F}(M),$$

as the relative doubling of all the compact objects in $\mu Sh^c_{\mathfrak{c}_F}(\mathfrak{c}_F)$ in this case will be sheaves with perfect stalks on M. This should be closer to the Fukaya categorical construction of the cup functor.

4.2.2. Doubling as adjoints of microlocalization

Following the Sato-Sabloff exact triangle Proposition 4.2.8, we may expect that the doubling functor w_Λ should be closely related to the left and right adjoint of microlocalization $m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$.

Recall from Section 3.4 that the tautological inclusion $\iota_\Lambda : Sh_\Lambda(M) \hookrightarrow Sh(M)$ admits both a left adjoint ι_Λ^* and a right adjoint $\iota_\Lambda^!$. They will be used in the following propositions.

Theorem 4.2.9. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian. Then the right adjoint of microlocalization is isomorphic to*

$$m_\Lambda^r = \iota_\Lambda^! \circ w_\Lambda : \mu Sh_\Lambda(\Lambda) \to Sh_\Lambda(M).$$

Proof. We know that for $\mathscr{F} \in Sh_\Lambda(M)$,

$$Hom(\mathscr{F}, \iota_\Lambda^!(\mathscr{G})) \simeq Hom(\mathscr{F}, \mathscr{G}).$$

Thus it suffices to show that for any $\mathscr{F} \in Sh_\Lambda(M)$, there is a canonical quasi-isomorphism

$$Hom(\mathscr{F}, w_\Lambda(\mathscr{G})) \simeq \Gamma(T^{*,\infty}M, \mu hom(m_\Lambda(\mathscr{F}), \mathscr{G})).$$

Again, following the proof of Theorem 4.0.7, we consider a good open covering $\mathscr{U}$ and a good family of refinements $\mathscr{V}^t$, and show that locally

$$Hom(\mathscr{F}|_{V_\alpha}, w_\Lambda(\mathscr{G})|_{V_\alpha}) \simeq \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F}|_{V_\alpha}, \mathscr{G}|_{V_\alpha})).$$

Writing down the definition of $w_\Lambda(\mathscr{G})_{V_\alpha}$, we have

$$\begin{aligned}
&Hom(\mathscr{F}|_{V_\alpha}, w_\Lambda(\mathscr{G})_{V_\alpha}) \\
&\qquad \simeq Hom\big(\mathscr{F}|_{V_\alpha}, \mathrm{Cone}(T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha})\big) \\
&\qquad \simeq \mathrm{Cone}\big(Hom(\mathscr{F}|_{V_\alpha}, T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}) \to Hom(\mathscr{F}|_{V_\alpha}, T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha})\big).
\end{aligned}$$

Given the non-characteristic condition for the good family of refinements $\mathscr{V}^t$, by Lemma 4.2.4 we know that

$$Hom(\mathscr{F}|_{V_\alpha}, T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}) \simeq Hom(j_{\alpha!}\mathscr{F}_\alpha, T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)),$$

$$Hom(\mathscr{F}|_{V_\alpha}, T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}) \simeq Hom(j_{\alpha!}\mathscr{F}_\alpha, T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)).$$

In addition, it is easy to show that the restriction functors above commute with the canonical map $T_{-\epsilon}(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha*}\mathscr{G}_\alpha)|_{V_\alpha}$. Therefore, by the Sato-Sabloff exact triangle Theorem 4.1.3 we can conclude that

$$\begin{aligned}&Hom(\mathscr{F}|_{V_\alpha}, w_\Lambda(\mathscr{G})_{V_\alpha})\\ &\qquad \simeq \mathrm{Cone}\big(Hom(\mathscr{F}|_{V_\alpha}, T_{-\epsilon}(j_{\alpha *}\mathscr{G}_\alpha)|_{V_\alpha}) \to Hom(\mathscr{F}|_{V_\alpha}, T_\epsilon(j_{\alpha *}\mathscr{G}_\alpha)|_{V_\alpha})\big)\\ &\qquad \simeq \mathrm{Cone}\big(Hom(j_{\alpha !}\mathscr{F}_\alpha, T_{-\epsilon}(j_{\alpha *}\mathscr{G}_\alpha)) \to Hom(j_{\alpha !}\mathscr{F}_\alpha, T_\epsilon(j_{\alpha *}\mathscr{G}_\alpha))\big)\\ &\qquad \simeq \Gamma(T^{*,\infty}U_\alpha, \mu hom(j_{\alpha !}\mathscr{F}_\alpha, j_{\alpha *}\mathscr{G}_\alpha)) \simeq \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F}_\alpha, \mathscr{G}_\alpha)),\end{aligned}$$

where the last inequality again follows from non-characteristic deformation in Lemma 4.2.5. Hence the proof is completed. □

Remark 4.2.4. *In fact, in the proof we have shown that for $\mathscr{F} \in Sh_\Lambda(M), \mathscr{G} \in \mu Sh_\Lambda(M)$,*

$$Hom(\mathscr{F}, w_\Lambda(\mathscr{G})) \simeq Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \simeq \Gamma(T^{*,\infty}M, \mu hom(m_\Lambda(\mathscr{F}), \mathscr{G})).$$

Note that this is also a direct corollary of Proposition 4.1.2.

Theorem 4.2.10. *Let $\Lambda \subset T^{*,\infty}M$ be a closed subanalytic Legendrian. Then the left adjoint of microlocalization is isomorphic to*

$$m_\Lambda^l = \iota_\Lambda^* \circ w_\Lambda[-1] : \mu Sh_\Lambda(\Lambda) \to Sh_\Lambda(M).$$

Proof. Similar to the proof of Theorem 4.2.9, it suffices to show that for any $\mathscr{F} \in Sh_\Lambda(M)$, there is a canonical quasi-isomorphism

$$Hom(w_\Lambda(\mathscr{F})[-1], \mathscr{G}) \simeq \Gamma(T^{*,\infty}M, \mu hom(\mathscr{F}, m_\Lambda(\mathscr{G}))).$$

Again, we consider a good open covering $\mathscr{U}$ and a good family of refinements $\mathscr{V}^t$, and show that locally

$$Hom(w_\Lambda(\mathscr{F})_{V_\alpha}[-1], \mathscr{G}|_{V_\alpha}) \simeq \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F}|_{V_\alpha}, G|_{V_\alpha})).$$

Writing down the definition of $w_\Lambda(\mathscr{F})_{V_\alpha}[-1]$, we have

$$\begin{aligned} &Hom(w_\Lambda(\mathscr{F})[-1]|_{V_\alpha}, \mathscr{G}|_{V_\alpha}) \\ &\quad \simeq Hom\big(\mathrm{Cone}(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha} \to T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha})[-1], \mathscr{G}|_{V_\alpha}\big) \\ &\quad \simeq \mathrm{Cone}\big(Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}, \mathscr{G}|_{V_\alpha}) \to Hom(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha)|_{V_\alpha}, \mathscr{G}|_{V_\alpha})\big). \end{aligned}$$

Then by non-characteristic deformation in Lemma 4.2.4, 4.2.5 and Sato-Sabloff exact triangle Corollary 4.1.3 we can conclude that

$$\begin{aligned} &Hom(w_\Lambda(\mathscr{F})|_{V_\alpha}, \mathscr{G}|_{V_\alpha}) \\ &\quad \simeq \mathrm{Cone}\big(Hom(T_\epsilon(j_{\alpha!}\mathscr{F}_\alpha), j_{\alpha*}\mathscr{G}|_{V_\alpha}) \to Hom(T_{-\epsilon}(j_{\alpha!}\mathscr{F}_\alpha), j_{\alpha*}\mathscr{G}|_{V_\alpha})\big) \\ &\quad \simeq \Gamma(T^{*,\infty}U_\alpha, \mu hom(j_{\alpha!}\mathscr{F}_\alpha, j_{\alpha*}\mathscr{G}|_{U_\alpha})) \simeq \Gamma(T^{*,\infty}V_\alpha, \mu hom(\mathscr{F}_\alpha, \mathscr{G}_\alpha)), \end{aligned}$$

which completes the proof of the theorem. □

Remark 4.2.5. *In fact, in the proof we have shown that for $\mathscr{F} \in \mu Sh_\Lambda(M), \mathscr{G} \in Sh_\Lambda(M)$,*

$$Hom(w_\Lambda(\mathscr{F})[-1], \mathscr{G}) \simeq Hom(w_\Lambda(\mathscr{F})[-1], T_{-\epsilon}(\mathscr{G})) \simeq \Gamma(T^{*,\infty}M, \mu hom(\mathscr{F}, m_\Lambda(\mathscr{G}))),$$

which is again an application of Proposition 4.1.2.

Remark 4.2.6. *Moreover, we remark that when we apply the doubling construction to a single connected component $\Lambda_i \subset \Lambda$, using the same argument, one can still show that*

$$m^l_{\Lambda_i} = \iota^*_\Lambda \circ w_{\Lambda_i}[-1], \;\; m^r_{\Lambda_i} = \iota^!_\Lambda \circ w_{\Lambda_i}.$$

The reader may compare it with the discussion in Remark 4.1.3.

Later in Section 6.1, we will explain the geometric interpretation of the adjoint functors ι^*_Λ and $\iota^!_\Lambda$ in terms of wrappings, so that we will see the difference between m^l_Λ and m^r_Λ is indeed whether we choose positive or negative wrappings for the doubling.

4.3. Doubling and Quantization of Noncompact Legendrians

Unlike the case of compact Lagrangians, for noncompact Lagrangians, without control on the Weinstein tubular neighbourhood, one may not get a uniform Reeb push-off $T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})$ for some fixed time $\epsilon > 0$, and then fail to connect the small Reeb push-off with some large Reeb push-off. Even though we could still find a Legendrian isotopy between small push-offs and large push-offs, the norm of the candidate Hamiltonian vector field may be unbounded and we may no longer get a Hamiltonian vector field. For conical Legendrian cobordisms for example, one can easily see that the negative end is exactly where the radius of the Weinstein tubular neighbourhood looses control.

Therefore, we set up the general theory of the doubling construction and sheaf quantization for noncompact Legendrians in the course of the proof, which will be applied to many interesting noncompact Legendrians, including those coming from exact Lagrangian fillings of Legendrians, exact Lagrangian cobordisms between Lagrangians (in the sense of Arnol'd), and exact Lagrangian cobordisms between Legendrians (in the sense of symplectic field theory).

The goal of this section is to generalize the doubling construction to certain noncompact Legendrian submanifolds. Our main theorem is the following.

Theorem 4.3.1. *Let $\Lambda \subset J^1(M)$ be any smooth Legendrian submanifold. When there exists a complete adapted metric on $J^1(M)$ such that Λ admits a tubular neighbourhood of positive radius $\epsilon_0 > 0$, then for any $0 < \epsilon < \epsilon_0$, there exists a fully faithful functor*

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M \times \mathbb{R}).$$

We remark that the assumption that Λ admits a tubular neighbourhood of positive radius is crucial, and indeed this is why there is not in general a doubling functor for conical Legendrian cobordisms with a uniform $\epsilon > 0$ (and why we need extra data near the negative end of the cobordisms).

4.3.1. Doubling functor for noncompact Legendrians

Given any $\mathscr{F} \in \mu Sh_\Lambda(\Lambda)$, for a sufficiently small open subsets $V \subset U \subset M \times \mathbb{R}$, Lemma 4.2.1 ensures that there exists a sheaf $\mathscr{F}_U \in Sh_\Lambda(U)$. Let

$$\Lambda_{\pm T} = \{(x,t,s;\xi,\tau,\sigma) \,|\, \exists\, s > 0, (x,t;\xi,\tau) \in T_{\pm s}(\Lambda), \sigma = -H_s(x,t;\xi,\tau)\}$$

be the Legendrian movies of Λ under the positive/negative Reeb flows. Let

$$T_\pm : Sh_\Lambda(M \times \mathbb{R}) \to Sh_{\Lambda_{\pm T}}(M \times \mathbb{R} \times (0, +\infty))$$

be the equivalences of sheaf categories induced by the corresponding Hamiltonian isotopies $T_\pm$ by Guillermou-Kashiwara-Schapira [**88**].

Then for $\epsilon > 0$ a sufficiently small number depending on U, V and Λ, we define

$$\widetilde{w}_\Lambda(\mathscr{F})_{V \times (0,\epsilon)} = \mathrm{Cone}\big(\Psi^0_{T_-}(j_{U*}\mathscr{F}_U) \to \Psi^0_{T_+}(j_{U*}\mathscr{F}_U)\big)|_{V \times (0,\epsilon)}.$$

When $U \cap \pi(\Lambda) = \varnothing$, note that we can choose $\mathscr{F}_U = 0_U$ and $\epsilon = +\infty$. When $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ is compact, we can choose an open covering such that only finitely many open subsets intersect $\pi(\Lambda) \subset M \times \mathbb{R}$. Then there exists a uniform positive number

$$0 < \epsilon < \min_{\alpha \in I} \epsilon_\alpha = \min_{\alpha \in I,\, U_\alpha \cap \Lambda \neq \varnothing} \epsilon_\alpha.$$

In other words, we have

$$M \times \mathbb{R} \times \{\epsilon\} \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha)$$

Hence by restricting to $M \times \mathbb{R} \times \{\epsilon\}$, we get Theorem 4.0.7.

In general, there may not exist any $\epsilon > 0$ such that

$$M \times \mathbb{R} \times \{\epsilon\} \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha).$$

Hence it is difficult to construct the doubling functor for a uniform Reeb pushoff $\epsilon > 0$.

While there may not be a uniform $\epsilon > 0$ so that $M \times \mathbb{R} \times \{\epsilon\} \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha)$, there always exists some smooth function $\rho : M \times \mathbb{R} \to (0, +\infty)$ such that

$$\mathrm{Graph}(\rho) \subseteq \bigcup_{\alpha \in I} V_\alpha \times (0, \epsilon_\alpha).$$

Instead of restricting to the slice $M \times \mathbb{R} \times \{\epsilon\}$, we will restrict to the slice $\mathrm{Graph}(\rho)$.

Let the contact Hamiltonian be $\rho(x, t)$. The induced Hamiltonian vector field is $X_\rho = \rho(x, t)\partial_t - (\partial_x \rho(x, t) + \xi\, \partial_t \rho(x, t))\partial_\xi$, and the Hamiltonian diffeomorphism is

$$T_\rho(x, t; \xi) = (x, t + \rho(x, t); \xi - \partial_x \rho(x, t) - \xi\, \partial_t \rho(x, t)).$$

We will show that by restricting the doubled sheaf to the slice $\mathrm{Graph}(\rho)$, we get a sheaf with singular support on $T_{-\rho}(\Lambda) \cup T_\rho(\Lambda)$.

Proposition 4.3.2. *Let $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be a subsanalytic Legendrian subset. Then for any C^1-small smooth function $\rho : M \times \mathbb{R} \to (0, +\infty)$, there exists a fully faithful functor*

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_{-\rho}(\Lambda) \cup T_\rho(\Lambda)}(M \times \mathbb{R}).$$

Proof. Since $\rho : M \times \mathbb{R} \to (0, +\infty)$ is C^1-small, we know that $\Lambda_{-T} \cup \Lambda_T$ is non-characteristic with respect to $\mathrm{Graph}(\rho) \subseteq M \times \mathbb{R} \times (0, +\infty)$. Hence by restricting $\widetilde{w}_\Lambda(\mathscr{F})$ to $\mathrm{Graph}(\rho)$, the singular support is contained in $T_{-\rho}(\Lambda) \cup T_\rho(\Lambda)$. Full faithfulness follows from the fact that up to the reparametrization

$$\phi_\rho : M \times \mathbb{R} \times (0, +\infty) \to M \times \mathbb{R} \times (0, +\infty),\ (x, t, \epsilon\rho(x, t)) \mapsto (x, t, \epsilon),$$

$\Lambda_{-T} \cup \Lambda_T$ is the Legendrian movie of a Hamiltonian flow. □

As explained at the beginning, for a general noncompact Legendrian, this is in fact the best we can do. In the following section, we will explain how to strengthen the result with the presence of a tubular neighbourhood of positive radius.

Proof of Theorem 4.3.1. Consider the tubular neighbourhood $J^1_{<\epsilon_0}(\Lambda)$ of radius $\epsilon_0 > 0$ of the noncompact Legendrian $\Lambda \subseteq J^1(M)$. We claim that there exists a contact Hamiltonian isotopy φ^1_H, $0 \leq t \leq 1$, such that

$$\varphi^1_H(T_{\pm\rho}(\Lambda)) = T_{\pm\epsilon}(\Lambda).$$

Consider the standard coordinates in $J^1_{<\epsilon_0}(\Lambda) \subseteq J^1(\Lambda)$. There exists a Legendrian isotopy from $T_{\pm\rho}(\Lambda)$ to $T_{\pm\epsilon}(\Lambda)$

$$\Lambda_{\pm u} = \{(x, (1-u)d\rho(x,t), ut + (1-u)\rho(x,t)) | (x,t) \in \Lambda \times \mathbb{R}\}.$$

Fix $\epsilon < \epsilon' < \epsilon_0$. Define a Hamiltonian function $H : \Lambda \times \mathbb{R} \to \mathbb{R}$ such that

$$H|_{\bigcup_{u\in[0,1]} \Lambda_{\pm u}} = \pm(\epsilon - \rho(x,t)),\ H|_{J^1(\Lambda)\backslash J^1_{<\epsilon'}(\Lambda)} \equiv 0.$$

Then the corresponding contact Hamiltonian vector field

$$X_H = H\frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i}$$

is has proper support with respect to the projection $T^*\Lambda \times \mathbb{R} \to \Lambda$ and is tangent to the fibers/leaves $T^*_x\Lambda \times \mathbb{R}$. Therefore, the integral flow is supported in $J^1_{<\epsilon'}(\Lambda)$ and preserves the fibers/leaves. Hence the flow is well defined for all time $s \geq 0$.

The compact support condition then allows us to extend the Hamiltonian flow trivially from the tubular neighbourhood $J^1_{<\epsilon'}(\Lambda)$ to the ambient manifold $J^1(M)$. Using Guillermou-Kashiwara-Schapira Theorem 3.3.1, we can then conclude that there is an equivalence of sheaf categories

$$Sh_{T_{-\rho}(\Lambda)\cup T_{\rho}(\Lambda)}(M \times \mathbb{R}) \xrightarrow{\sim} Sh_{T_{-\epsilon}(\Lambda)\cup T_{\epsilon}(\Lambda)}(M \times \mathbb{R}).$$

Then the result follows immediately from Proposition 4.3.2. □

Remark 4.3.1. *Assume that there exists an open covering* $\{U_\alpha\}_{\alpha\in I}$ *and a refinement of* $\{V_\alpha\}_{\alpha\in I}$ *of* $M \times \mathbb{R}$ *with a collection of time intervals* $\epsilon_\alpha > 0$ *with a uniform lower bound. Then by restricting to small open subsets, one can easily show that the doubling functor constructed using Proposition 4.3.2 agrees with Theorem 4.0.7.*

There are definitely examples of noncompact Legendrians that no not admit a tubular neighbourhood of a positive radius with respect to any complete adapted metric. Conical Legendrian cobordisms are one class of such examples. This is why we need some extra data. We will deal with them in the next section.

4.3.2. Separation of double copies of the Legendrian

Given the doubling on $T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)$ for some $\epsilon > 0$, to separate the double copies of the Legendrian, we need to apply some Hamiltonian isotopy from $\Lambda \cup T_\epsilon(\Lambda)$ to $\Lambda \cup T_u(\Lambda)$ for any $u > 0$. We show that this can again be done once there exists some tubular neighbourhood of positive radius.

Proposition 4.3.3. *Let $\Lambda \subset J^1(M)$ be any smooth Legendrian submanifold. When there exists a complete adapted metric on $J^1(M)$ such that Λ admits a neighbourhood of positive radius $\epsilon/2 > 0$ disjoint from $\bigcup_{u \geq \epsilon} T_u(\Lambda)$, then for any $u > \epsilon$, there exists a fully faithful functor*

$$Sh_{\Lambda \cup T_\epsilon(\Lambda)}(M \times \mathbb{R}) \xrightarrow{\sim} Sh_{\Lambda \cup T_u(\Lambda)}(M \times \mathbb{R}).$$

Proof. Denote the Weinstein tubular neighbourhood of Λ by $U_\epsilon(\Lambda)$. By the assumption, we know that $U_{\epsilon/2}(\Lambda)$ and $\bigcup_{u \geq \epsilon} T_u(\Lambda)$ have a positive distance. Choose a cut-off function $H : J^1(M) \to \mathbb{R}$ such that

$$H|_{U_{\epsilon/2}(\Lambda)} \equiv 0,\ H|_{J^1(M) \setminus U_\epsilon(\Lambda)} \equiv 1,\ |dH| \leq 3/\epsilon < +\infty.$$

Since the metric on $J^1(M)$ is adapted, we know that the Hamiltonian vector field $|X_H| \leq (1 + 9/\epsilon^2)^{1/2} < +\infty$. Since in addition that the metric on $J^1(M)$ is complete, we know that the Hamiltonian flow φ_H^s exists for any $s \in \mathbb{R}$. Moreover, when $u \geq 0$,

$$\varphi_H^u\big(\Lambda \cup T_{2\epsilon/3}(\Lambda)\big) = \widetilde{L} \cup T_{\epsilon+u}(\widetilde{L}).$$

Therefore, by Guillermou-Kashiwara-Schapira Theorem 3.3.1, we can conclude that there is a canonical equivalence as in the statement of the proposition. ☐

Corollary 4.3.4. *Let $\Lambda \subset J^1(M)$ be any smooth Legendrian submanifold. When there exists a complete adapted metric on $J^1(M)$ such that Λ admits a tubular neighbourhood of positive radius $\epsilon > 0$ disjoint from $\bigcup_{s \geq \epsilon} T_s(\Lambda)$, then for any $s > \epsilon$, there exists a fully faithful functor*

$$w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{\Lambda \cup T_s(\Lambda)}(M \times \mathbb{R}).$$

In particular, when the Lagrangian projection $\pi_{Lag}(\Lambda) \subset T^*M$ admits a tubular neighbourhood of positive radius, we always get the above functor.

Using the above results, we can in fact prove the sheaf quantization theorem for certain noncompact embedded Lagrangians that admit a tubular neighbourhood of a positive radius for some adapted metric. In particular, we believe that we can recover the result of Jin-Treumann [**94**].

4.3.3. Sheaf quantization for closed Lagrangians

We recall the sheaf quantization result of closed exact Lagrangians of Guillermou [**84**]. Here is Guillermou's sheaf quantization theorem, from which he deduces results on nearby Lagrangians (see also Jin [**93**] who applies the result to sheaves over ring spectra and proves additional properties on nearby Lagrangians).

Theorem 4.3.5 (Guillermou [**84**])**.** *Let $L \subset T^*M$ be an embedded closed exact Lagrangian submanifold , and $\widetilde{L} \subset J^1(M) \cong S^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be its Legendrian lift. Then for any $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$, there exists a sheaf $\mathscr{F} \in Sh_{\widetilde{L}}(M \times \mathbb{R})$ with zero stalk at $M \times \{-\infty\}$ such that*

$$m_{\widetilde{L}}(\mathscr{F}) = \mathscr{L}$$

which determines a fully faithful functor $\Psi_L : \mu Sh_{\widetilde{L}}(\widetilde{L}) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R})_0$.

Proof. For any $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$, by Theorem 4.0.7 and Corollary 4.3.4, for any $u > 0$, there exists a sheaf $\mathscr{F} \in Sh_{\widetilde{L} \cup T_u(\widetilde{L})}(M \times \mathbb{R})$ such that

$$m_{\widetilde{L}}(\mathscr{F}_{dbl}) = \mathscr{L}.$$

Since $L \subset T^*M$ is closed, the front projection $\pi_{\text{front}}(\widetilde{L}) \subset M \times \mathbb{R}$ is compact. In particular, there exists $C > 0$ such that $\pi_{\text{front}}(\widetilde{L}) \subset M \times (-\infty, C-1)$, and there exists $C' > 0$ such that $\pi_{\text{front}}(T_{C'}(\widetilde{L})) \subset M \times (C+1, +\infty)$. Let $j_C : M \times (-\infty, C) \hookrightarrow M \times \mathbb{R}$ and

$$\varphi_C : M \times \mathbb{R} \xrightarrow{\sim} M \times (-\infty, C)$$

a diffeomorphism such that $\varphi_C|_{M \times (-\infty, C)} = \mathrm{id}_{M \times (-\infty, C)}$. Then

$$\mathscr{F} = \varphi_C^{-1} j_C^{-1} \mathscr{F}_{dbl} \in Sh_{\widetilde{L}}(M \times \mathbb{R})$$

is the sheaf that satisfies the property.

For the full faithfulness property, writing $\mathscr{F}_{dbl} = w_{\widetilde{L}}(\mathscr{L})$, we can compute

$$Hom(\mathscr{F}, \mathscr{F}) = Hom(j_C^{-1} w_{\widetilde{L}}(\mathscr{L}), j_C^{-1} w_{\widetilde{L}}(\mathscr{L})) = Hom(w_{\widetilde{L}}(\mathscr{L}), j_{C*} j_C^{-1} w_{\widetilde{L}}(\mathscr{L})).$$

By adjunction property in Theorem 4.2.10, we know that

$$Hom(w_{\widetilde{L}}(\mathscr{L}), j_{C*} j_C^{-1} w_{\widetilde{L}}(\mathscr{L})) = \Gamma(\widetilde{L}, \mu hom(\mathscr{L}, m_{\widetilde{L}}(w_{\widetilde{L}}(\mathscr{L})))) = \Gamma(\widetilde{L}, \mu hom(\mathscr{L}, \mathscr{L}).$$

Finally, we show that the essential image is $Sh_{\widetilde{L}}(M \times \mathbb{R})_0$. We simply need to notice that for $\mathscr{F} \in Sh_{\widetilde{L}}(M \times \mathbb{R})_0$, $\mathscr{F}_{dbl} = w_{\widetilde{L}}(\mathscr{L}) = \mathrm{Cone}(\mathscr{F} \to T_u(\mathscr{F}))$ and hence $\Psi(\mathscr{F}) = \mathscr{F}$. □

In the proof, we have in fact seen that the sheaf quantization functor is the left adjoint to the microlocalization functor, namely $m_{\widetilde{L}}(\mathscr{F}) = \mathscr{L}$ if $\mathscr{F}$ is the sheaf quantization of $\mathscr{L}$.

4.3.4. Sheaf quantization of Lagrangian fillings

We recall the sheaf quantization result of exact Lagrangian fillings of Legendrian submanifolds of Jin-Treumann [**94**].

Given an exact Lagrangian filling $L \subset T^*M$ of a Legendrian submanifold $\Lambda \subset T^{*,\infty}M$, Theorem 4.3.7 below defines a fully faithful functor $Loc(L) \hookrightarrow Sh_\Lambda(M)$, which realizes exact Lagrangian fillings, endowed with local systems, as objects in the constructible sheaf category associated to Λ. First, a technical lemma:

Lemma 4.3.6. *Let $L_t \subset T^*M$, $t \in \mathbb{D}^k$, be a family of exact Lagrangian fillings of a family of Legendrian submanifold $\Lambda_t \subset T^{*,\infty}M$, $t \in \mathbb{D}^k$. Then L_t is Hamiltonian isotopic to a family of exact Lagrangians L_t' whose primitive $f_{L_t'}$ is proper and bounded from below (where $\lambda_{st}|_{L_t'} = df_{L_t'}$).*

Proof. The proof is close to the non-parametric version of the lemma in [**95**, Section 3.6]. Since L_t, $t \in \mathbb{D}^k$, is a family of exact Lagrangian fillings of Λ_t, $t \in \mathbb{D}^k$, for any $t \in \mathbb{D}^k$ there is some $r_t \gg 0$ sufficiently large such that $L_t \cap \{(x, \xi) \in T^*M | |\xi| > r_t\} = \Lambda \times (r_t, +\infty)$. Since $\mathbb{D}^k$ is compact one can find $r_0 \gg 0$ such that for any $t \in \mathbb{D}^k$, $L_t \cap \{(x, \xi) \in T^*M | |\xi| > r_0\} = \Lambda \times (r_0, +\infty)$.

Let $\beta : [0, +\infty) \to \mathbb{R}$ be a function such that $\beta(r) = 0$ when r is sufficiently small and $\beta(r) = -1/r$ when $r > r_0$ is sufficiently large, and the Hamiltonian $H(r) = \beta(r)$. Then consider $L_t' = \varphi_H^\epsilon(L_t)$. One can check that when $r > r_0$, $df_{L_t'} = \epsilon/r$ and thus are proper and bounded from below. □

Theorem 4.3.7 (Jin-Treumann [**95**])**.** *Let $L \subset T^*M$ be an exact Lagrangian filling of a Legendrian submanifold $\Lambda \subset T^{*,\infty}M$ whose primitive f_L is proper and bounded from below (where $\lambda_{st}|_L = df_L$). Let $\widetilde{L} \subset J^1(M) \cong T^{*,\infty}_{\tau<0}(M \times \mathbb{R})$ be the Legendrian lift of L. Then for any $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$, there exists a sheaf $F \in Sh_{\widetilde{L}}(M \times \mathbb{R})$ with zero stalk at $M \times \{-\infty\}$ such that*

$$m_{\widetilde{L}}(\mathscr{F}) = \mathscr{L}$$

which determines a fully faithful functor $\Psi_L : \mu Sh_{\widetilde{L}}(\widetilde{L}) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R})_0$.

Following the discussion in the previous section, the main technical preparation we need is the tubular neighbourhood theorem for exact Lagrangian fillings, which is Lemma 2.2.3.

Proof of Theorem 4.3.7. For any $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$, by Corollary 4.3.4, for any $u > 0$, there exists a sheaf $\mathscr{F} \in Sh_{\widetilde{L} \cup T_u(\widetilde{L})}(M \times \mathbb{R})$ such that

$$m_{\widetilde{L}}(\mathscr{F}_{dbl}) = \mathscr{L}.$$

Choose $C' \gg C \gg 0$ such that (i) $T^{*,\infty}(M \times (-\infty, C)) \cap T_{C'}(\widetilde{L}) = \emptyset$, and (ii) there exists a diffeomorphism

$$\varphi_C : M \times (-\infty, +\infty) \xrightarrow{\sim} M \times (-\infty, C)$$

such that $\varphi_C^* : T^{*,\infty}(M \times (-\infty, C)) \xrightarrow{\sim} T^{*,\infty}(M \times \mathbb{R})$ sends $\widetilde{L} \cap T^{*,\infty}_{\tau<0}(M \times (-\infty, C))$ to $\widetilde{L}$. Write $j_C : M \times (-\infty, C) \hookrightarrow M \times \mathbb{R}$ for the inclusion map. Then

$$\mathscr{F} = \varphi_C^{-1} j_C^{-1} \widetilde{\mathscr{F}_{dbl}} \in Sh_{\widetilde{L}}(M \times \mathbb{R})$$

is the sheaf that satisfies the property. The full faithfulness and essential surjectivity are proved in the same way. □

4.3.5. Sheaf quantization of Arnol'd Lagrangian cobordisms

Our main result in this section is the existence of sheaf quantization for exact Lagrangian cobordisms between Lagrangian submanifolds in the sense of Arnol'd [**10**].

For the result presented here, the author has benefitted a lot from the discussion with Asano and Ike in our joint work in progress. In fact, it was them who attracted the author's attention to the sheaf quantization problem of Lagrangian cobordisms between Lagrangians.

Recall that An exact Lagrangian cobordism $V \subset T^*(M \times \mathbb{R})$ between $L_1, \dots, L_p$ and $K_1, \dots, K_q$ is an exact Lagrangian submanifold such that

$$V \cap T^*(M \times (-\infty, 0)) = \bigcup_{i=1}^{p} L_i \times (-\infty, 0) \times \{i\},$$
$$V \cap T^*(M \times (1, +\infty)) = \bigcup_{j=1}^{q} K_j \times (0, +\infty) \times \{j\}.$$

Our goal in this section is to prove Theorem 4.0.8 on sheaf quantization of Arnol'd Lagrangian cobordisms.

As in the previous cases, the main technical preparation we need is the tubular neighbourhood theorem for exact Lagrangian cobordisms between Lagrangians, which is Lemma 2.2.2.

Then one may try to separate the images of the projections of $\widetilde{V} \cup T_u(\widetilde{V})$ on $M \times \mathbb{R} \times \mathbb{R}$. This is in general not possible when the Lagrangians on both ends have multiple components. However, we can always separate the images of their projections in a horizontally bounded region $T^*(M \times \mathbb{R} \times [-R, R])$. Thus, on such a bounded region, we do get a sheaf quantization.

Lemma 4.3.8. *Let $V \subset T^*(M \times \mathbb{R})$ be an exact Lagrangian cobordism between $L_1, \dots, L_p$ and $K_1, \dots, K_q$. Then for any $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$, when $R \gg 0$, there exists* *a sheaf* $\mathscr{F} \in Sh_{\widetilde{V}}(M \times \mathbb{R} \times [-R, R])$ *with zero stalk at* $M \times \{-\infty\} \times [-R, R]$ *such that*

$$m_{\widetilde{V}}(\mathscr{F}_{[-R,R]}) = \mathscr{L}_{V \cap T^*(M \times [-R,R])}.$$

Proof. For any $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$, by Corollary 4.3.4, for any $u > 0$, there exists a sheaf quantization

$$\mathscr{F}_{dbl,[-R,R]} \in Sh_{\widetilde{V} \cup T_u(\widetilde{V})}(M \times \mathbb{R} \times \mathbb{R}).$$

Since $\widetilde{V} \cap T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times [-R, R])$ is compact, so is its image under the projection onto $M \times \mathbb{R} \times [-R, R]$. Therefore, there exists a sufficiently large $C > 0$ and $C' > C$ such that

$$\pi_{M \times \mathbb{R} \times [-R,R]}(\widetilde{V} \cap T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times [-R, R])) \subset M \times (-\infty, C - 1) \times [-R, R],$$

$$\pi_{M \times \mathbb{R} \times [-R,R]}(T_u(\widetilde{V}) \cap T^*(M \times \mathbb{R} \times [-R, R])) \subset M \times (C + 1, +\infty) \times [-R, R].$$

Write $j_C : M \times (-\infty, C) \times [-R, R] \hookrightarrow M \times \mathbb{R} \times [-R, R]$ the inclusion map. Consider a diffeomorphism

$$\varphi_C : M \times (-\infty, C) \times [-R, R] \to M \times \mathbb{R} \times [-R, R]$$

such that $\varphi_C = \mathrm{id}$ on $M \times (-\infty, c-1) \times [-R, R]$. Then $\mathscr{F}_{[-R,R]} = \varphi_C^{-1} j_C^{-1} \mathscr{F}_{dbl,[-R,R]}$.

□

Then we claim that the sheaf $\mathscr{F}_{V \cap T^*(M \times [-R,R])}$ can be uniquely extended to $M \times \mathbb{R} \times \mathbb{R}$ when R is sufficiently large, by proving that the two ends of Λ_V are simply conical Lagrangian movies of $\bigcup_{i=1}^p \widetilde{L}_i$ and $\bigcup_{j=1}^q \widetilde{K}_j$.

For any $\Lambda \subset T^{*,\infty}(M \times \mathbb{R})$, we define

$$\Lambda_{\pi_i} = \{(x, t + ir, r; \xi, \tau, -i\tau) | (x, t; \xi, \tau) \in \Lambda, \, r \in \mathbb{R}\}.$$

Lemma 4.3.9. *Let $L_1, \dots, L_p$ be closed exact Lagrangians in T^*M. Suppose that for any $1 \leq i < j \leq p$, there exists $c_{ij} \in \mathbb{R}$,*

$$\pi_{front}(\widetilde{L}_i) \subset M \times (c_{ij} + 1, +\infty), \; \pi_{front}(\widetilde{L}_j) \subset M \times (-\infty, c_{ij} - 1).$$

Then the conical Lagrangian $\bigcup_{i=1}^p \widetilde{L}_{i,\pi_i} \cap T^*(M \times \mathbb{R} \times \mathbb{R}_{\leq 0})$ is the conical Lagrangian movie of $\bigcup_{i=1}^p \widetilde{L}_i$ under a homogeneous Hamiltonian isotopy.

Proof. We can in fact define a diffeotopy $\phi_r : M \times \mathbb{R} \to M \times \mathbb{R}$ such that the Hamiltonian diffeomorphism $\phi_r^* : T^*(M \times \mathbb{R}) \to T^*(M \times \mathbb{R})$ satisfies the condition. Define ϕ_r such that

$$\phi_r(x, t) = (x, t + ir), \; c_{i-1,i} + 1 < t < c_{i,i+1} - 1.$$

Then by the assumption, clearly $\bigcup_{i=1}^p \widetilde{L}_{i,\pi_i} \cap T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times \mathbb{R}_{\leq 0})$ is the conical Lagrangian movie of $\bigcup_{i=1}^p \widetilde{L}_i$ under ϕ_r^* $(r \in (-\infty, 0])$. □

With these preparations, we are now able to finish the proof of our main theorem.

Proof of Theorem 4.0.8. By Lemma 4.3.8, for any $R > 0$, there exists a sheaf quantization

$$\mathscr{F}_{[-R,R]} \in Sh_{\widetilde{V}}(M \times \mathbb{R} \times [-R, R]).$$

We know that $V \cap T^*(M \times (-\infty, 0]) = \bigcup_{i=1}^{p} L_i \times (-\infty, 0] \times \{i\}$, and their Legendrian lifts are $\bigcup_{i=1}^{p} \widetilde{L}_{i,\pi_i} \cap T^{*,\infty}(M \times \mathbb{R} \times \mathbb{R}_{\leq 0})$. For each $R > 0$,

$$\bigcup_{i=1}^{p} \widetilde{L}_{i,\pi_i} \cap T^{*,\infty}(M \times \mathbb{R} \times \{-R\}) = \bigcup_{i=1}^{p} T_{-iR}(\widetilde{L}_i).$$

Since $\widetilde{L}_i$ is compact, we know that when $R > 0$ is sufficiently large, for any $1 \leq i < j \leq p$, there exists $c_{ij} \in \mathbb{R}$,

$$\pi_{\mathrm{front}}(T_{-iR}(\widetilde{L}_i)) \subset M \times (c_{ij} + 1, +\infty),\ \pi_{\mathrm{front}}(T_{-jR}(\widetilde{L}_j)) \subset M \times (-\infty, c_{ij} - 1).$$

Fix this $R > 0$. Then we can apply Lemma 4.3.9, and get an equivalence of categories

$$Sh_{\bigcup_{i=1}^{p} T_{-iR}(\widetilde{L}_i)}(M \times \mathbb{R}) \xrightarrow{\sim} Sh_{\bigcup_{i=1}^{p} \widetilde{L}_{i,\pi_i}}(M \times \mathbb{R} \times (-\infty, -R]).$$

Given the sheaf quantization $\mathscr{F}_{V \cap T^*(M \times [-R,R])} \in Sh_{\widetilde{V}}(M \times \mathbb{R} \times [-R-1, R+1])$, by Theorem 3.3.1 extend it to $\mathscr{F}_{V \cap T^*(M \times (-\infty,R])} \in Sh_{\Lambda_V}(M \times \mathbb{R} \times (-\infty, R+1])$. For the other end, we apply the same argument and thus we get $\mathscr{F}_V \in Sh_{\widetilde{V}}(M \times \mathbb{R} \times \mathbb{R})$ which completes the proof. Essential surjectivity is proved in the same way as in the compact case. $\square$

4.4. Doubling and Quantization of Lagrangian Cobordisms

Given an exact Lagrangian cobordism L between Legendrian from Λ_- to $\Lambda_+ \subset J^1(M)$, following Section 2.1.2, we can identify it with a conical Legendrian cobordism $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$.

For conical Lagrangian cobordisms, we do not have a sheaf quantization functor in general, due to lack of control on the size of the tubular neighbourhood of the Legendrian at the negative end. In fact, we need the prescribed data near the negative end Λ_-. The main theorem is thus a conditional sheaf quantization theorem Theorem 4.0.9.

Let $T_t : J^1(M \times \mathbb{R}_{>0}) \to J^1(M \times \mathbb{R}_{>0})$ is the Reeb flow. Similar to the sheaf quantization theorem of Guillermou and Jin-Treumann [**84**, **94**], we first construct a doubling functor using a small Reeb push-off in a Weinstein tubular neighbourhood

$$Loc(L) \hookrightarrow Sh_{T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})}(M \times \mathbb{R})$$

and then push one of the copies to infinity through a Legendrian isotopy and get

$$Loc(L) \to Sh_{\widetilde{L}}(M \times \mathbb{R}).$$

For conical Legendrian cobordisms, one can easily see that at the negative end the radius of the Weinstein tubular neighbourhood $T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})$ loses control, so one may not get a uniform Reeb push-off for some fixed time $\epsilon > 0$, and then fail to connect the small Reeb push-off with some large Reeb push-off. This is why the doubling construction does not provide a sheaf quantization without extra conditions.

Therefore, our strategy is to construct the doubling separately near the negative end and away from the negative end. Near the negative end, using the sheaf singularly supported on a single copy of the Legendrian, one can immediately define a sheaf supported on a double copy of the Legendrian by hand, while away from the negative

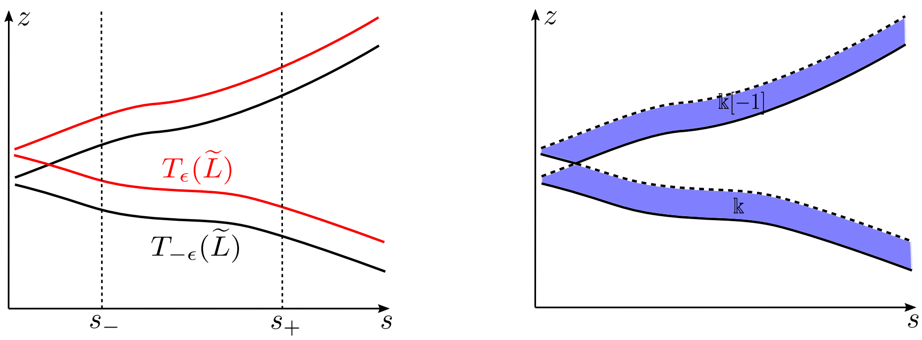

Figure 4.2. The conditional doubling construction which define sheaves in $T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})$ for a conical Legendrian cobordism.

end, we have good control on the radius of the Weinstein tubular neighbourhood theorem and the original doubling construction works. Then we show that one can push off one of the copies to infinity.

4.4.1. Doubling for conical Legendrian cobordisms

In this section, we will mainly follow the strategy in the previous section to construct the doubling functor. Recall that $Sh_\Lambda(M \times \mathbb{R})_0$ consists of sheaves with acyclic stalks at $M \times \{-\infty\}$.

Theorem 4.4.1. *Let $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$. Then there exists a fully faithful conditional doubling functor*

$$w_{\widetilde{L}} : Sh_{\Lambda_-}(M \times \mathbb{R})_0 \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_{\widetilde{L}}(\widetilde{L}) \hookrightarrow Sh_{T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})}(M \times \mathbb{R} \times \mathbb{R}_{>0})_0.$$

4.4.1.1. Doubling near the negative end. We turn to the doubling construction on conical Legendrian cobordisms. We construct the doubling functor near the negative end of the Legendrian cobordism, using the information $Sh_{\Lambda_-}(M)$.

More precisely, assume that $\widetilde{L} \cap J^1(M \times (0, s_-))$ is conical. Consider $c(\Lambda_-)$ the length of the shortest Reeb chord on Λ_-. Then for $\epsilon > 0$, pick $s_0 > 0$ such that $\epsilon < s_0 c(\Lambda_-) < s_- c(\Lambda_-)$. We show the existence of the doubling functor on the conical end $M \times \mathbb{R} \times (0, s_0)$.

Lemma 4.4.2. *Let $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$ that is conical on $J^1(M \times (0, s_-))$ for $\epsilon < s_0 c(\Lambda_-) < s_- c(\Lambda_-)$. Then there exists a doubling functor*

$$w_{\widetilde{L}}^{(0,s_-)} : Sh_{\Lambda_-}(M \times \mathbb{R}) \to Sh_{T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})}(M \times \mathbb{R} \times (0, s_-))$$

such that $i_{s_0}^{-1} w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F}) = w_{\Lambda_-} \circ m_{\Lambda_-}(\mathscr{F})$.

Proof. Define the projection $\pi : M \times \mathbb{R} \times (0, t_0) \to M \times \mathbb{R}$ by $\pi(x, z, t) = (x, z/t)$, and let

$$w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F}) = \mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{F}) \to T_\epsilon(\pi^{-1}\mathscr{F})).$$

Then it is clear that when $SS^\infty(\mathscr{F}) \subset \Lambda_-$, $SS^\infty\big(w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F})\big) \subset (T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})) \cap T^{*,\infty}(M \times \mathbb{R} \times (0, s_-))$. Finally, the identity

$$i_{s_0}^{-1} w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F}) = w_{\Lambda_-} \circ m_{\Lambda_-}(\mathscr{F})$$

follows from the exact triangle of functors $T_{-\epsilon} \to T_\epsilon \to w_\Lambda \circ m_\Lambda$. $\square$

Remark 4.4.1. *We explain why it is necessary to assume the existence of a sheaf* $\mathscr{F} \in Sh_{\Lambda_-}(M \times \mathbb{R})_0$ *in the construction. In fact, this is the obstruction to go from Proposition 4.3.2 to Theorem 4.3.1. Lack of a Weinstein tubular neighbourhood of* $\widetilde{L}$ *with positive radius at the negative end makes it difficult to connect* $T_{-\rho}(\widetilde{L}) \cup T_\rho(\widetilde{L})$ *and* $T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})$*, following the notation in Proposition 4.3.2.*

For example, consider the trivial 1-dimensional conical Legendrian cobordism, by the reparametrization identifying $J^1(\mathrm{pt} \times (0, +\infty))$ *with* $J^1(\mathrm{pt} \times \mathbb{R})$ *as in Figure 2.2 (middle), one may assume that*

$$\widetilde{L} = \{(s, \pm e^s, \pm e^s) | s \in \mathbb{R}\}.$$

One can easily check that $\inf_{x,x' \in \widetilde{L}} d(x, x') = 0$ *under the standard complete adapted metric. Then lack of a Weinstein tubular neighbourhood of positive radius with respect to the standard metric will prevent the Legendrian isotopy between* $T_{-\rho}(\widetilde{L}) \cup T_\rho(\widetilde{L})$ *and* $T_{-\epsilon}(\widetilde{L}) \cup T_\epsilon(\widetilde{L})$ *from being extended to a Hamiltonian isotopy.*

Next, we prove full faithfulness of the functor near negative end. Recall that $Sh_\Lambda(M \times \mathbb{R})_0$ is the subcategory of sheaves with acyclic stalks at $M \times \{-\infty\}$. This will be used frequently in the statements.

Lemma 4.4.3. *Let* $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ *be a conical Legendrian cobordism from* $\Lambda_- \subset J^1(M)$ *to* $\Lambda_+ \subset J^1(M)$ *that is conical on* $J^1(M \times (0, s_-))$*. Then for* $\mathscr{F}, \mathscr{G} \in Sh_{\Lambda_-}(M \times \mathbb{R})_0$*,*

$$Hom(T_\epsilon(\pi^{-1}\mathscr{F}), T_{-\epsilon}(\pi^{-1}\mathscr{G})) \simeq 0.$$

Proof. Since $SS^\infty(T_{-\epsilon}(\pi^{-1}\mathscr{F})) \cap SS^\infty(T_\epsilon(\pi^{-1}\mathscr{G})) = \varnothing$, by Proposition 3.1.7, we know that

$$SS^\infty\big(\mathscr{H}om(T_\epsilon(\pi^{-1}\mathscr{F}), T_{-\epsilon}(\pi^{-1}\mathscr{G}))\big) \subset -T_\epsilon(\Lambda \times (0, s_-)) + T_{-\epsilon}(\Lambda_- \times (0, s_-)).$$

Therefore, a point $(x, z, s; y, 0, \sigma) \in SS^\infty\big(\mathscr{H}om(T_\epsilon(\pi^{-1}\mathscr{F}), T_{-\epsilon}(\pi^{-1}\mathscr{F}))\big)$ means there are points $(x, t; \xi, 1)$ and $(x, t'; \xi, 1) \in \Lambda_-$ such that $(x, z, s) = (x, st + \epsilon, s) = (x, st' - \epsilon, s)$ and

$$(x, z, s; y, 0, \sigma) = -(x, st + \epsilon, s; s\xi, 1, t) + (x, st' - \epsilon, s; s\xi, 1, t').$$

In other words, $s(t' - t) = 2\epsilon$ and thus $\sigma = t' - t > 0$. By microlocal Morse lemma 3.1.3, we know that

$$Hom(T_\epsilon(\pi^{-1}\mathscr{F}), T_{-\epsilon}(\pi^{-1}\mathscr{G})) = Hom(T_\epsilon(\pi^{-1}\mathscr{F})|_{M\times\mathbb{R}\times(0,s'_-)}, T_{-\epsilon}(\pi^{-1}\mathscr{G})|_{M\times\mathbb{R}\times(0,s'_-)}).$$

Write $\pi^{-1}\mathscr{F}|_{M\times\mathbb{R}(0,s'_-)} = \pi^{-1}\mathscr{F}|_{(0,s'_-)}$. For $s'_- < s_-$ sufficiently small, since $\pi^{-1}\mathscr{F}|_{(0,s'_-)}$ has acyclic stalk at $-\infty$, $T_{-\epsilon}(\pi^{-1}\mathscr{G})|_{(0,s'_-)}$ is a local system on $\mathrm{supp}(T_\epsilon(\pi^{-1}\mathscr{G})|_{(0,s'_-)})$. Then we know that

$$SS^\infty(\mathscr{H}om(T_\epsilon(\pi^{-1}\mathscr{F})|_{(0,s'_-)}, T_{-\epsilon}(\pi^{-1}\mathscr{G})|_{(0,s'_-)})) \subset -SS^\infty(T_\epsilon(\pi^{-1}\mathscr{F})|_{(0,s'_-)})$$

consisting of points $(x, t, s; y, \tau, \sigma)$ such that $\tau < 0$. Therefore, by microlocal Morse lemma again,

$$Hom(T_\epsilon(\pi^{-1}\mathscr{F})|_{(0,s'_-)}, T_{-\epsilon}(\pi^{-1}\mathscr{G})|_{(0,s'_-)}) = 0.$$

This proves our claim. □

Lemma 4.4.4. *Let $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$ that is conical on $J^1(M \times (0, s_-))$. Then*

$$Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), T_\epsilon(\pi^{-1}\mathscr{G})) \simeq Hom(\mathscr{F}, \mathscr{G}).$$

Proof. We know that $(x, z, s; y, 0, \sigma) \in SS^\infty\big(\mathscr{H}om(T_{-\epsilon}(\pi^{-1}\mathscr{F}), T_\epsilon(\pi^{-1}\mathscr{F}))\big)$ if there are points $(x, t; \xi, 1)$ and $(x, t'; \xi, 1) \in \Lambda_-$ such that $(x, z, s) = (x, st - \epsilon, s) = (x, st' + \epsilon, s)$ and

$$(x, z, s; y, 0, \sigma) = (x, st - \epsilon, s; s\xi, 1, t) + (x, st' + \epsilon, s; s\xi, 1, t').$$

In other words, $s(t' - t) = -2\epsilon$ and thus $\sigma = t' - t < 0$. By microlocal Morse lemma 3.1.3, we know that

$$Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), T_\epsilon(\pi^{-1}\mathscr{G})) = Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F})|_{M\times\mathbb{R}\times(s'_-, s_-)}, T_\epsilon(\pi^{-1}\mathscr{G})|_{M\times\mathbb{R}\times(s'_-, s_-)}).$$

Then for $s'_- < s_-$ sufficiently close, the fronts $T_{-\epsilon}(\Lambda_- \times (s'_-, s_-))$ and $T_\epsilon(\Lambda_- \times (s'_-, s_-))$ are Legendrian movies of a Legendrian isotopy. Hence by Theorem 3.3.1 and Proposition 4.1.2

$$\begin{aligned} &Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F})|_{(s'_-, s_-)}, T_\epsilon(\pi^{-1}\mathscr{G})|_{(s'_-, s_-)}) \\ &\quad = Hom(T_{-\epsilon/s_-}\mathscr{F}, T_{\epsilon/s_-}\mathscr{G}) = Hom(\mathscr{F}, \mathscr{G}). \end{aligned}$$

This proves our claim. □

Remark 4.4.2. *Alternatively, one can use the fact that* $T_u\mathscr{H} = \Bbbk_{[0,+\infty)} \star \mathscr{H}$ *to show that*

$$
\begin{aligned}
Hom(\pi^{-1}\mathscr{F}, \pi^{-1}\mathscr{G}) &= \varinjlim_{\epsilon>0} Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), T_{\epsilon}(\pi^{-1}\mathscr{G})) \\
&= Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), T_{\epsilon}(\pi^{-1}\mathscr{G})).
\end{aligned}
$$

Then the result follows from Theorem 3.3.1.

Proposition 4.4.5. *Let* $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ *be a conical Legendrian cobordism from* $\Lambda_- \subset J^1(M)$ *to* $\Lambda_+ \subset J^1(M)$ *that is conical on* $J^1(M \times (0, s_-))$*. Then the doubling functor is fully faithful*

$$
w_{\widetilde{L}}^{(0,s_-)} : Sh_{\Lambda_-}(M \times \mathbb{R})_0 \hookrightarrow Sh_{T_{-\epsilon}(\widetilde{L}) \cup T_{\epsilon}(\widetilde{L})}(M \times \mathbb{R} \times (0, s_-))_0.
$$

Proof. It suffices to show that for any $\mathscr{F}, \mathscr{G} \in Sh_{\Lambda_-}(M \times \mathbb{R})_0$ with acyclic stalks at $-\infty$,

$$
Hom(\mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{F}) \to T_{\epsilon}(\pi^{-1}\mathscr{F})), \mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{G}) \to T_{\epsilon}(\pi^{-1}\mathscr{G}))) \simeq Hom(\mathscr{F}, \mathscr{G}).
$$

First, we prove that

$$
Hom(T_{\epsilon}(\pi^{-1}\mathscr{F}), \mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{G}) \to T_{\epsilon}(\pi^{-1}\mathscr{G}))) = Hom(\mathscr{F}, \mathscr{G}).
$$

By Lemma 4.4.3 we know that for $s'_- < s_-$ sufficiently small,

$$
Hom(T_{\epsilon}(\pi^{-1}\mathscr{F}), T_{-\epsilon}(\pi^{-1}\mathscr{G})) = Hom(T_{\epsilon}(\pi^{-1}\mathscr{F})|_{(0,s'_-)}, T_{-\epsilon}(\pi^{-1}\mathscr{G})|_{(0,s'_-)}) = 0.
$$

On the other hand, we know by Theorem 3.3.1 that

$$Hom(T_\epsilon(\pi^{-1}\mathscr{F}), T_\epsilon(\pi^{-1}\mathscr{G})) = Hom(T_\epsilon(\pi^{-1}\mathscr{F})|_{(0,s'_-)}, T_\epsilon(\pi^{-1}\mathscr{G})|_{(0,s'_-)}) = Hom(\mathscr{F}, \mathscr{G}).$$

Since the natural continuation map given by the Reeb flow T_t factors through the restrictions to $M \times \mathbb{R} \times (0, s'_-)$. This implies that

$$Hom(T_\epsilon(\pi^{-1}\mathscr{F}), \mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{G}) \to T_\epsilon(\pi^{-1}\mathscr{G}))) = Hom(\mathscr{F}, \mathscr{G}).$$

Next, we prove that

$$Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), \mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{G}) \to T_\epsilon(\pi^{-1}\mathscr{G}))) = 0.$$

By Lemma 4.4.4, we know that for $s'_- < s_-$ sufficiently close to each other,

$$\begin{aligned} Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), T_\epsilon(\pi^{-1}\mathscr{G})) &= Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F})|_{(s'_-,s_-)}, T_\epsilon(\pi^{-1}\mathscr{G})|_{(s'_-,s_-)}) \\ &= Hom(\mathscr{F}, \mathscr{G}). \end{aligned}$$

On the other hand, we know by Theorem 3.3.1 that

$$\begin{aligned} Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), T_{-\epsilon}(\pi^{-1}\mathscr{G})) &= Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F})|_{(s'_-,s_-)}, T_{-\epsilon}(\pi^{-1}\mathscr{G})|_{(s'_-,s_-)}) \\ &= Hom(\mathscr{F}, \mathscr{G}) \end{aligned}$$

Since the natural continuation map given by the Reeb flow T_t factors through the restrictions to $M \times \mathbb{R} \times (s'_-, s_-)$. This implies that

$$Hom(T_{-\epsilon}(\pi^{-1}\mathscr{F}), \mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{G}) \to T_{\epsilon}(\pi^{-1}\mathscr{G}))) = 0.$$

Combining the two equalities, we can therefore conclude that the doubling functor is fully faithful near the negative end. $\square$

Proposition 4.4.6. *Let $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$ that is conical on $J^1(M \times (0, s_-))$. Then $\iota_{\widetilde{L}}^* \circ w_{\widetilde{L}}^{(0,s_-)}$ is the left adjoint of the restriction*

$$i_-^{-1} : Sh_{\widetilde{L}}(M \times \mathbb{R} \times (0, s_-))_0 \hookrightarrow Sh_{\Lambda_-}(M \times \mathbb{R})_0.$$

Proof. It suffices to show that for any $\mathscr{F}, \mathscr{G} \in Sh_{\Lambda_-}(M \times \mathbb{R})_0$ with acyclic stalks at $-\infty$

$$Hom(\mathrm{Cone}(T_{-\epsilon}(\pi^{-1}\mathscr{F}) \to T_{\epsilon}(\pi^{-1}\mathscr{F})), \pi^{-1}\mathscr{G}) \simeq Hom(\mathscr{F}, \mathscr{G}).$$

This follows from the Lemma 4.4.3 and 4.4.4. $\square$

4.4.1.2. Doubling away from the negative end. We construct the doubling functor away from the negative end using the doubling construction with some uniform Reeb pushoff.

In fact, by Lemma 2.2.5, there exists a complete adapted metric on $J^1(M \times (s_0, +\infty))$ such that $\widetilde{L} \cap J^1(M \times (s_0, +\infty))$ admits a tubular neighbourhood of positive radius. Therefore, we get a fully faithful doubling functor

$$w_{\widetilde{L}}^{(s_0,+\infty)} : \mu Sh_{\widetilde{L}}(\widetilde{L}) \to Sh_{T_{-\epsilon}(\widetilde{L}) \cup T_{\epsilon}(\widetilde{L})}(M \times \mathbb{R} \times (s_0, +\infty)).$$

Lemma 4.4.7. *Let $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$ that is conical on $J^1(M \times (0, s_-))$ for $\epsilon < s_0 c(\Lambda_-) < s_- c(\Lambda_-)$. Given $\mathscr{F} \in Sh_{\Lambda_-}(M \times \mathbb{R})$ and $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$, suppose*

$$m_{\Lambda_-}(\mathscr{F}) = i_{\Lambda_-}^{-1} \mathscr{L}.$$

Then for the inclusions $i_{(s_0,s_-)}^- : M \times \mathbb{R} \times (s_0, s_-) \hookrightarrow M \times \mathbb{R} \times (0, s_-)$ and $i_{(s_0,s_-)}^+ : M \times \mathbb{R} \times (s_0, s_-) \hookrightarrow M \times \mathbb{R} \times (s_0, +\infty)$, we have a canonical isomorphism

$$(i_{(s_0,s_-)}^-)^{-1} w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F}) = (i_{(s_0,s_-)}^+)^{-1} w_{\widetilde{L}}^{(s_0,+\infty)}(\mathscr{L}).$$

Proof. Since $\widetilde{L}$ is conical on $J^1(M \times (s_0, s_-))$, there is a canonical equivalence by Guillermou-Kashiwara-Schapira that

$$Sh_{\Lambda_-}(M \times \mathbb{R}) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R} \times (s_0, s_-)),$$

it suffices to show that for any $s_1 \in (s_0, s_-)$ and the corresponding inclusions $i_{s_1}^- : M \times \mathbb{R} \times \{s_1\} \hookrightarrow M \times \mathbb{R} \times (0, s_-)$ and $i_{s_1}^+ : M \times \mathbb{R} \times \{s_1\} \hookrightarrow M \times \mathbb{R} \times (s_0, +\infty)$, there is an isomorphism

$$(i_{s_1}^-)^{-1} w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F}) = (i_{s_1}^+)^{-1} w_{\widetilde{L}}^{(s_0,+\infty)}(\mathscr{L}).$$

On the other hand, as our construction of $w_{\widetilde{L}}^{(s_0,+\infty)}$ is local with respect to small open subsets $U \times I \times J \subset M \times \mathbb{R} \times (s_0, s_-)$, it is compatible with the construction on a single slice w_{Λ_-} and there is an obvious isomorphism that

$$(i_{s_1}^+)^{-1} w_{\widetilde{L}}^{(s_0,+\infty)}(\mathscr{L}) = w_{\Lambda_-}(i_{\Lambda_-}^{-1}\mathscr{L}) = w_{\Lambda_-} \circ m_{\Lambda_-}(\mathscr{F}).$$

Then the isomorphism follows from Lemma 4.4.2 that $i_{s_1}^{-1} w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F}) = w_{\Lambda_-} \circ m_{\Lambda_-}(\mathscr{F})$.

□

Remark 4.4.3. *Instead of defining the doubling functor for the Legendrian cobordism in $J^1(M \times \mathbb{R}_{>0})$, one may consider defining the doubling functor on the truncated cobordism in $J^1(M \times (s_0, +\infty))$, where a Weinstein neighbourhood of positive radius always exist. However, then there will be obstruction to apply Reeb pushoff to send one copy of the Legendrian to infinity, see Remark 4.4.4. As we will explain, actually both the construction of doubling and the construction of pushing one copy to infinity come down to the question about the tubular neighbourhoods.*

Using the above lemma, we can prove the well-definedness of the doubling functor in Theorem 4.4.1 which is

$$w_{\widetilde{L}} : Sh_{\Lambda_-}(M \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_{\widetilde{L}}(\widetilde{L}) \hookrightarrow Sh_{T_{-\epsilon}(\widetilde{L}) \cup T_{\epsilon}(\widetilde{L})}(M \times \mathbb{R} \times \mathbb{R}_{>0}).$$

Next, we will need to address the full faithfulness of the doubling functor. This follows immediately from full faithfulness on the negative end Proposition 4.4.5 and full faithfulness away from the negative end Theorem 4.3.1.

Moreover, we can show the adjunction property in the following proposition.

Theorem 4.4.8. *Let $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$. Then when restricted to the subcategory of sheaves with compact supports, $\iota_{\widetilde{L}}^* \circ w_{\widetilde{L}}$ is the left adjoint of the functor*

$$(i_-, m_{\widetilde{L}}) : Sh_{\widetilde{L}}(M \times \mathbb{R} \times (0, +\infty))_0 \to Sh_{\Lambda_-}(M \times \mathbb{R})_0 \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_{\widetilde{L}}(\widetilde{L}).$$

Proof. On the negative end, for any $\mathscr{F} \in Sh_{\Lambda_-}(M \times \mathbb{R})_0$ and $\mathscr{G} \in Sh_{\widetilde{L}}(M \times \mathbb{R} \times (0, s_-))_0$, by Proposition 4.4.6, we know that

$$Hom(w_{\widetilde{L}}^{(0,s_-)}(\mathscr{F}), \mathscr{G}) = Hom(\mathscr{F}, i_-^{-1}\mathscr{G}).$$

Away from the negative end, for any $\mathscr{L} \in \mu Sh_{\widetilde{L}}(\widetilde{L})$ and $\mathscr{G} \in Sh_{\widetilde{L}}(M \times \mathbb{R} \times (s_0, +\infty))_0$, by Theorem 4.2.10, we know that

$$Hom(w_{\widetilde{L}}^{(s_0,+\infty)}(\mathscr{L}), \mathscr{G}) = \Gamma(\widetilde{L}, \mu hom(\mathscr{L}, m_{\widetilde{L}}(\mathscr{G}))).$$

On the overlap region, suppose $m_{\Lambda_-}(\mathscr{F}) = i_-^{-1}\mathscr{L}$. We also have

$$Hom(w_{\Lambda_-}(\mathscr{L}), i_-^{-1}\mathscr{G}) = \Gamma(\Lambda_-, \mu hom(i_-^{-1}\mathscr{L}, m_{\Lambda_-}(i_-^{-1}\mathscr{G}))).$$

Therefore, we can conclude that globally the adjunction holds. □

4.4.2. Separation of double copied Legendrians

Given the doubling construction in the previous section, now we will pull the double copies $\widetilde{L}$ and $T_\epsilon(\widetilde{L})$ apart and get the sheaf quantization functor for the conical Legendrian cobordism $\widetilde{L}$.

Proposition 4.4.9. *Let $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from $\Lambda_- \subset J^1(M)$ to $\Lambda_+ \subset J^1(M)$. Then for any $\epsilon, \epsilon' > 0$, there exists a canonical equivalence*

$$Sh_{\widetilde{L} \cup T_\epsilon(\widetilde{L})}(M \times \mathbb{R} \times \mathbb{R}_{>0}) \xrightarrow{\sim} Sh_{\widetilde{L} \cup T_{\epsilon'}(\widetilde{L})}(M \times \mathbb{R} \times \mathbb{R}_{>0}).$$

Proof. Following Proposition 4.3.3, we need to show that there exists some neighbourhood $U_{\epsilon'}(\Lambda)$ of Λ that is disjoint from $\bigcup_{u \geq \epsilon} T_u(\Lambda)$.

First, we find a neighbourhood of the negative end of $\widetilde{L}$. Recall $h(\Lambda_-) = \max_{(x,\xi,t)\in\Lambda_-} t - \min_{(x,\xi,t)\in\Lambda_+} t$. Assume that $\widetilde{L}$ is conical on $J^1(M \times (0, s_0))$, and moreover fix $e^{\epsilon/2} s_0 h(\Lambda_-) < \epsilon/2$. Then there exists a neighbourhood of $\widetilde{L} \cap J^1(M \times (0, s_0))$ of radius $\epsilon/2$ that is disjoint from $\bigcup_{u\geq\epsilon} T_u(\widetilde{L})$. This is because for points in $\bigcup_{u\geq\epsilon} T_u(\widetilde{L}) \cap J^1(M \times (0, e^{\epsilon/2} s_0))$, the distance between the t coordinates is at least $\epsilon/2$, while for the other points, the distance between the s coordinates is at least $\epsilon/2$.

Then, we find a tubular neighbourhood away from the negative end of $\widetilde{L}$. In fact, by Lemma 2.2.4, the Lagrangian projection $L \cap T^*(M \times (s_0, +\infty))$ admits a tubular neighbourhood of positive radius ϵ_1. Therefore, following Lemma 2.2.1, consider a tubular neighbourhood of $\widetilde{L} \cap J^1(M \times (s_0, +\infty))$ of radius $\epsilon' = \min(\epsilon, \epsilon_1)$. Then

$\widetilde{L} \cap J^1(M \times (s_0, +\infty))$ is disjoint from $\bigcup_{u \geq \epsilon} T_u(\widetilde{L})$. Indeed, for points in $\bigcup_{u \geq \epsilon} T_u(\widetilde{L}) \cap J^1(M \times (0, e^{\epsilon/2} s_0))$, the distance between the z coordinates is at least ϵ', while for the other points, the distance between the z coordinates is again at least ϵ'. By considering the union of the two neighbourhoods, we complete the proof. □

Remark 4.4.4. *Suppose one starts by working on $J^1(M \times (s_0, +\infty))$ where the doubling construction exists for some uniform $\epsilon > 0$. Then there will be serious difficulty when one tries to pushoff one of the copies by the Reeb flow. This is because by choosing a complete adapted metric on $J^1(M \times (s_0, +\infty))$, different from the restriction of the one on $J^1(M \times \mathbb{R}_{>0})$, the negative end becomes asymptotically horizontal and there will no longer be a tubular neighbourhood of Λ with positive radius that is disjoint from $\bigcup_{u \geq \epsilon} T_u(\Lambda)$.*

For example, consider the trivial 1-dimensional conical Legendrian cobordism as in Figure 2.2 (right), by the reparametrization identifying $J^1(\mathrm{pt} \times (1, +\infty))$ with $J^1(\mathrm{pt} \times \mathbb{R})$, one may assume that

$$\widetilde{L} = \{(s, \pm e^s, \pm t_0/2 \pm e^s) | s \in \mathbb{R}\}.$$

One can easily check that $\inf_{x, x' \in \widetilde{L}} d(x, T_{t_0}(x')) = 0$ under the standard complete adapted metric. Lack of control on the tubular neighbourhood will forbid us to connect the obvious Legendrian isotopy from $\widetilde{L} \cup T_\epsilon(\widetilde{L})$ to $\widetilde{L} \cup T_{t_0+\epsilon}(\widetilde{L})$ by a Hamiltonian isotopy.

Based on the proposition, we can prove finish the proof of Theorem 4.0.9. Let $t_{\max}(\Lambda_+) = \max_{(x,\xi,t)\in\Lambda_+} t$, $t_{\min}(\Lambda_-) = \min_{(x,\xi,t)\in\Lambda_+} t$ and the height be $h(\Lambda_+) = t_{\max}(\Lambda_+) - t_{\min}(\Lambda_+)$. Suppose that $\widetilde{L}$ is conical on $J^1(M \times (s_+, +\infty))$. We choose $\epsilon' > 0$ and $s'_+ > 0$ such that $s_+ h(\Lambda_+) < s'_+ h(\Lambda_+) < \epsilon'$.

Proof of Theorem 4.0.9. Using Theorem 4.4.1 and Proposition 4.4.9, we know that there exists a doubling functor (fully faithful on the subcategory of compactly supported sheaves)

$$w'_{\widetilde{L}} : Sh_{\Lambda_-}(M \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_{\widetilde{L}}(\widetilde{L}) \to Sh_{\widetilde{L} \cup T_{\epsilon'}(\widetilde{L})}(M \times \mathbb{R} \times \mathbb{R}_{>0}).$$

Then by restricting to $M \times (-\infty, s'_+ + s'_+ t_{\max}(\Lambda_+)) \times (0, s'_+)$, we get a functor

$$Sh_{\Lambda_-}(M \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_{\widetilde{L}}(\widetilde{L}) \to Sh_{\widetilde{L}}(M \times (-\infty, s'_+ + s'_+ t_{\max}(\Lambda_+)) \times (0, s'_+)).$$

Choose a diffeomorphism $\varphi : M \times (-\infty, s'_+ + s'_+ t_{\max}(\Lambda_+)) \times (0, s'_+) \xrightarrow{\sim} M \times \mathbb{R} \times (0, s'_+)$ that is the identity on $M \times (-\infty, s'_+/2 + s'_+ t_{\max}(\Lambda_+)) \times (0, s'_+)$. We will get the first equivalence

$$Sh_{\widetilde{L}}(M \times (-\infty, s'_+ + s'_+ t_{\max}(\Lambda_+)) \times (0, s'_+)) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R} \times (0, s'_+)).$$

Then since $\widetilde{L}$ is conical on $J^1(M \times (s_+, +\infty))$, consider the equivalence induced by Guillermou-Kashiwara-Schapira that

$$Sh_{\widetilde{L}}(M \times \mathbb{R} \times (s_+, s'_+)) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R} \times (s_+, +\infty)).$$

We will therefore get the second equivalence

$$Sh_{\widetilde{L}}(M \times \mathbb{R} \times (0, s'_+)) \xrightarrow{\sim} Sh_{\widetilde{L}}(M \times \mathbb{R} \times (0, +\infty)).$$

Therefore, combing the first equivalence and the second equivalence, we can conclude that there exists a conditional sheaf quantization functor.

Now, it suffices to show that the sheaf quantization functor is fully faithful when restricted to the subcategory of sheaves with compact supports at the negative end $Sh_{\Lambda_-}(M \times \mathbb{R})_0$. Let $j : M \times (-\infty, s'_+ + s'_+ t_{\max}(\Lambda_+)) \times (0, s'_+) \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$ and $\varphi : M \times (-\infty, s'_+ + s'_+ t_{\max}(\Lambda_+)) \times (0, s'_+) \xrightarrow{\sim} M \times \mathbb{R} \times (0, s'_+)$ be the diffeomorphism. By Theorem 4.4.8, we have

$$\begin{aligned} Hom(\Psi_L(\mathscr{F}, \mathscr{L}), \Psi_L(\mathscr{F}, \mathscr{L})) &= Hom(\varphi^{-1} j^{-1} w'_{\widetilde{L}}(\mathscr{F}, \mathscr{L}), \varphi^{-1} j^{-1} w'_{\widetilde{L}}(\mathscr{F}, \mathscr{L})) \\ &= Hom(w'_{\widetilde{L}}(\mathscr{F}, \mathscr{L}), j_* \varphi_* \varphi^{-1} j^{-1} w'_{\widetilde{L}}(\mathscr{F}, \mathscr{L})) \\ &\simeq Hom((\mathscr{F}, \mathscr{L}), (i_{\Lambda_-}^{-1}, m_{\widetilde{L}})(w'_{\widetilde{L}}(\mathscr{F}, \mathscr{L}))) \\ &= Hom((\mathscr{F}, \mathscr{L}), (\mathscr{F}, \mathscr{L})). \end{aligned}$$

This concludes the proof of the full faithfulness property.

Finally, it suffices to prove essential surjectivity in order to conclude that this is an equivalence. In fact, for any $\mathscr{F} \in Sh_{\widetilde{L}}(M \times \mathbb{R})_0$, we can easily show that $\Psi_L(i_-^{-1}\mathscr{F}, m_{\widetilde{L}}(\mathscr{F})) = \mathscr{F}$. Actually, when constructing the doubling $w_{\widetilde{L}}(i_-^{-1}\mathscr{F}, m_{\widetilde{L}}(\mathscr{F}))$, we have

$$w_{\widetilde{L}}(i_-^{-1}\mathscr{F}, m_{\widetilde{L}}(\mathscr{F})) = \mathrm{Cone}(T_{-\epsilon}(\mathscr{F}) \to T_{\epsilon}(\mathscr{F})).$$

At the negative end, this identity follows from the definition, while away from the negative end, this identity follows from the definition and the exact triangle of functors in Corollary 4.2.8. □

CHAPTER 5

Action Filtration, Persistence and Reeb Chords

Estimating the number of Reeb chords has been a basic question on Legendrian submanifolds since Arnold's time [**9**]. When the contact manifold is $(Y, \xi) = (P \times \mathbb{R}_t, \ker(dt - \theta_P))$ where $(Y, d\theta_P)$ is an exact symplectic manifold, one can pick the contact form $\alpha = dt - \theta_P$, and then the Reeb vector field is $\partial/\partial t$. For Λ a closed Legendrian, consider the Lagrangian projection

$$\pi_{\text{Lag}} : \Lambda \hookrightarrow P \times \mathbb{R} \to P.$$

The Reeb chords between Legendrian submanifolds correspond bijectively to intersection points of their Lagrangian projections.

For the number of self Reeb chords, when n is even, there is a topological lower bound coming from $[\pi_{\text{Lag}}(\Lambda)] \cdot [\pi_{\text{Lag}}(\Lambda)] = \chi(\Lambda)/2$. Some flexibility results tell us that this is sometimes the best bound one can expect [**53**]. However, under some extra assumptions, there are rigid behaviours beyond this purely algebraic topological bound.

Using pseudo-holomorphic curves, a number of celebrated theorems on the number of self Reeb chords have been found [**39**, **115**, **131**]. In particular, for Legendrians $\Lambda \subset P \times \mathbb{R}$, using Legendrian contact homology, works by Ekholm-Etnyre-Sullivan, Ekholm-Etnyre-Sabloff and Dimitroglou Rizell-Golovko [**45**, **56**, **58**] showed that, under some assumptions, the number of self Reeb chords is bounded from below by half of the sum of Betti numbers.

Other than estimating self Reeb chords, estimating the number of Reeb chords between Λ and some Hamiltonian pushoff $\varphi_H^1(\Lambda)$ has also been an important question. When the contact Hamiltonian comes from a symplectic Hamiltonian on P, this question reduces to the Arnold conjecture for (immersed) Lagrangian submanifolds $\pi_{\mathrm{Lag}}(\Lambda)$ [**9**].

Many Legendrians can be displaced from themselves so that there are no Reeb chords between Λ and $\varphi_H^1(\Lambda)$. However, when the norm of the Hamiltonian is sufficiently small, one can get estimates on the number of Reeb chords between Λ and $\varphi_H^1(\Lambda)$ using pseudo-holomorphic curves [**7**, **35**, **46**, **110**]. In particular a recent result by Dimitroglou Rizell-Sullivan [**47**], using the persistence of Legendrian contact homology, showed that for Legendrians $\Lambda \subset P \times \mathbb{R}$, under certain assumptions, there is a lower bound of the number of Reeb chords in terms of Betti numbers, when the oscillation norm of the Hamiltonian is small comparing to the length of Reeb chords.

The main purpose of this chapter is to set up the correspondence and estimate the number of Reeb chords using microlocal sheaf theory.

For self Reeb chords of a Legendrian $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, we have the following results analogous to Ekholm-Etnyre-Sullivan [**56**], Ekholm-Etnyre-Sabloff [**58**] and Dimitroglou Rizell-Golovko [**45**], where they showed the same inequality under the existence of a finite dimensional representation of the Chekanov-Eliashberg dg algebra, or Sabloff-Traynor [**136**] where they used generating families.

A Legendrian submanifold $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ is chord generic, if the Lagrangian projection $\pi_{\mathrm{Lag}}(\Lambda)$ is immersed with only transverse double points. Let $\mathcal{Q}(\Lambda)$ be the set of Reeb chords on Λ. Assume that the Maslov class $\mu(\Lambda) = 0$. Then there is a grading on Reeb chords of Λ given by the Conley-Zehnder index; see Section 2.3.2. Let $\mathcal{Q}_i(\Lambda)$ be the set of degree i Reeb chords on Λ.

Theorem 5.0.10. *Let M be orientable, $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be a closed chord generic Legendrian submanifold and $\Bbbk$ be a field. If there exists a $\Bbbk$-coefficient pure sheaf $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ with microlocal rank r such that* $\mathrm{supp}(\mathscr{F})$ *is compact, then*

$$|\mathcal{Q}_i(\Lambda)| + |\mathcal{Q}_{n-i}(\Lambda)| \geq b_i(\Lambda; \Bbbk).$$

In particular, the number of Reeb chords

$$|\mathcal{Q}(\Lambda)| \geq \frac{1}{2}\sum_{i=0}^{n} b_i(\Lambda; \Bbbk).$$

Here $b_i(\Lambda; \Bbbk) = \dim_{\Bbbk} H^i(\Lambda; \Bbbk)$.

Theorem 5.0.11 (Theorem 1.5.1)**.** *Let M be orientable, $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be a closed chord generic Legendrian submanifold and $\Bbbk$ be a field. If there exists a $\Bbbk$-coefficient sheaf $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ with perfect stalk such that* $\mathrm{supp}(\mathscr{F})$ *is compact, then*

$$|\mathcal{Q}(\Lambda)| \geq \frac{1}{2}\sum_{i=0}^{n} b_i(\Lambda; \Bbbk).$$

Here $b_i(\Lambda;\Bbbk) = \dim_{\Bbbk} H^i(\Lambda;\Bbbk)$.

Remark 5.0.5. *The condition that* $\mathrm{supp}(\mathscr{F})$ *is compact may be thought of as an analogue of the linear at infinity condition on generating families* [**136**]*. If we drop this condition, then there will be counterexamples. Consider the positive conormal* $\nu^{*,\infty}_{M,\tau>0}(M \times \mathbb{R}) \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ *(which is just the zero section* $M \subset J^1(M)$*). There is an obvious sheaf* $\Bbbk_{M\times[0,+\infty)}$ *with the prescribed singular support. However that Legendrian has no Reeb chords.*

Remark 5.0.6. *When there is a sheaf* $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ *with perfect stalk, then one can show that* [**84**] *necessarily the Maslov class* $\mu(\Lambda) = 0$*. However this condition is not necessary to get estimates on number of Reeb chords. In general, one can consider the triangulated orbit category* $Sh^b_\Lambda(M \times \mathbb{R})_{/[1]}$ *consisting of sheaves of* 1*-cyclic complexes (see* [**100**] *and* [**84**, Section 3]*). When there is a sheaf* $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})_{/[1]}$*, then we still expect that*

$$|\mathcal{Q}(\Lambda)| \geq \frac{1}{2}\sum_{i=0}^{n} b_i(\Lambda;\Bbbk),$$

but we do not work out the details here.

Remark 5.0.7. *In* [**45**, **56**, **58**] *they imposed the condition that the Legendrian* Λ *is horizontally displaceable, meaning that there exists a Hamiltonian isotopy* φ^s_H $(s \in I)$ *such that there are no Reeb chords between* Λ *and* $\varphi^1_H(\Lambda)$*. In Section 5.3.3 we show that if* Λ *is horizontally displaceable, then any* $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ *necessarily has compact support.*

However, there are Legendrians that are not horizontally displaceable but admit sheaves with compact support. For example let

$$T_c : M \times \mathbb{R} \to M \times \mathbb{R},\ (x,t) \mapsto (x, t+c)$$

be the vertical translation. Then the double copy of positive conormals $\nu^{*,\infty}_{M \cup T_c(M), \tau>0}(M \times \mathbb{R}) \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ *(which is the zero section and its Reeb pushoff in* $J^1(M)$*) is not horizontal displaceable but it admits a nontrivial sheaf with compact support. This means that our theorems work in a slightly more general setting.*

Remark 5.0.8. *Conjecturally* r *dimensional representations of the Chekanov-Eliashberg dg algebra should be equivalent to microlocal rank* r *pure sheaves (see* [**30**]*). Therefore Theorem 5.0.10 is just an analogue of* [**45**, **56**, **58**]*. However, Theorem 5.0.11 has no direct analogue in the literature to our knowledge.*

For Reeb chords between a Legendrian Λ and its Hamiltonian pushoff $\varphi^1_H(\Lambda)$, we have the following results, analogous to Dimitroglou Rizell and Sullivan [**47**]. Define the oscillation norm of the Hamiltonian to be

$$\|H_s\|_{\text{osc}} = \int_0^1 \left(\max_{x \in P \times \mathbb{R}} H_s - \min_{x \in P \times \mathbb{R}} H_s \right) ds.$$

Denote by $l(\gamma)$ the length of a Reeb chord γ. Assume that the Maslov class $\mu(\Lambda) = 0$, which ensures the existence of a grading on chords of Λ (see Section 2.3.2), and let

$$c_i(\Lambda) = \min\{l(\gamma) | \gamma \text{ is a Reeb chord, } \deg(\gamma) = i \text{ or } n-i\}.$$

Order them so that $c_{j_0}(\Lambda) \geq c_{j_1}(\Lambda) \geq ... \geq c_{j_n}(\Lambda)$.

Theorem 5.0.12 (Theorem 1.5.2)**.** *Let M be orientable, $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be a closed Legendrian submanifold of dimension n, and $\Bbbk$ be a field. Suppose there exists a $\Bbbk$-coefficient pure sheaf $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ such that $\mathrm{supp}(\mathscr{F})$ is compact. Let H_s be any compactly supported Hamiltonian $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ such that for some $0 \leq k \leq n$*

$$\|H_s\|_{osc} < c_{j_k}(\Lambda)$$

and $\varphi^1_H(\Lambda)$ is transverse to the Reeb flow applied to Λ. Then the number of Reeb chords between Λ and $\varphi^1_H(\Lambda)$ is

$$\mathcal{Q}(\Lambda, \varphi^1_H(\Lambda)) \geq \sum_{i=0}^{k} b_{j_i}(\Lambda; \Bbbk).$$

Here $b_j(\Lambda; \Bbbk) = \dim H^j(\Lambda; \Bbbk)$.

Remark 5.0.9. *It is shown* [**47**] *that this bound is sharp for Legendrian unknotted spheres with a single Reeb chord.*

Remark 5.0.10. *Dimitroglou Rizell-Sullivan considered* [**47**] *Legendrians that only admit augmentations over a subalgebra of the Chekanov-Eliashberg dg algebra $\mathcal{A}^l(\Lambda) \subset \mathcal{A}(\Lambda)$. We conjecture that, by combining our technique and Asano-Ike's technique* [**12**]*, if there exists $\mathscr{F} \in Sh^b_{\Lambda_q \cup \Lambda_r}(M \times \mathbb{R} \times (0, l))$, one might get analogous results.*

We are also able to recover the nonsqueezing result of Legendrians admitting sheaves into a stablized/loose Legendrian [**47**] as a byproduct.

5.1. Action Filtration on Sheaf Homomorphisms

We recall the definitions we made in the introduction and prove some basic properties. As is explained in the introduction, we consider to add an extra $\mathbb{R}$ factor in order to see the Reeb chords. We follow the construction of Shende's lecture notes, which goes back to Tamarkin [**153**, Chapter 3]. Similar constructions can also been found in Guillermou [**84**, Section 13 & 16], Nadler-Shende [**124**, Section 6] and Kuo [**104**].

Definition 5.1.1. *Let $q : M \times \mathbb{R}^2 \to M \times \mathbb{R}$ be $q(x, t, u) = (x, t)$ and $r : M \times \mathbb{R}^2 \to M \times \mathbb{R}$ be $r(x, t, u) = (x, t - u)$. For a Legendrian submanifold $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, let*

$$\Lambda_q = \{(x, \xi, t, \tau, u, 0)|(x, \xi, t, \tau) \in \Lambda\},$$

$$\Lambda_r = \{(x, \xi, t + u, \tau, u, -\tau)|(x, \xi, t, \tau \in \Lambda)\}.$$

For a sheaf $\mathscr{F} \in Sh^b(M \times \mathbb{R})$, let

$$\mathscr{F}_q = q^{-1}\mathscr{F}, \;\; \mathscr{F}_r = r^{-1}\mathscr{F}.$$

Here, Λ_q is the movie of Λ under the identity contact isotopy, while Λ_r is the movie of Λ under the vertical translation defined by the Reeb flow. It is not hard to

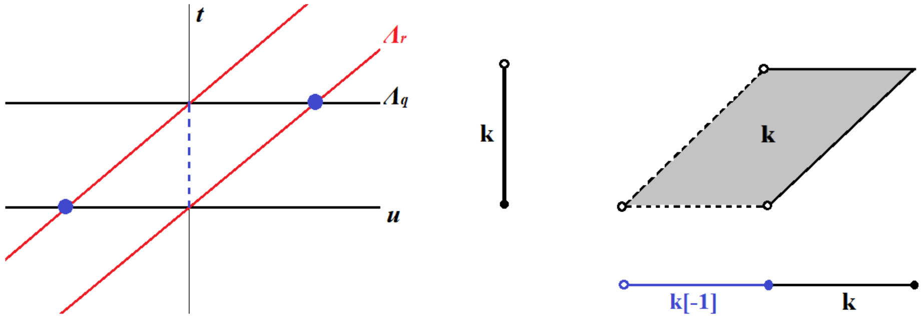

Figure 5.1. When M is a point, $\Lambda \subset \mathbb{R}$ consists of two points 0 and 1, the front of the Legendrians Λ_q and Λ_r are shown on the left. For $\mathscr{F} = \Bbbk_{[0,1)}$, the sheaf $\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)$ and its projection $u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)$ are shown on the right. The blue points are coming from the Reeb chord corresponding to the dashed blue line.

observe that every intersection point for some Λ and Reeb translation $T_c(\Lambda)$ where

$$T_c : T^{*,\infty}_{\tau>0}(M \times \mathbb{R}) \to T^{*,\infty}_{\tau>0}(M \times \mathbb{R});\;\; (x, \xi, t, \tau) \mapsto (x, \xi, t + c, \tau)$$

comes from a Reeb chord of Λ. Lemma 4.1.4 shows that those are all covectors pointing toward du direction (i.e. in $M \times \mathbb{R}_t \times T^*\mathbb{R}_u$) that lie in the singular support of $\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)$, indeed,

$$SS^\infty(\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \cap \mathrm{Graph}(du) = \varnothing.$$

$$SS^\infty(\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \cap \mathrm{Graph}(-du) \hookrightarrow \mathcal{Q}_\pm(\Lambda).$$

The following corollary produces an acyclic complex, which will be used to deduce Sabloff duality. The reader may compare it to the acyclic complex produced in generating family (co)homology [**136**, Section 3.1].

Lemma 5.1.1. *For* $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ *and* $\mathscr{F}, \mathscr{G} \in Sh^b_\Lambda(M \times \mathbb{R})$ *such that* $\mathrm{supp}(\mathscr{F}), \mathrm{supp}(\mathscr{G})$ *are compact,*

$$\Gamma(M \times \mathbb{R}^2, \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \simeq 0.$$

Proof. Since $SS^\infty(\mathscr{F}_q) \cap SS^\infty(\mathscr{G}_r) = \Lambda_q \cap \Lambda_r = \emptyset$, by Proposition 3.1.7

$$\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r) \simeq \mathscr{D}M\mathscr{F}_q \otimes \mathscr{G}_r.$$

Since $\mathrm{supp}(\mathscr{F}), \mathrm{supp}(\mathscr{G})$ are compact, we know that for sufficiently large $c > 0$, $T_{\pm c}(\Lambda) \cap \Lambda = \varnothing$. Hence for large $c > 0$,

$$\mathrm{supp}(\mathrm{D}'\mathscr{F}_q \otimes \mathscr{G}_r) \subset q^{-1}(\mathrm{supp}(\mathscr{F})) \cap r^{-1}(\mathrm{supp}(\mathscr{G})) \subset M \times [-c, c]^2.$$

Therefore consider the function $\varphi_+(x,t,u) = u$, $\varphi_+|_{\mathrm{supp}(\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r))}$ is proper and

$$SS(\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \cap \mathrm{Graph}(d\varphi_+) = \varnothing.$$

One can apply microlocal Morse lemma 3.1.3 and see that

$$\Gamma(M \times \mathbb{R}^2, \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \simeq \Gamma(M \times \mathbb{R} \times (-\infty, -c), \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) = 0.$$

This completes the proof. □

Similar to the case in Legendrian contact homology, where people defines two $\mathcal{A}_\infty$-categories $\mathcal{A}ug_-$ and $\mathcal{A}ug_+$, here we also define two dg categories of sheaves. The idea comes from the definition of the generating family cohomology.

From now on, the projection $M \times \mathbb{R}^2$, $(x,t,u) \mapsto u$ will be denoted by u.

Definition 5.1.2. *For $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ and $\mathscr{F}, \mathscr{G} \in Sh_\Lambda(M \times \mathbb{R})$, let*

$$Hom_-(\mathscr{F},\mathscr{G}) = \Gamma(u^{-1}([0,+\infty)), Hom(\mathscr{F}_q, \mathscr{G}_r)),$$

$$Hom_+(\mathscr{F},\mathscr{G}) = \Gamma(u^{-1}((0,+\infty)), Hom(\mathscr{F}_q, \mathscr{G}_r)).$$

The main theorem in this section is the following:

Theorem 5.1.2. *For $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ and $\mathscr{F}, \mathscr{G} \in Sh_\Lambda(M \times \mathbb{R})$ such that* $\mathrm{supp}(\mathscr{F})$ *and* $\mathrm{supp}(\mathscr{G})$ *are compact. Then*

$$Hom_-(\mathscr{F},\mathscr{G}) \simeq Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G})),\ Hom_+(\mathscr{F},\mathscr{G}) \simeq Hom(\mathscr{F}, T_{\epsilon}(\mathscr{G}))$$

when $\epsilon < c(\Lambda)$. Consequently, by Theorem 4.0.6,

$$Hom_-(\mathscr{F},\mathscr{G}) \simeq \Gamma(M, \Delta^* \mathcal{H}om(\pi_1^{-1}\mathscr{F}, \pi_2^{-1}\mathscr{G})),\ Hom_+(\mathscr{F},\mathscr{G}) = Hom(\mathscr{F},\mathscr{G}).$$

Example 5.1.1. *Let M be a point, $\Lambda \subset \mathbb{R}$ consists of two points 0 and 1 (see Figure 5.1). For $\mathscr{F} = \Bbbk_{[0,1)}$, the sheaf*

$$u_* \mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r) \simeq \Bbbk_{(-1,0]}[-1] \oplus \Bbbk_{(0,1]}.$$

Therefore as the projection $u : \mathbb{R}^2 \to \mathbb{R}$ is proper on $\mathrm{supp}(\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))$, *we have*

$$Hom_-(\mathscr{F},\mathscr{F}) = \Gamma([0,+\infty), \Bbbk_0[-1] \oplus \Bbbk_{(0,1]}) = \Bbbk[-1],$$

$$Hom_+(\mathscr{F},\mathscr{F}) = \Gamma((0,+\infty), \Bbbk_{(0,1]}) = \Bbbk.$$

Now we prove Theorem 5.1.2. The first part of the proof

$$\Gamma(u^{-1}((0,+\infty)), Hom(\mathscr{F}_q, \mathscr{G}_r)) \simeq Hom(\mathscr{F}, \mathscr{G})$$

is essentially due to Guillermou [**84**, Corollary 16.6]. Here we adapt the proof of Jin-Treumann [**94**, Proposition 3.16].

Proof of Theorem 5.1.2. Consider $Hom_+(\mathscr{F}, \mathscr{G})$. Let C be the minimal length of chords $\gamma \in \mathcal{Q}(\Lambda)$. As in the proof of Corollary 5.1.1, we can choose $\varphi_+(x,t,u) = u$, and by microlocal Morse lemma 3.1.3, when $c_0 < c(\Lambda)$,

$$\Gamma(M \times \mathbb{R} \times (0, c_0), \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \simeq \Gamma(M \times \mathbb{R} \times (0, +\infty), \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)).$$

Now it suffices to show that for $0 < c < c_0$

$$\Gamma(M \times \mathbb{R} \times (0, c_0), \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \simeq Hom(\mathscr{F}, T_{c,*}(\mathscr{G})).$$

This follows from Guillermou's result which we now recall. Note that when $0 < c < c_0$ there are no intersection points between Λ and $T_c(\Lambda)$. Hence $(\Lambda_q \cup \Lambda_r) \cap T^{*,\infty}(M \times \mathbb{R} \times (0,c))$ is the movie of a Legendrian isotopy (one can consider a Hamiltonian supported away from a neighbourhood of Λ that is equal to 1 near $\bigcup_{\epsilon < c < c(\Lambda)} T_c(\Lambda)$). By Guillermou-Kashiwara-Schapira's Theorem 3.3.1, we know for any $0 < c < c_0$

$$\Gamma(M \times \mathbb{R} \times (0, c_0), \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)) \simeq Hom(\mathscr{F}, T_{c,*}(\mathscr{G})).$$

This proves the assertion.

Then consider $Hom_-(\mathscr{F},\mathscr{G})$. First of all note that for sufficiently small $\epsilon > 0$, there are no Reeb chords of length less than ϵ, in order words (by Lemma 4.1.4), no points in $(-\Lambda_q + \Lambda_r) \cap \mathrm{Graph}(-du)$. Hence applying microlocal Morse lemma 3.1.3 to $u^{-1}((-\epsilon,+\infty))$ and $u^{-1}([0,+\infty))$ we know

$$\begin{aligned}\Gamma(M\times\mathbb{R}\times[0,+\infty),\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r)) &\simeq \varprojlim_{\epsilon>0}\Gamma(M\times\mathbb{R}\times(-\epsilon,+\infty),\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r))\\ &\simeq \varprojlim_{\epsilon>0}\Gamma(M\times\mathbb{R}\times(-\epsilon,0),\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r))\\ &\simeq \Gamma(M\times\mathbb{R}\times(-c_0,0),\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r)).\end{aligned}$$

where $i_{u>-\epsilon} : u^{-1}((-\epsilon,+\infty)) \hookrightarrow M\times\mathbb{R}^2$ is the inclusion. Similarly, $(\Lambda_q\cup\Lambda_r)\cap T^{*,\infty}(M\times\mathbb{R}\times(-c_0,0))$ is the movie of a Legendrian isotopy. By Guillermou-Kashiwara-Schapira's Theorem 3.3.1, we know for any $0<c<c_0$

$$\Gamma(M\times\mathbb{R}\times(-c_0,0),\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r)) \simeq Hom(\mathscr{F},T_{-c,*}(\mathscr{G})).$$

This proves the assertion. □

Remark 5.1.2. *The reason* $Hom(\mathscr{F},\mathscr{G}) \not\simeq Hom_-(\mathscr{F},\mathscr{G})$ *is that for the homomorphism*

$$i_{u=0}^{-1}\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r) \not\simeq \mathscr{H}om(i_{u=0}^{-1}\mathscr{F}_q, i_{u=0}^{-1}\mathscr{G}_r).$$

(Using the language in Nadler-Shende [**124**, Section 2]*, this is because the gapped condition fails for* Λ_r *and* Λ_q *as there exist Reeb chords whose lengths shrink to zero*

when $u \to 0$.*) However, for tensor products we can easily get*

$$i_{u=0}^{-1}(\mathrm{D}'\mathscr{F}_q \otimes \mathscr{G}_r) \simeq i_{u=0}^{-1}(\mathrm{D}'\mathscr{F}) \otimes i_{u=0}^{-1}\mathscr{G}.$$

The following corollary can be viewed as a version of degeneration to Morse flow trees in Legendrian contact homology (that certain pseudoholomorphic curves degenerate to Morse gradient flows) in for example [**58**, Theorem 3.6, Part (4)]. It says that certain sheaf homomorphisms descend to Morse theory. A similar result in sheaf theory can also been found in [**92**, Section 4.3].

Corollary 5.1.3. *For* $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ *and* $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ *a microlocal rank* r *sheaf such that* $\mathrm{supp}(\mathscr{F})$ *is compact, then*

$$\Gamma(u^{-1}(0), \Gamma_{u\leq 0}(\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)))[1] \simeq C^*(\Lambda; \Bbbk^{r^2}).$$

Proof. Note that we have an exact triangle

$$\begin{aligned} i_{u\geq 0}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r) &\to i'_{u>0,*}i_{u>0}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r) \\ &\to \Gamma_{u=0}(i_{u\geq 0}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))[1] \xrightarrow{+1} . \end{aligned}$$

Here $i_{u\geq 0} : u^{-1}([0, +\infty)) \hookrightarrow M \times \mathbb{R}^2$ and $i'_{u>0} : u^{-1}((0, +\infty)) \hookrightarrow u^{-1}([0, +\infty))$ are the inclusions. Taking global sections and compare it with the exact triangle in Theorem 4.0.6, we know that

$$\Gamma_{u=0}(M \times \mathbb{R}^2, i_{u\geq 0}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))[1] \simeq C^*(\Lambda; \Bbbk^{r^2}).$$

However, write $i_{u\geq -\epsilon}$ to be the inclusion $u^{-1}((-\epsilon,+\infty))\hookrightarrow M\times\mathbb{R}^2$. We also have

$$\begin{aligned}\Gamma_{u=0}(M\times\mathbb{R}^2, i_{u\geq 0}^{-1}\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r)) &\simeq Hom(\Bbbk_{u=0}, i_{u\geq 0}^{-1}\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r))\\ &\simeq \varprojlim_{\epsilon>0} Hom(i_{u\geq -\epsilon}^{-1}\Bbbk_{u\leq 0}, i_{u\geq -\epsilon}^{-1}\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r))\end{aligned}$$

This isomorphism is because for sufficiently small $\epsilon>0$, there are no Reeb chords of length less than ϵ, and thus (by Lemma 4.1.4), no points in $((-\Lambda_q)+\Lambda_r)\cap \mathrm{Graph}(-du)$. Therefore by microlocal Morse lemma the sections on $u^{-1}([0,+\infty))$ are the same as $u^{-1}((-\epsilon,+\infty))$ for small $\epsilon>0$.

$$\begin{aligned}\Gamma_{u=0}(M\times\mathbb{R}^2, i_{u\geq 0}^{-1}\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r)) &\simeq \varprojlim_{\epsilon>0} Hom(i_{u\geq -\epsilon}^{-1}\Bbbk_{u\leq 0}, i_{u\geq -\epsilon}^{-1}\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r))\\ &\simeq \varprojlim_{\epsilon>0} Hom(i_{|u|<\epsilon}^{-1}\Bbbk_{u\leq 0}, i_{|u|<\epsilon}^{-1}\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r))\\ &\simeq \varprojlim_{\epsilon>0} \Gamma(u^{-1}((-\epsilon,\epsilon)), \Gamma_{u\leq 0}(\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r)))\\ &\simeq \Gamma(u^{-1}(0), \Gamma_{u\leq 0}(\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r))).\end{aligned}$$

Here $i_{|u|<\epsilon}: u^{-1}((-\epsilon,\epsilon))\hookrightarrow M\times\mathbb{R}^2$ is the inclusion. The second equality holds because Lemma 4.1.4 enables us to apply microlocal Morse lemma restrict from $u^{-1}((-\epsilon,+\infty))$ to $u^{-1}((-\epsilon,\epsilon))$. This proves our assertion. □

5.2. Persistence Modules and Hamiltonian Isotopy

5.2.1. Persistence Modules and Sheaves

A persistent module is roughly speaking an $\mathbb{R}$-direct system of modules. It has been studied by a number of people, for example in [**32**, **33**].

Definition 5.2.1. *Let* $\Bbbk$ *be a ring. A persistence module* $M_{\mathbb{R}}$ *is a family* $\{M_\alpha\}_{\alpha\in\mathbb{R}}$ *of graded* $\Bbbk$*-modules, together with a family* $\{f_{\alpha_0\alpha_1} : M_{\alpha_0} \to M_{\alpha_1}\}_{\alpha_0\leq\alpha_1}$ *such that* $f_{\alpha_1\alpha_2} \circ f_{\alpha_0\alpha_1} = f_{\alpha_0\alpha_2}$ *and* $f_{\alpha\alpha} = \mathrm{id}_{M_\alpha}$*.* $M_{\mathbb{R}}$ *is tame if for any* $\alpha \in \mathbb{R}$, $\dim M_\alpha < \infty$*.*

Definition 5.2.2. *Let* $M_{\mathbb{R}}, N_{\mathbb{R}}$ *be two persistence modules. They are* (ϵ, ϵ')*-interleaved if there exists*

$$\phi_\alpha : M_\alpha \to N_{\alpha+\epsilon},\ \phi'_\alpha : M_\alpha \to N_{\alpha+\epsilon'},$$

$$\psi_\alpha : N_\alpha \to M_{\alpha+\epsilon},\ \psi'_\alpha : N_\alpha \to M_{\alpha+\epsilon'}$$

such that the following diagrams commute

$$f^M_{\alpha,\alpha+\epsilon+\epsilon'} = \psi_{\alpha+\epsilon} \circ \phi_\alpha,\ \ f^N_{\alpha,\alpha+\epsilon+\epsilon'} = \phi'_{\alpha+\epsilon'} \circ \psi'_\alpha.$$

The interleaving distance between $M_{\mathbb{R}}, N_{\mathbb{R}}$ *is*

$$d(M_{\mathbb{R}}, N_{\mathbb{R}}) = \inf\{\epsilon + \epsilon' | M_{\mathbb{R}}, N_{\mathbb{R}} \text{ are } (\epsilon, \epsilon')\text{-interleaved}\}.$$

One of the origins of the study of persistence modules is to study real functions on a manifold. Let $f \in C^\infty(X)$ and $X^\alpha_f = f^{-1}((\alpha, +\infty))$. Then $\{H^*(X^\alpha_f)\}_{\alpha\in\mathbb{R}}$ is a persistence module. A crucial result in [**32**] is that the distance of a family of persistence modules $\{H^*(X^\alpha_f)\}_{\alpha\in\mathbb{R}}$ when f changes is controlled by the C^0-norm of f:

$$d(H^*(X^\alpha_f), H^*(X^\alpha_g)) \leq d_{C^0}(f, g).$$

Remark 5.2.1. *In* [**32**] *the authors were assuming that* $\phi = \psi, \phi' = \psi'$ *and only got the bound by* $2d_{C^0}(f,g)$. *However, when Usher-Zhang* [**158**]*, or Asano-Ike* [**11**] *were trying to define an analogue of the interleaving distance and apply that to symplectic topology, they found that one had to allow the case where* $\phi \neq \psi, \phi' \neq \psi'$ *in order to get a better bound* $d_{C^0}(f,g)$. *Therefore we adapt their definition here.*

In this paper, we will use the language of constructible sheaves on $\mathbb{R}$ instead of persistence modules. Here is the classification result of these sheaves.

Theorem 5.2.1 (Guillermou [**86**, Corollary 7.3]; Kashiwara-Schapira [**89**, Theorem 1.17])**.** *Let* $\mathscr{F} \in Sh^b_{\nu<0}(\mathbb{R})$ *be a constructible sheaf. Then there exists a finite (index) set* A *such that*

$$\mathscr{F} \simeq \bigoplus_{\alpha \in A} \Bbbk^{r_\alpha}_{(u_\alpha, v_\alpha]}[n_\alpha].$$

Each interval $(u_\alpha, v_\alpha]$ *is called a bar.*

Note that for any constructible sheaf $\mathscr{F} \in Sh^b_{\nu<0}(\mathbb{R})$, we can associate a tame persistence module by $M_\alpha = H^*\Gamma((-\infty, \alpha), \mathscr{F})$. All definitions and results in persistence modules can be stated in 1-dimensional sheaf theory easily. In fact, one can probably show that the category of tame persistence modules is equivalent to the full subcategory of constructible sheaves in $Sh^b_{\nu<0}(\mathbb{R})$. However we won't discuss it here.

Now we define the interleaving distance for sheaves in arbitrary dimensions.

Definition 5.2.3 (Asano-Ike [**11**])**.** *Let* $\mathscr{F}, \mathscr{G} \in Sh^b_{\tau>0}(M \times \mathbb{R})$ *be two constructible sheaves. Let* $T_c : \mathbb{R} \to \mathbb{R}$ *be the translation* $T_c(x,t) = (x, t+c)$. *They*

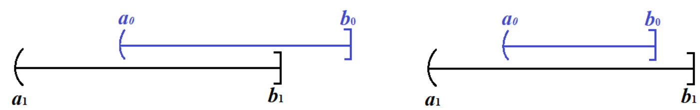

Figure 5.2. The sheaves $\Bbbk_{(a_0,b_0]}$ and $\Bbbk_{(a_1,b_1]}$ in two different cases.

are (ϵ,ϵ')-interleaved if there exists

$$\phi : \mathscr{F} \to T_{\epsilon,*}\mathscr{G},\ \psi : \mathscr{G} \to T_{\epsilon',*}\mathscr{F},$$

$$\phi' : \mathscr{G} \to T_{\epsilon,*}\mathscr{F},\ \psi' : \mathscr{F} \to T_{\epsilon',*}\mathscr{G}$$

such that the following diagrams commute

$$t^{\mathscr{F}}_{0,\epsilon+\epsilon'} = T_{\epsilon,*}\psi \circ \phi,\ \ t^{\mathscr{G}}_{0,\epsilon+\epsilon'} = T_{\epsilon',*}\phi' \circ \psi'$$

where $t^{\mathscr{H}}_{a,b} : \mathscr{H} \to T_{a+b,*}\mathscr{H}$ is the natural map. The interleaving distance between $\mathscr{F}, \mathscr{G}$ is

$$d(\mathscr{F},\mathscr{G}) = \inf\{\epsilon+\epsilon' | \mathscr{F},\mathscr{G} \text{ are } (\epsilon,\epsilon')\text{-interleaved}\}.$$

Example 5.2.2. *Consider the sheaves $\Bbbk_{(a_0,b_0]}$ and $\Bbbk_{(a_1,b_1]}$ in $Sh^b_{\nu<0}(\mathbb{R})$. Since their singular supports satisfy $\nu < 0$, we need to choose the translation in the negative direction $U_c : \mathbb{R} \to \mathbb{R}$, $x \mapsto x - c$. Then if a, a', b, b' are distinct, by Proposition 3.1.7*

$$\mathscr{H}om(\Bbbk_{(a,b]}, \Bbbk_{(a',b']}) = \Bbbk_{[a,b)\cap(a',b']}.$$

There exists a degree zero non-vanishing map iff $a' < a$ and $b' < b$. Now we estimate the distance between $\Bbbk_{(a_0,b_0]}$ and $\Bbbk_{(a_1,b_1]}$ in two specific cases.

Suppose $a_0 > a_1, b_0 > b_1$ and $a_0 < b_1$ (Figure 5.2 left). When $\epsilon + \epsilon' > b_1 - a_1$, the natural map

$$\tau_{0,\epsilon+\epsilon'} : \mathbb{k}_{(a_1,b_1]} \to \mathbb{k}_{(a_1-\epsilon-\epsilon',b_1-\epsilon-\epsilon']}$$

becomes zero, so we can choose all the maps to be zero. Now we assume that $\epsilon + \epsilon' < b_1 - a_1$, which means the natural map as a composition

$$\tau_{0,\epsilon+\epsilon'} : \mathbb{k}_{(a_1,b_1]} \to \mathbb{k}_{(a_0-\epsilon',b_0-\epsilon']} \to \mathbb{k}_{(a_1-\epsilon-\epsilon',b_1-\epsilon-\epsilon']}$$

is nonzero. For the second map to be nonzero, we require $a_0 - \epsilon' < a_1$ and $b_0 - \epsilon' < b_1$, i.e. $\epsilon' > \max\{a_0 - a_1, b_0 - b_1\}$. Now we choose any

$$\epsilon > 0, \;\; \epsilon' > \max\{a_0 - a_1, b_0 - b_1\}.$$

Then maps in the composition

$$\tau_{0,\epsilon+\epsilon'} : \mathbb{k}_{(a_1,b_1]} \to \mathbb{k}_{(a_0-\epsilon',b_0-\epsilon']} \to \mathbb{k}_{(a_1-\epsilon-\epsilon',b_1-\epsilon-\epsilon']}$$

can be chosen to be nonzero. For the other composition

$$\tau_{0,\epsilon+\epsilon'} : \mathbb{k}_{(a_0,b_0]} \to \mathbb{k}_{(a_1-\epsilon,b_1-\epsilon]} \to \mathbb{k}_{(a_0-\epsilon-\epsilon',b_0-\epsilon-\epsilon']},$$

we have $a_0 - \epsilon - \epsilon' < a_1 - \epsilon, b_0 - \epsilon - \epsilon' < b_1 - \epsilon$. Therefore the maps can also be chosen to be nonzero. Therefore we can show that the distance is

$$d(\mathbb{k}_{(a_0,b_0]}, \mathbb{k}_{(a_1,b_1]}) = \inf\{\epsilon + \epsilon'\} = \max\{a_1 - a_0, b_1 - b_0\}.$$

Suppose $a_0 > a_1, b_0 < b_1$. Then $(a_0, b_0] \subset (a_1, b_1]$ (Figure 5.2 right). Without loss of generality, we may still assume that $\epsilon + \epsilon' < b_1 - a_1$, which means the composition

$$\tau_{0,\epsilon+\epsilon'} : \mathbb{k}_{(a_1,b_1]} \to \mathbb{k}_{(a_0-\epsilon',b_0-\epsilon']} \to \mathbb{k}_{(a_1-\epsilon-\epsilon',b_1-\epsilon-\epsilon']}$$

is nonzero. For the first map to be nonzero, we require $a_0 - \epsilon' < a_1 < b_0 - \epsilon'$, i.e. $\epsilon' > a_0 - a_1$. For the second map to be nonzero, we require $a_0 - \epsilon' < b_1 - \epsilon - \epsilon' < b_0 - \epsilon'$, i.e. $\epsilon > b_1 - b_0$. Therefore one can show that

$$d(\mathbb{k}_{(a_0,b_0]}, \mathbb{k}_{(a_1,b_1]}) = \inf\{\epsilon + \epsilon'\} = (a_0 - a_1) + (b_1 - b_0).$$

For the other two cases, one has similar results. In conclusion, one can see that the persistence distance is measuring how far the bars differ from each other (in fact it is the Gromov-Hausdorff distance between the intervals).

Here is a basic property we're going to use from time to time. It basically says that the persistence distance is a pseudo metric.

Lemma 5.2.2. *Suppose $\mathscr{F}, \mathscr{G}$ are (a_0, b_0)-interleaved, and $\mathscr{G}, \mathscr{H}$ are (a_1, b_1)-interleaved. Then $\mathscr{F}, \mathscr{H}$ are $(a_0 + a_1, b_0 + b_1)$-interleaved. In particular,*

$$d(\mathscr{F}, \mathscr{H}) \leq d(\mathscr{F}, \mathscr{G}) + d(\mathscr{G}, \mathscr{H}).$$

Proof. We have the following commutative diagrams that give the natural maps τ_{0,a_0+b_0} and τ_{0,a_1+b_1}:

$$\mathscr{F} \xrightarrow{\phi} T_{a_0,*}\mathscr{G} \xrightarrow{T_{a_0,*}\psi} T_{a_0+b_0,*}\mathscr{F},\ \mathscr{G} \xrightarrow{\phi'} T_{b_0,*}\mathscr{F} \xrightarrow{T_{b_0,*}\psi'} T_{a_0+b_0,*}\mathscr{G},$$

$$\mathscr{G} \xrightarrow{\gamma} T_{a_1,*}\mathscr{H} \xrightarrow{T_{a_0,*}\delta} T_{a_1+b_1,*}\mathscr{G},\ \mathscr{H} \xrightarrow{\gamma'} T_{b_1,*}\mathscr{G} \xrightarrow{T_{b_0,*}\delta'} T_{a_1+b_1,*}\mathscr{H}.$$

Therefore we can construct the following maps that give the natural map $\tau_{0,a_0+a_1+b_0+b_1}$:

$$\mathscr{F} \xrightarrow{T_{a_0,*}\gamma\circ\phi} T_{a_0+a_1,*}\mathscr{H} \xrightarrow{T_{a_0+a_1+b_1,*}\psi\circ T_{a_0+a_1,*}\delta} T_{a_0+b_0+b_0+b_1,*}\mathscr{F},$$

$$\mathscr{H} \xrightarrow{T_{a_1,*}\phi'\circ\gamma'} T_{a_1+a_0,*}\mathscr{F} \xrightarrow{T_{a_1+a_0+b_0,*}\delta'\circ T_{a_1+a_0,*}\psi'} T_{a_1+a_0+b_1+b_0,*}\mathscr{H}.$$

This proves the assertion. □

5.2.2. Continuity under Hamiltonian Isotopy

Given a Hamiltonian isotopy $\varphi_H^s\,(s \in I)$ on $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, Guillermou-Kashiwara-Schapira defined an equivalence functor called sheaf quantization $\Phi_H^s : Sh^b_{\tau>0}(M \times \mathbb{R}) \to Sh^b_{\tau>0}(M \times \mathbb{R})$ (Theorem 3.3.1). Asano and Ike studied how the quantization of a Hamiltonian isotopy changes the interleaving distance. Recall that

$$\|H\|_{\mathrm{osc}} = \int_0^1 (\max H_s - \min H_s)\ ds.$$

Theorem 5.2.3 (Asano-Ike [**11**])**.** *Let H be a compactly supported Hamiltonian on $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ and $\Phi_H^s\,(s \in I)$ be its sheaf quantization functor. Then for $\mathscr{F} \in Sh_{\tau>0}(M \times \mathbb{R})$,*

$$d(\mathscr{F}, \Phi_H^1(\mathscr{F})) \leq \|H\|_{osc}.$$

To make the section self-contained, we give a proof of the theorem (the version we're going to use is a little bit weaker as we will add the proper assumption in the following lemma, but that's unnecessary). Denote by $\gamma_{a,b}$ the following cone in $\mathbb{R}^2$:

$$\gamma_{a,b} = \{(\tau, \sigma) | -a\tau < \sigma < b\tau\} \subset \mathbb{R}^2.$$

Lemma 5.2.4 (Guillermou-Schapira [**89**, Proposition 5.9]; [**11**, Proposition 4.3])**.** *For* $\mathscr{H} \in Sh_{\tau>0}(M \times \mathbb{R} \times I)$ *and* $s_0 < s_1 \in I$, *if there exists* $a, b, r \in \mathbb{R}_{>0}$ *such that*

$$SS(\mathscr{H}) \cap T^*(M \times \mathbb{R} \times (s_0 - r, s_1 + r)) \subset T^*M \times ((\mathbb{R} \times I) \times \gamma_{a,b}),$$

Suppose the projection $\pi_{M\times\mathbb{R}} : M \times \mathbb{R} \times I \to M \times \mathbb{R}$ *is proper on* $\mathrm{supp}(\mathscr{H})$. *Then the natural morphisms*

$$\tau_{0,a(s_1-s_0)+\epsilon} : \pi_{M\times\mathbb{R},*}(\mathscr{H}|_{M\times\mathbb{R}\times[s_0,s_1)}) \to T_{a(s_1-s_0)+\epsilon,*}\pi_{M\times\mathbb{R},*}(\mathscr{H}|_{M\times\mathbb{R}\times[s_0,s_1)}),$$

$$\tau_{0,b(s_1-s_0)+\epsilon} : \pi_{M\times\mathbb{R},*}(\mathscr{H}|_{M\times\mathbb{R}\times(s_0,s_1]}) \to T_{b(s_1-s_0)+\epsilon,*}\pi_{M\times\mathbb{R},*}(\mathscr{H}|_{M\times\mathbb{R}\times(s_0,s_1]})$$

both vanish.

Proof. We will only check the first assertion. Without loss of generality, we may assume that $I = \mathbb{R}$. Write $\pi = \pi_{M\times\mathbb{R}}$. Consider the diagram

$$\begin{array}{ccc} \{x\} \times \mathbb{R}^2 & \xrightarrow{\widehat{x}} & M \times \mathbb{R}^2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \{x\} \times \mathbb{R} & \xrightarrow{x} & M \times \mathbb{R} \end{array}$$

Since π is proper on $\mathrm{supp}(\mathscr{H})$, by proper base change formula we have

$$x^{-1}\pi_*(\mathscr{H}|_{[u_0,u_1)}) \simeq \pi_*\widehat{x}^{-1}(\mathscr{H}|_{[u_0,u_1)}).$$

Hence we may in fact assume that M is a point.

Recall $(\gamma_{a,b}^\vee)^\circ = \{(t,s)| - b^{-1}t < s < a^{-1}s\}$. By microlocal cut-off lemma, we know that

$$\mathscr{H} \simeq \widehat{s}_*(\widehat{p_1}^{-1}\Bbbk_{(\gamma_{a,b}^\vee)^\circ} \otimes \widehat{p_2}^{-1}\mathscr{H}) \simeq s_*\Gamma_{(\gamma_{a,b}^\vee)^\circ\times\mathbb{R}^2}(\widehat{p_2}^{-1}\mathscr{H}),$$

where $\widehat{s}(t,s,t',s') = (t+t', s+s')$, $\widehat{p}_1(t,s,t',s') = (t,s)$ and $\widehat{p}_2(t,s,t',s') = (t',s')$. Also, note that $SS^\infty(\Bbbk_{\mathbb{R}\times[u_0,u_1)}) \cap SS^\infty(\mathscr{H}) = \emptyset$. Hence

$$\begin{aligned}\pi_*(\mathscr{H}|_{\mathbb{R}\times[u_0,u_1)}) &\simeq \pi_*\Gamma_{\mathbb{R}\times(u_0,u_1]}\mathscr{H} \simeq \pi_*\widehat{s}_*\Gamma_D(\widehat{p_2}^{-1}\mathscr{H}) \\ &\simeq \pi_*\widehat{s}_*\mathscr{H}om(\Bbbk_D, \widehat{p_2}^{-1}\mathscr{H}),\end{aligned}$$

where $D = ((\gamma_{a,b}^\vee)^\circ \times \mathbb{R}^2) \cap \{(t,s,t',s')|s_0 < s+s' \leq s_1\}$. Let $\widehat{T}_c(t,s,t',s') = (t+c,s,t',s')$. Then

$$T_{c,*}\pi_*\widehat{s}_*\mathscr{H}om(\Bbbk_D, \widehat{p_2}^{-1}\mathscr{H}) \simeq \pi_*\widehat{s}_*\mathscr{H}om(\Bbbk_{\widehat{T}_c(D)}, \widehat{p_2}^{-1}\mathscr{H}),$$

and the natural map $\tau_{0,c}$ is induced by $\Bbbk_D \to \Bbbk_{\widehat{T}_c(D)}$.

Now we consider to decompose $\widehat{p}_2(t,s,t',s') = (t',s')$ as $\widehat{p}(t,s,t',s') = (t,t',s')$ and $p_2(t,t',s') = (t',s')$. Then we know

$$\mathscr{H}om(\Bbbk_D, \widehat{p_2}^{-1}\mathscr{H}) \simeq \mathscr{H}om(\Bbbk_D, \widehat{p}^{-1}p_2^{-1}\mathscr{H}) \simeq \mathscr{H}om(\Bbbk_D, \widehat{p}^{!}p_2^{-1}\mathscr{H})[-1]$$

$$\simeq p_*\mathscr{H}om(\widehat{p}_!\,\Bbbk_D, p_2^{-1}\mathscr{H})[-1],$$

$$\mathscr{H}om(\Bbbk_{\widehat{T}_c(D)}, \widehat{p_2}^{-1}\mathscr{H}) \simeq p_*\mathscr{H}om(\widehat{p}_!\,\Bbbk_{\widehat{T}_c(D)}, p_2^{-1}\mathscr{H})[-1].$$

Hence it suffices to show that $\widehat{p}_!\,\Bbbk_D \to \widehat{p}_!\Bbbk_{\widehat{T}_c(D)}$ is zero. However, when $t<0$, the support of the sheaf $\Bbbk_D$ in the fiber $\widehat{p}^{-1}(t,t',s') \cap D = \emptyset$; when $t \geq 0$,

$$\widehat{p}^{-1}(t,t',s') \cap D = (s_0 - s', s_1 - s'] \cap (-b^{-1}t, a^{-1}t).$$

When the support of $\Bbbk_D$ in the fiber of $\widehat{p}$ is empty or a half closed half open interval, the stalk $(\widehat{p}_!\,\Bbbk_D)_{(t,t',s')} = 0$; when it is an open interval, then the stalk $(\widehat{p}_!\,\Bbbk_D)_{(t,t',s')} \neq 0$. Hence

$$\mathrm{supp}(\widehat{p}_!\,\Bbbk_D) = \{(t,t',s') | t > 0, s_0 < s' + a^{-1}t \leq s_1\}.$$

Therefore when $c > a(s_1 - s_0)$ we know $\mathrm{supp}(\widehat{p}_!\,\Bbbk_D) \cap \mathrm{supp}(\widehat{p}_!\,\Bbbk_{\widehat{T}_c(D)}) = \emptyset$ (see Figure 5.3). This completes the proof. □

Proof of Theorem 5.2.3. The movie of a subset $\Lambda \subset T^*(M \times \mathbb{R})$ under the Hamiltonian isotopy $\varphi_{\widehat{H}}^s$ $(s \in I)$ is

$$\Lambda_H = \{(x,t,s,\xi,\tau,\sigma) | (x,t,\xi,\tau) = \varphi_{\widehat{H}}^s(x_0,t_0,\xi_0,\tau_0),\, \nu = -\tau H_s \circ \widehat{\varphi}_H^s(x,t,\xi/\tau)\}.$$

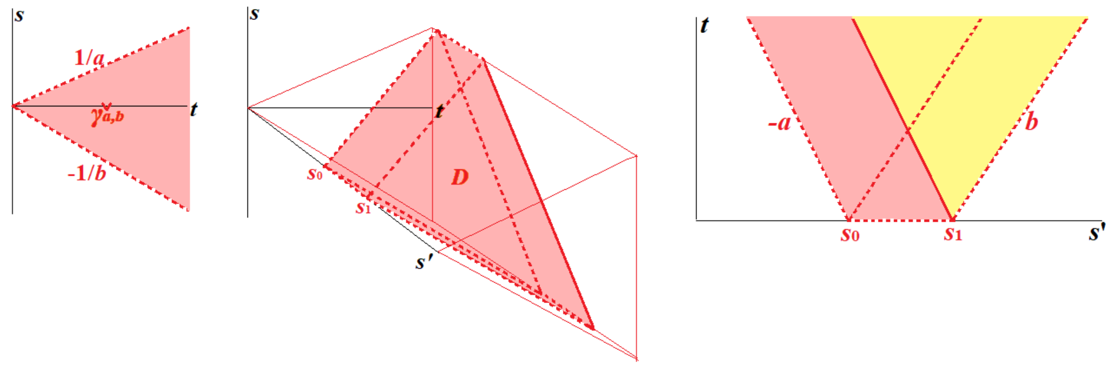

Figure 5.3. The figure on the left is the open cone $(\gamma_{a,b}^{\vee})^{\circ}$; the one in the middle is the subset D forgetting the t' coordinate; the one on the right is the projection $p(D)$ forgetting the t' coordinate, where the fibers in the yellow region are half closed half open intervals and the fibers in the red region are open intervals.

Therefore it follows immediately that in an interval $[s_{i-1}, s_i]$, one can choose $r > 0$ small such that

$$SS(\mathscr{H}) \cap T^*(M \times \mathbb{R} \times (s_{i-1} - r, s_i + r)) \subset T^*M \times ((\mathbb{R} \times I) \times \gamma_{a_i,b_i}),$$

where $a_i = \max_{s\in(s_{i-1}-r,s_i+r)} H_s$, $b_i = -\min_{s\in(s_{i-1}-r,s_i+r)} H_u$. This will enable us to apply Lemma 5.2.4 later.

Write $\pi = \pi_{M\times\mathbb{R}} : M \times \mathbb{R}^2 \to M \times \mathbb{R}$. To connect $\mathscr{H}|_{M\times\mathbb{R}\times\{s_{i-1}\}}$ and $\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}$, we consider the following exact triangles

$$\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i)}) \to \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]}) \to \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) \xrightarrow{+1},$$

$$\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times(s_{i-1},s_i]}) \to \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]}) \to \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_{i-1}\}}) \xrightarrow{+1}.$$

Consider the commutative diagram given by natural morphisms under translation

$$\begin{array}{ccccccc}
\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i)}) & \longrightarrow & \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]}) & \longrightarrow & \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) & \xrightarrow{+1} & \\
\downarrow{\scriptstyle \tau_{0,c}} & & \downarrow{\scriptstyle \tau_{0,c}} & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle \tau_{0,c}} & & \\
T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i)}) & \to & T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]}) & \to & T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) & \xrightarrow{+1} & .
\end{array}$$

By Lemma 5.2.4, when $c = a_i(s_i - s_{i-1}) + \epsilon$, the left vertical arrow is zero. Hence by the commutative diagram

$$\begin{array}{ccccc}
 & & \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) & \xrightarrow{+1} & \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i)})[1] \\
 & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle \tau_{0,c}} & & \downarrow{\scriptstyle 0} \\
T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]}) & \longrightarrow & T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) & \xrightarrow{+1} & T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i)})[1],
\end{array}$$

the composition

$$\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) \to T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) \to T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i)})[1]$$

is zero. In other words, there exists a morphism

$$\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}) \xrightarrow{\phi} T_{c,*}\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]})$$

that makes the diagram commute. This shows that

$$\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}}), \quad \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]})$$

are $(0, a_i(s_i - s_{i-1}) + \epsilon)$-interleaved. Similarly,

$$\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_{i-1}\}}),\ \ \pi_*(\mathscr{H}|_{M\times\mathbb{R}\times[s_{i-1},s_i]})$$

are $(b_i(s_i - s_{i-1}) + \epsilon, 0)$-interleaved. By Lemma 5.2.2, this means $\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_{i-1}\}})$ and $\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{s_i\}})$ will be $(b_i(s_i - s_{i-1}) + \epsilon, a_i(s_i - s_{i-1}) + \epsilon)$-interleaved.

Now we choose a division of $[0, 1]$, then by Lemma 5.2.2, we know $\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{0\}})$ and $\pi_*(\mathscr{H}|_{M\times\mathbb{R}\times\{1\}})$ are (a, b)-interleaved where

$$a = \sum_{i=1}^{N} a_i(s_i - s_{i-1}) + N\epsilon,\ \ b = \sum_{i=1}^{N} b_i(s_i - s_{i-1}) + N\epsilon$$

are the Riemann sums. Therefore by letting $\epsilon \ll 1/N$ we know that

$$d(\mathscr{H}, \Phi_H^1(\mathscr{H})) \leq \inf_{0=s_0<\ldots<s_N=1} \left\{ \sum_{i=1}^{N} \Big(\max_{(s_{i-1}-r,s_i+r)} H_s - \min_{(s_{i-1}-r,s_i+r)} H_s \Big)(s_i - s_{i-1}) \right\},$$

so the result follows. □

Using this machinery, we now study our sheaf $\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)$ for $\mathscr{F}, \mathscr{G} \in Sh^b(M \times \mathbb{R})$. As we have seen in previous sections, the last $\mathbb{R}$ component encodes the length of all Reeb chords on Λ. Hence in order to get information on how the Reeb chords change under Hamiltonian isotopies, we project the sheaf to the last component $\mathbb{R}$ via $u : M \times \mathbb{R}^2 \to \mathbb{R},\ (x, t, u) \mapsto u$ and estimate the persistence structure on

$$u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r).$$

By Lemma 4.1.4, this is a constructible sheaf in $Sh^b_{\nu<0}(\mathbb{R})$. Here is our main result in this section.

Definition 5.2.4. *Let* $q : M\times\mathbb{R}^2 \to M\times\mathbb{R}$ *be* $q(x,t,u) = (x,t)$ *and* $r : M\times\mathbb{R}^2 \to M\times\mathbb{R}$ *be* $r(x,t,u) = (x,t-u)$*. For sheaves* $\mathscr{F},\mathscr{G} \in Sh^b(M\times\mathbb{R})$*, let*

$$\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F},\mathscr{G}) = u_*\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r).$$

Theorem 5.2.5. *Let* $\Lambda \subset T^{*,\infty}_{\tau>0}(M\times\mathbb{R})$ *be a compact Legendrian,* H *be a Hamiltonian on* $T^{*,\infty}_{\tau>0}(M\times\mathbb{R})$ *and* Φ^s_H $(s\in I)$ *be its sheaf quantization. Then for* $\mathscr{F},\mathscr{G} \in Sh_\Lambda(M\times\mathbb{R})$ *with compact support,*

$$d(\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F},\mathscr{G}),\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F},\Phi^1_H(\mathscr{G}))) \leq \|H\|_{osc}.$$

Proof. First of all we extend H to a compactly supported Hamiltonian on $T^{*,\infty}_{\tau>0}(M\times\mathbb{R}^2)$. Namely choose a compactly supported cutoff function β_0 on $T^{*,\infty}_{\tau>0}(M\times\mathbb{R})$ such that

$$\beta_0|_{\bigcup_{s\in I}\varphi^s_H(\Lambda)} \equiv 1.$$

Let $H_0 = \beta_0 H$ be a compactly supported Hamiltonian on $T^{*,\infty}_{\tau>0}(M\times\mathbb{R})$. Then we can define $\widehat{H}_0(x,t,u,\xi,\tau,\nu) = \beta_0(x,t-u,\xi,\tau)H(x,t-u,\xi,\tau)$. Since $\operatorname{supp}(\mathscr{F}),\operatorname{supp}(\mathscr{G})$ are compact, we may assume that there exists $c>0$,

$$q^{-1}(\operatorname{supp}(\mathscr{F}))\cap r^{-1}\Big(\bigcup_{s\in I}\pi\big(\varphi^s_H(\pi^{-1}(\operatorname{supp}(\mathscr{G})))\big)\Big) \subset M\times[-c,c]^2,$$

where $\pi : T^{*,\infty}(M \times \mathbb{R}) \to M \times \mathbb{R}$ is the projection. Choose a compactly supported cutoff function $\widehat{\beta}_1$ on $T^{*,\infty}_{\tau>0}(M \times \mathbb{R}^2)$ such that

$$\widehat{\beta}_1|_{M\times[-c,c]^2} \equiv 1.$$

Then let $\widehat{H}(x,t,u,\xi,\tau,\nu) = \widehat{\beta}_1(x,t,u)\widehat{H}_0(x,t,u,\xi,\tau,\nu)$. One can see that

$$\mathscr{H}om(\mathscr{F}_q, (\Phi^s_H \mathscr{G})_r) = \mathscr{H}om(\mathscr{F}_q, \Phi^s_{\widehat{H}_0}(\mathscr{G}_r)) = \mathscr{H}om(\mathscr{F}_q, \Phi^s_{\widehat{H}}(\mathscr{G}_r)).$$

We try to show that

$$d(u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r), u_*\mathscr{H}om(\mathscr{F}_q, (\Phi^1_H \mathscr{G})_r) \leq d(\mathscr{G}_r, \Phi^1_H(\mathscr{G}_r)).$$

Namely, if $\mathscr{G}_r, \mathscr{G}'_r$ are (ϵ, ϵ')-interleaved, then $u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)$, $u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{G}'_r)$ will also be (ϵ, ϵ')-interleaved. Let $T_c(x,t,u) = (x, t+c, u)$ and $U_c(x,t,u) = (x,t,u-c)$. Then since $r \circ T_c = r \circ U_c$ and $q = q \circ U_c$,

$$\mathscr{H}om(\mathscr{F}_q, T_{c,*}\mathscr{G}_r) = \mathscr{H}om(U_{c,*}\mathscr{F}_q, U_{c,*}\mathscr{G}_r) = U_{c,*}\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r).$$

For any morphism $\mathscr{G}_r \to T_{c,*}\mathscr{G}'_r$ there is a canonical morphism

$$\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r) \to \mathscr{H}om(\mathscr{F}_q, T_{c,*}\mathscr{G}'_r).$$

Therefore there is always a canonical morphism

$$\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r) \to U_{c,*}\mathscr{H}om(\mathscr{F}_q, \mathscr{G}'_r).$$

By abuse of notations, we also write $U_c : \mathbb{R} \to \mathbb{R}$, $u \mapsto u - c$. Note that $u \circ U_c = U_c$, so one will have a canonical morphism

$$u_* \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r) \to U_{c,*} u_* \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r').$$

This shows that if $\mathscr{G}_r, \mathscr{G}_r'$ are (ϵ, ϵ')-interleaved, then $u_* \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)$, $u_* \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r')$ will also be (ϵ, ϵ')-interleaved, and hence completes the proof. □

Here are two examples about $\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(x_0,t_0)}, \mathscr{F})$ for the skyscraper sheaf $\Bbbk_{(x_0,t_0)}$ and $\mathscr{F} \in Sh^b_\Lambda(\mathbb{R}^2)$. We will see that $\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(x_0,t_0)}, \mathscr{F})$ detects Reeb chords between Λ and the cotangent fiber $T^{*,\infty}_{(x_0,t_0)}(M \times \mathbb{R})$.

Note that although $\Bbbk_{(x_0,t_0)} \notin Sh_{\tau>0}(\mathbb{R}^2)$, one can still apply the same argument in Proposition 3.1.7 and Lemma 4.1.4, and find that $\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(x_0,t_0)}, \mathscr{F}) \in Sh_{\nu<0}(\mathbb{R})$.

Example 5.2.3. *The first example is about birth-death of Reeb chords (Figure 5.4 right). We consider a family of Legendrians* $\Lambda_s = \{(x, \pm 3(x+s)^{1/2}/2, (x+s)^{3/2}) | x + s \geq 0\} \subset J^1(\mathbb{R})$ *whose front projections are standard cusps* $\{(x,t) | t^2 = (x+s)^3\}$. *Consider Reeb chords from* Λ_s *to the fiber* $T^{*,\infty}_{(0,1)}\mathbb{R}^2$. *At* $s = 0$, *a pair of Reeb chords are created.*

For $F \in Mod(\Bbbk)$, *consider the sheaf*

$$\mathscr{F}_s = F_{\{(x,t)|0 \leq t < (x+s)^{3/2} \ or \ (x+s)^{3/2} \leq t < 0\}}.$$

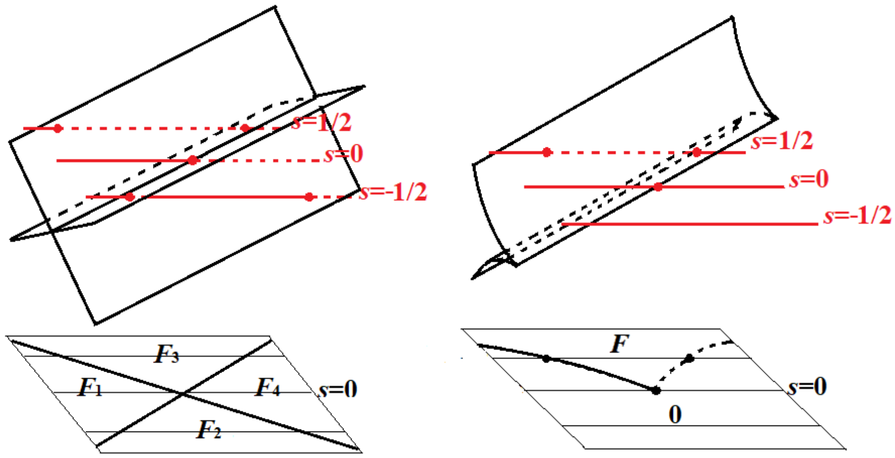

Figure 5.4. Birth-death of Reeb chords (on the right) and swapping of Reeb chords (on the left). On the top, the black Legendrians are $(\Lambda_s)_r$ while the red curves are $(T^{*,\infty}_{(0,1)}\mathbb{R}^2)_q$. The u-axis is horizontal, the t-axis is vertical, while the s-axis is pointing into the blackboard.

Then consider $u_*\mathscr{H}om((\Bbbk_{(0,1)})_q, \mathscr{F}_r)$. One can see that

$$u_*\mathscr{H}om((\Bbbk_{(0,1)})_q, (\mathscr{F}_s)_r)_{u=c} = \Gamma(\mathbb{R}, \mathscr{H}om(\Bbbk_{(0,1)}, T_{c,*}\mathscr{F}_s)) = \mathscr{F}_s|_{(x,t)=(0,1-c)}.$$

Therefore when $s \leq 0$, we have $\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(0,1)}, \mathscr{F}_s) = 0$. When $s > 0$,

$$\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(0,1)}, \mathscr{F}_s) = F_{(1-s^{3/2}, 1+s^{3/2}]}.$$

In other words, the birth of Reeb chords creates a new bar.

When the Hamiltonian isotopy swaps the length of two Reeb chords, the behaviour of the sheaf $\mathscr{H}om_{(-\infty,+\infty)}(-,-)$ under the isotopy may be more complicated. However, there are still very specific cases where the behaviour is relatively clear.

Example 5.2.4. *The second example is a specific case of swapping of Reeb chords (Figure 5.4 left). We consider a family of Legendrians* $\Lambda_s = \{(x, \pm 1, \pm(x+s))|x \in \mathbb{R}\} \subset J^1(\mathbb{R})$ *whose front projections are standard crossings* $\{(x,t)|t = \pm(x+s)\}$*. Consider Reeb chords from* Λ_s *to the fiber* $T^{*,\infty}_{(0,1)}\mathbb{R}^2$*. At* $s = 0$*, a pair of Reeb chords are swapped.*

For $F_1, F_2, F_3, F_4 \in Mod(\Bbbk)$*, suppose for* $\mathscr{F} = \mathscr{F}_0$*,*

$$\mathscr{F}|_{\{(x,y)|t\geq|x|\}} = F_1|_{\{(x,y)|t\geq|x|\}},\ \mathscr{F}|_{\{(x,y)|t<-|x|\}} = F_4|_{\{(x,y)|t<-|x|\}}$$

$$\mathscr{F}|_{\{(x,y)|x<0,-t<x\leq t\}} = F_2|_{\{(x,y)|x<0,-t<x\leq t\}},\ \mathscr{F}|_{\{(x,y)|x>0,-t<x\leq t\}} = F_1|_{\{(x,y)|x>0,-t<x\leq t\}}.$$

The sheaf $\mathscr{F}$ *is characterized by the diagram (see Example 3.1.5 or* [**148**, Section 3.3]*)*

$$\begin{array}{ccc} F_1 & \longrightarrow & F_3 \\ \downarrow & & \downarrow \\ F_2 & \longrightarrow & F_4. \end{array}$$

such that this is a (homotopy) push out diagram (since we consider complexes of sheaves, this means that $\mathrm{Tot}\,(F_1 \to F_2 \oplus F_3 \to F_4) \simeq 0$*; see* [**148**, Section 3.3 & 3.4]*). Then* $u_*\mathscr{H}om((\Bbbk_{(0,1)})_q, \mathscr{F}_r)_{u=c} = \mathscr{F}_s|_{(x,t)=(0,1-c)}$*. For* $s < 0$*,* $\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(0,1)}, \mathscr{F}_s)$ *is determined by the diagram*

$$F_1 \longrightarrow F_2 \longrightarrow F_4.$$

When $s > 0$*,* $\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(0,1)}, \mathscr{F}_s)$ *is characterized by the diagram*

$$F_1 \longrightarrow F_3 \longrightarrow F_4.$$

Decomposing the sheaf as $\bigoplus_{\alpha\in A} \Bbbk^{r_\alpha}_{(a_\alpha,b_\alpha]}[n_\alpha]$*, we have for* $s < 0$*,*

$$\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(0,1)}, \mathscr{F}_s) \simeq V_{(-\infty,+\infty)} \oplus V_{(-\infty,-s]} \oplus V_{(-\infty,s]} \\ \oplus V_{(-s,s]} \oplus V_{(-s,+\infty)} \oplus V_{(s,+\infty)}.$$

When $s > 0$*,*

$$\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(0,1)}, \mathscr{F}_s) \simeq U_{(-\infty,+\infty)} \oplus U_{(-\infty,-s]} \oplus U_{(-\infty,s]} \\ \oplus U_{(-s,s]} \oplus U_{(-s,+\infty)} \oplus U_{(s,+\infty)}.$$

Using the condition $\mathrm{Tot}\,(F_1 \to F_2 \oplus F_3 \to F_4) \simeq 0$*, one can show that*

$$V_{(-s,s]} \simeq U_{(-s,s]} \simeq 0,$$
$$V_{(-\infty,-s]} \simeq U_{(-\infty,s]},\ V_{(-s,+\infty)} \simeq U_{(s,+\infty)},$$
$$V_{(-\infty,s]} \simeq U_{(-\infty,-s]},\ V_{(s,+\infty)} \simeq U_{(-s,+\infty)},$$
$$V_{(-\infty,+\infty)} \simeq U_{(-\infty,+\infty)}.$$

Hence in this specific case, swapping of Reeb chords swaps starting/ending points of bars (Caution: this may not be true in general).

5.3. Application to Estimations of Reeb Chords

Our goal in this section is to relate the number of Reeb chords with $Hom_+(\mathscr{F}, \mathscr{F})$ and $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \mathscr{F})$, and hence finish the proof of Theorem 5.0.10, 5.0.11 and 5.0.12.

5.3.1. Local Calculation for Microstalks

By Lemma 4.1.4, we know that certain covectors in the microsupport of $\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)$ correspond to Reeb chords. The microlocal Morse inequality (Proposition 3.1.4) relates the global section of sheaves to its microstalks. Hence it suffices to determine if the ranks of the microstalks

$$\Gamma_{u \leq u_i}(\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))_{(x_i,t_i,u_i)}$$

are as expected. This will follow from concrete local calculations. Here is the main result.

Proposition 5.3.1. *For $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ a chord generic Legendrian and $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ a sheaf with perfect microstalk F, let $\{(x_i, 0, t_i, 0, u_i, \nu_i)\}_{i \in I}$ be the set*

$$((-\Lambda_q) + \Lambda_r) \cap \{(x, 0, t, 0, u, \nu) | u > 0, \nu < 0\}.$$

Suppose (x_i, t_i, u_i) corresponds to a degree d_i Reeb chord in Lemma 4.1.4. Then

$$\Gamma_{u \leq u_i}(\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))_{(x_i,t_i,u_i)} \simeq Hom(F, F)[-d_i].$$

First of all, let's recall from Section 2.3.2 that the degree of a Reeb chord $\gamma \in \mathcal{Q}_+(\Lambda)$ is defined as follows. Suppose at $a = (x, \xi, t, \tau)$ and $b = (x, \xi, t+u, \tau)$ $(u > 0)$,

$$n - \deg(\gamma) = d(a) - d(b) + \mathrm{ind}(D^2 h_{ab}) - 1,$$

where $d(b), d(a)$ are Maslov potentials at b, a, and $h_{ab} = h_b - h_a$ for h_b, h_a whose graphs at b, a are $\pi_{\mathrm{front}}(\Lambda)$. By Morse lemma, we assume that in local coordinates

$$h_b(x) = u, \;\; h_a(x) = -\sum_{i \leq k} x_i^2 + \sum_{j \geq k+1} x_j^2.$$

Next, by microlocal Morse lemma as in Example 3.2.4 we consider

$$\Gamma_{u \leq u_i}(\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))_{(x_i, t_i, u_i)} = \mathrm{Cone}(\Gamma(U_{x_i,t_i} \times (u_i - \epsilon, u_i + \epsilon), \mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)) \\ \to \Gamma(U_{x_i,t_i} \times (u_i, u_i + \epsilon), \mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)))[-1].$$

Since $((-\Lambda_q) + \Lambda_r) \cap \{(x, 0, t, 0, u, \nu) | u > 0, \nu > 0\} = \emptyset$, we know that

$$\Gamma(U_{x_i,t_i} \times (u_i - \epsilon, u_i + \epsilon), \mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)) \simeq \Gamma(U_{x_i,t_i} \times (u_i - \epsilon, u_i), \mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)).$$

Hence it suffices to calculate

$$\mathrm{Cone}\big(\Gamma(U_{x_i,t_i} \times (u_i - \epsilon, u_i), \mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r)) \\ \to \Gamma(U_{x_i,t_i} \times (u_i, u_i + \epsilon), \mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))\big)[-1].$$

Note that $(\Lambda_q \cup \Lambda_r) \cap T^{*,\infty}(U_{x_i,t_i} \times (u_i - \epsilon, u_i))$ and $(\Lambda_q \cup \Lambda_r) \cap T^{*,\infty}(U_{x_i,t_i} \times (u_i, u_i + \epsilon))$ are movies of Legendrian isotopies. Hence by Guillermou-Kashiwara-Schapira Theorem

3.3.1, it suffices to compute

$$\mathrm{Cone}\big(\Gamma(U_{x_i,t_i}\times\{u_i-\epsilon/2\},\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r)))$$
$$\to\Gamma(U_{x_i,t_i}\times\{u_i+\epsilon/2\},\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r))\big)[-1]$$

(as long as we can keep track of the restriction map). From now on, we write

$$U^-=U_{x_i,t_i}\times\{u_i-\epsilon/2\},\ \ U^+=U_{x_i,t_i}\times\{u_i+\epsilon/2\}.$$

Since $(-\Lambda_q)\cap\Lambda_r=\emptyset$, by Proposition 3.1.7

$$\mathscr{H}om(\mathscr{F}_q,\mathscr{F}_r)\simeq \mathrm{D}'\mathscr{F}_q\otimes\mathscr{F}_r,$$

where $\mathrm{D}'\mathscr{F}_q\in Sh_{-\Lambda_q}(M\times\mathbb{R}^2)$. Now write

$$U^\pm\cap\{(x,t)|t>h_b(x)\}=U_{q,0},\ U^\pm\cap\{(x,t)|t\leq h_b(x)\}=U_{q,1},$$

$$U^\pm\cap\{(x,t)|t<h_a(x)+u_i\pm\epsilon/2\}=U^\pm_{r,0},\ U^\pm\cap\{(x,t)|t\geq h_a(x)+u_i\pm\epsilon/2\}=U^\pm_{r,1}.$$

Without loss of generality by microlocal Morse lemma, as in Example 3.1.5 or [**148**, Section 3.3] we may assume

$$\mathrm{D}'\mathscr{F}_q|_{U_{q,0}}\simeq Q_0|_{U_{q,0}},\ \mathrm{D}'\mathscr{F}_q|_{U_{q,1}}\simeq Q_1|_{U_{q,1}},$$
$$\mathscr{F}_r|_{U^\pm_{r,0}}\simeq R_0|_{U^\pm_{r,0}},\ \mathscr{F}_r|_{U^\pm_{r,1}}\simeq R_1|_{U^\pm_{r,1}}.$$

In addition here we claim that

$$\mathrm{Cone}(Q_1\to Q_0)\simeq \mathrm{D}'F[-d(b)],\ \ \mathrm{Cone}(R_1\to R_0)\simeq F[d(a)+1].$$

Lemma 5.3.2. *Let* $\mathscr{F} \in Sh^b_{T^{*,\infty}_{\mathbb{R}^n \times \mathbb{R}_{>0},-}\mathbb{R}^{n+1}}(\mathbb{R}^{n+1})$ *and* $\varphi(x,t) = t$. *Then*

$$R\Gamma_{\varphi \leq 0}(\mathrm{D}'\mathscr{F})_{(0,\dots,0)} = \mathrm{D}' R\Gamma_{\varphi \geq 0}(\mathscr{F})_{(0,\dots,0)}[-1].$$

Proof. We assume that $\mathscr{F}|_{\mathbb{R}^n \times [0,+\infty)} = F_1|_{\mathbb{R}^n \times [0,+\infty)}$ and $\mathscr{F}|_{\mathbb{R}^n \times (-\infty,0)} = F_0|_{\mathbb{R}^n \times (-\infty,0)}$. Then we have an exact triangle

$$\Gamma_{\varphi \geq 0}(\mathscr{F})_{(0,0)} \to F_1 \to F_0 \xrightarrow{+1}.$$

Therefore by taking the dual we have

$$\mathrm{D}'F_0 \to \mathrm{D}'F_1 \to \mathrm{D}'\Gamma_{\varphi \geq 0}(\mathscr{F})_{(0,0)} \xrightarrow{+1}.$$

However, we claim $\mathrm{D}'\mathscr{F}|_{\mathbb{R}^n \times (0,+\infty)} = \mathrm{D}'F_1|_{\mathbb{R}^n \times (0,+\infty)}$ and $\mathrm{D}'\mathscr{F}|_{\mathbb{R}^n \times (-\infty,0]} = \mathrm{D}'F_0|_{\mathbb{R}^n \times (-\infty,0]}$. We will only check the stalk at $\mathbb{R}^n \times \{0\}$ (other stalks can be computed easily). In fact

$$\mathrm{D}'\mathscr{F}_{(0,0)} = \Gamma(\{(0,...,0)\}, \mathscr{H}om(\mathscr{F}, \Bbbk)) \simeq \Gamma(\mathbb{R}^{n+1}, \mathscr{H}om(\mathscr{F}, \Bbbk)),$$

and the latter consists of morphisms $\phi_1 : F_1 \to \Bbbk, \phi_0 : F_0 \to \Bbbk$ such that the following diagram commutes:

$$\begin{array}{ccccc} F_1 & \xleftarrow{\sim} & F_1 & \longrightarrow & F_0 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ \Bbbk & \xleftarrow{\sim} & \Bbbk & \xrightarrow{\sim} & \Bbbk. \end{array}$$

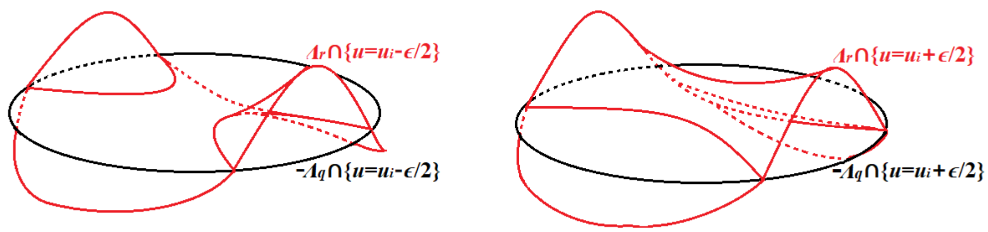

Figure 5.5. When $n = 2$ and $k = 1$, the open subsets U^- (on the left) and U^+ (on the right).

Such pairs (ϕ_1, ϕ_0) corresponds bijectively to $\phi_0 : F_0 \to \Bbbk$ (ϕ_1 will just be the composition of ϕ_0 and the restriction map $F_1 \to F_0$), so

$$Hom(\mathscr{F}, \Bbbk) \simeq Hom(F_0, \Bbbk) = \mathrm{D}' F_0.$$

Therefore we know that

$$\Gamma_{\varphi \leq 0}(\mathrm{D}'\mathscr{F})_{(0,\dots,0)} = \mathrm{Cone}(\mathrm{D}'F_0 \to \mathrm{D}'F_1)[-1] \simeq \mathrm{D}'\Gamma_{\varphi \geq 0}(\mathscr{F})_{(0,\dots,0)}[-1].$$

This proves the assertion. $\square$

Suppose first $0 < k < n$. Now we compute $R\Gamma(U_{(x_i,t_i)} \times \{u_i \pm \epsilon/2\}, R\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))$ separately. At $u = u_i - \epsilon/2$, we know that

$$U_{q,1} \cap U_{r,0}^- \cong D^k \times D^{n-k} \times (0,1], U_{q,0} \cap U_{r,1}^- \cong D^k \times D^{n-k} \times [0,1),$$

$$U_{q,1} \cap U_{r,1}^- \cong D^k \times D^{n-k} \times [0,1],\ U_{q,0} \cap U_{r,0}^- \cong D^k \times (S^{n-k-1} \times (0,1)) \times (0,1).$$

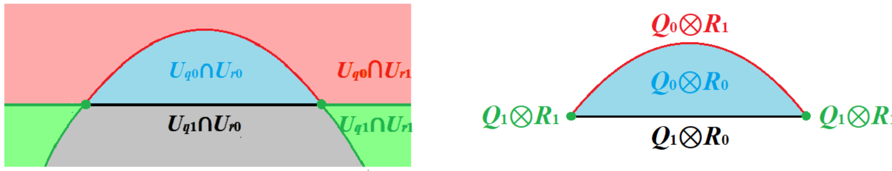

Figure 5.6. When $n = 2$ and $k = 1$, the stratification on $\overline{U_{q,0} \cap U^+_{r,0}} \subset U^+$.

At $u = u_i + \epsilon/2$, we know that

$$U_{q,1} \cap U^+_{r,0} \cong D^k \times D^{n-k} \times (0, 1],\ U_{q,0} \cap U^+_{r,1} \cong D^k \times D^{n-k} \times [0, 1),$$

$$U_{q,1} \cap U^+_{r,1} \cong (S^{k-1} \times [0, 1)) \times D^{n-k} \times [0, 1], U_{q,0} \cap U^+_{r,0} \cong D^k \times D^{n-k} \times (0, 1),$$

and the boundary regions around $\overline{U_{q,0} \cap U^-_{r,0}}$ are (Figure 5.6)

$$U_{q,1} \cap U^-_{r,0} \cap \overline{U_{q,0} \cap U^-_{r,0}} = D^k_- \times D^{n-k},$$

$$U_{q,0} \cap U^-_{r,1} \cap \overline{U_{q,0} \cap U^-_{r,0}} = D^k_+ \times D^{n-k},$$

$$U_{q,1} \cap U^-_{r,1} \cap \overline{U_{q,0} \cap U^-_{r,0}} = S^{k-1} \times D^{n-k},$$

where $D^k_- \subset S^k$ is the lower hemisphere and $D^k_+ \subset S^k$ is the upper hemisphere.

Proof of Proposition 5.3.1. Suppose first that $0 < k < n$. At $u = u_i - \epsilon/2$, since $(-\Lambda_q) \cap \Lambda_r \cap \nu^{*,\infty}_{U_{q,1}\cap U^-_{r,1},+}(M \times \mathbb{R}) = \emptyset$ (recall $\nu^{*,\infty}_{U_{q,1}\cap U^-_{r,1},+}(M \times \mathbb{R})$ is the outward conormal), we know by microlocal Morse lemma that

$$\Gamma(U^-, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r) = \Gamma(U_{q,1} \cap U^-_{r,1}, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r) \simeq Q_1 \otimes R_1.$$

At $u = u_i + \epsilon/2$, since $(-\Lambda_q) \cap \Lambda_r \cap \nu^{*,\infty}_{U_{q,0} \cap U^+_{r,0},+}(M \times \mathbb{R}) = \emptyset$, we also know that (see Figure 5.6)

$$\Gamma(U^+, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r) = \Gamma(\overline{U_{q,0} \cap U^+_{r,0}}, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r).$$

Here $\overline{U_{q,0} \cap U^+_{r,0}} = \overline{D}^{k+1} \times D^{n-k}$ with a statification $D^{k+1} \times D^{n-k}, D^k_\pm \times D^{n-k}$ and $S^{k-1} \times D^{n-k}$. In addition

$$\mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r|_{D^{k+1} \times D^{n-k}} = Q_0 \otimes R_0,$$
$$\mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r|_{D^k_+ \times D^{n-k}} = Q_0 \otimes R_1,\ \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r|_{D^k_- \times D^{n-k}} = Q_1 \otimes R_1,$$
$$\mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r|_{S^{k-1} \times D^{n-k}} = Q_1 \otimes R_1.$$

It suffices for us to do calculations on $\overline{D}^{k+1}$, so from now on we will drop all the D^{n-k} terms. In order to calculate the (derived) global sections using Čech cohomology, we need to consider a refinement of the current stratification on $\overline{U_{q,0} \cap U^+_{r,0}}$, whose stars give a good cover (meaning that any finite intersection is contractible) of the region. We consider the stratification of S^{k-1} by $\partial\Delta^k$, whose stars are

$$\mathrm{St}\Delta_{[i_1 i_2 \dots i_v]} = \bigcup_{\{i_1, i_2, \dots, i_v\} \subset \{i'_1, i'_2, \dots, i'_u\} \subset \{0,1,\dots,k+1\}} \Delta_{[i'_1 i'_2 \dots i'_u]} = \bigcap_{1 \le j \le v} \mathrm{St}\Delta_{[i_j]}.$$

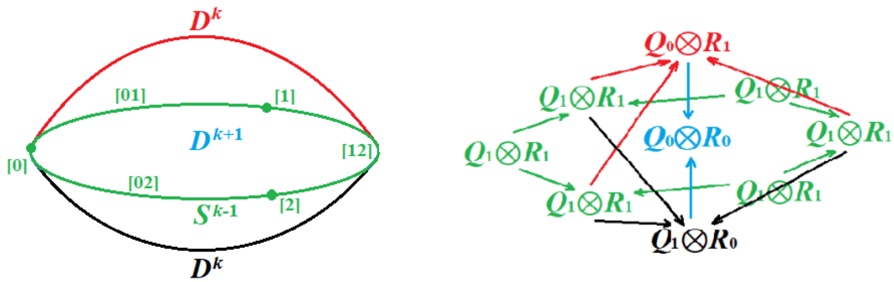

Figure 5.7. The stratification in the case $k = 2$ (left), and the restriction maps pointing from lower dimensional strata to higher dimensional strata (right). These are restriction maps because given the triangulation, the stars of lower dimensional ones contain stars of higher dimensional ones. The green indices on the left are labels of the simplices $\partial\Delta^k$.

Consider the stars which give a good cover (Figure 5.7 left). Therefore the (derived) global section is (Figure 5.7 right)

$$\Gamma\big(\overline{U_{q,0} \cap U_{r,0}^{+}}, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r\big)$$
$$\simeq \mathrm{Colim}\big((Q_1 \otimes R_1)^{\oplus k+1} \to (Q_1 \otimes R_1)^{\oplus (k+1)k/2} \to \ldots$$
$$\to (Q_1 \otimes R_1)^{\oplus k+1} \to (Q_0 \otimes R_1) \oplus (Q_1 \otimes R_0) \to (Q_0 \otimes R_0)\big).$$

Before starting to compute the microstalk, we need to keep track of the restriction functor

$$Q_1 \otimes R_1 \to \mathrm{Colim}\big((Q_1 \otimes R_1)^{\oplus k+1} \to (Q_1 \otimes R_1)^{\oplus (k+1)k/2} \to \ldots$$
$$\to (Q_1 \otimes R_1)^{\oplus k+1} \to (Q_0 \otimes R_1) \oplus (Q_1 \otimes R_0) \to (Q_0 \otimes R_0)\big).$$

Note that in $U = U_{(x_i,t_i)} \times (u_i - \epsilon, u_i + \epsilon)$, the $Q_1 \otimes R_1$ term is supported on $V \simeq D^k \times D^{n-k} \times [0,1] \times (0,1]$, where the restriction map is the one induced by

$$C^*(D^k \times D^{n-k} \times [0,1] \times (0,1]; \mathbb{K}) \to C^*((S^{k-1} \times [0,1)) \times D^{n-k} \times [0,1]; \mathbb{K}),$$

which is homotopic to the restriction map $C^*(\Delta^k; \mathbb{K}) \to C^*(\partial\Delta^k; \mathbb{K})$, where $\mathbb{K} = Q_1 \otimes R_1$. Hence the restriction map is just the diagonal map

$$\begin{aligned} \delta: \quad Q_1 \otimes R_1 &\to (Q_1 \otimes R_1)^{\oplus k+1}, \\ x &\mapsto (x, x, ..., x). \end{aligned}$$

Since the cone of the restriction map is

$$\mathrm{Cone}\big(C^*(\Delta^k; \mathbb{K}) \to C^*(\partial\Delta^k; \mathbb{K})\big) \simeq C^*(\Delta^k, \partial\Delta^k; \mathbb{K}) \simeq \Bbbk[-k],$$

we are able to calculate the microstalk:

$$
\begin{aligned}
&\Gamma_{u\le u_i}(\mathrm{D}'\mathscr{F}_q\otimes\mathscr{F}_r)_{(x_i,t_i,u_i)}\\
&\quad\simeq \mathrm{Cone}\big((Q_1\otimes R_1)\to \mathrm{Colim}\big((Q_1\otimes R_1)^{\oplus k+1}\to ...\\
&\qquad\qquad \to (Q_1\otimes R_1)^{\oplus k+1}\to (Q_0\otimes R_1)\oplus(Q_1\otimes R_0)\to(Q_0\otimes R_0)\big)\big)[-1]\\
&\quad\simeq \ \mathrm{Tot}\,\big((Q_1\otimes R_1)\to (Q_1\otimes R_1)^{\oplus k+1}\to ...\\
&\qquad\qquad \to (Q_1\otimes R_1)^{\oplus k+1}\to (Q_0\otimes R_1)\oplus(Q_1\otimes R_0)\to(Q_0\otimes R_0)\big)\\
&\quad\simeq \ \mathrm{Tot}\,\big(0\to 0\to ...\to (Q_1\otimes R_1)\to\\
&\qquad\qquad \to (Q_0\otimes R_1)\oplus(Q_1\otimes R_0)\to(Q_0\otimes R_0)\big)\\
&\quad\simeq \ \mathrm{Tot}\,\big((Q_1\otimes R_1)\to (Q_0\otimes R_1)\oplus(Q_1\otimes R_0)\to(Q_0\otimes R_0)\big)[-k].
\end{aligned}
$$

By Kunneth's formula, we can conclude that

$$
\begin{aligned}
&\mathrm{Tot}\,\big((Q_1\otimes R_1)\to (Q_0\otimes R_1)\oplus(Q_1\otimes R_0)\to(Q_0\otimes R_0)\big)[-k]\\
&\quad\simeq \mathrm{Tot}\big(Q_1\to Q_0\big)\otimes\mathrm{Tot}\big(R_1\to R_0\big)[-k]\\
&\quad\simeq (\mathrm{D}'F[-d(b)-1]\otimes F[d(a)])\,[-k]=\mathrm{D}'F\otimes F[d_i].
\end{aligned}
$$

Finally the only case left is the case when $k=0$ or n. The strategy is the same. When $k=n$, the sections at $u=u_i-\epsilon/2$ are

$$
\Gamma(U^-,\mathrm{D}'\mathscr{F}_q\otimes\mathscr{F}_r)=\Gamma(\overline{U_{q,1}\cap U_{r,1}^-},\mathrm{D}'\mathscr{F}_q\otimes\mathscr{F}_r)\simeq Q_1\otimes R_1.
$$

The sections at $u = u_i + \epsilon/2$ are

$$\Gamma(U^+, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r) = \Gamma(\overline{U_{q,0} \cap U^+_{r,0}}, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r)$$

$$\simeq \mathrm{Colim}\big((Q_1 \otimes R_1)^{\oplus n+1} \to (Q_1 \otimes R_1)^{\oplus (n+1)n/2} \to ...$$

$$\to (Q_1 \otimes R_1)^{\oplus n+1} \to (Q_0 \otimes R_1) \oplus (Q_1 \otimes R_0) \to (Q_0 \otimes R_0)\big)[-1].$$

Hence by the same argument using Kunneth's formula, the microstalk is

$$\Gamma_{u \le u_i}(\mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r)_{(x_i,t_i,u_i)}$$

$$\simeq \mathrm{Tot}\,\big((Q_1 \otimes R_1) \to (Q_1 \otimes R_1)^{\oplus n+1} \to ...$$

$$\to (Q_1 \otimes R_1)^{\oplus n+1} \to (Q_0 \otimes R_1) \oplus (Q_1 \otimes R_0) \to (Q_0 \otimes R_0)\big)$$

$$\simeq \mathrm{Tot}\,\big((Q_1 \otimes R_1) \to (Q_0 \otimes R_1) \oplus (Q_1 \otimes R_0) \to (Q_0 \otimes R_0)\big)[-n]$$

$$\simeq \mathrm{Tot}\big(Q_1 \to Q_0\big) \otimes \mathrm{Tot}\big(R_1 \to R_0\big)[-n]$$

$$\simeq (\mathrm{D}'F[-d(b)-1] \otimes F[d(a)])\,[-n] = \mathrm{D}'F \otimes F[d_i].$$

When $k = 0$, the sections at $u = u_i - \epsilon/2$ are

$$\Gamma(U^-, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r) = \Gamma(\overline{U_{q,1} \cap U^-_{r,1}}, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r) \simeq Q_1 \otimes R_1.$$

The sections at $u = u_i + \epsilon/2$ are

$$\Gamma(U^+, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r) = \Gamma(\overline{U_{q,0} \cap U^+_{r,0}}, \mathrm{D}'\mathscr{F}_q \otimes \mathscr{F}_r)$$

$$\simeq \mathrm{Colim}\left((Q_1 \otimes R_0) \oplus (Q_0 \otimes R_1) \to (Q_0 \to R_0)\right).$$

Therefore the microstalk is

$$\begin{aligned}
&\Gamma_{u\le u_i}(\mathrm{D}'\mathscr{F}_q\otimes\mathscr{F}_r)_{(x_i,t_i,u_i)}\\
&\qquad\simeq \mathrm{Cone}\big(Q_1\otimes R_1\to \mathrm{Colim}\left((Q_1\otimes R_0)\oplus(Q_0\otimes R_1)\to(Q_0\to R_0)\right)\big)[-1]\\
&\qquad\simeq \mathrm{Tot}\big(Q_1\otimes R_1\to(Q_1\otimes R_0)\oplus(Q_0\otimes R_1)\to(Q_0\to R_0)\big)\\
&\qquad\simeq \mathrm{Tot}\big(Q_1\to Q_0\big)\otimes\mathrm{Tot}\big(R_1\to R_0\big)\\
&\qquad\simeq \mathrm{D}'F[-d(b)-1]\otimes F[d(a)]=\mathrm{D}'F\otimes F[-d_i].
\end{aligned}$$

Hence the proof is completed. □

When $u<0$, we consider $\{(x_i,0,t_i,0,u_i,\nu_i)\}_{i\in I}$ be the set

$$((-\Lambda_q)+\Lambda_r)\cap\{(x,0,t,0,u,\nu)|u<0,\nu<0\}.$$

The calculation in Proposition 5.3.1 still holds, except that we have to be careful about the gradings.

We always assume that in our local model, when u increases, the point a is moving up in the horizontal u-direction passing through b. In the case of $u>0$, the point $(x_i,0,t_i,0,u_i,\nu_i)$ comes from a Reeb chord connecting a to b where b is above a, and as $u>0$ increases from 0, b is fixed and a is moving up. $\mathrm{Graph}(h_b),\mathrm{Graph}(h_a)$ are local models of $\pi_{\mathrm{front}}(\Lambda)$ at b,a, and in local coordinates

$$h_b(x)=u_i>0,\ \ h_a(x)=-\sum_{i\le k}x_i^2+\sum_{j\ge k+1}x_j^2.$$

However in the case of $u < 0$, the point $(x_i, 0, t_i, 0, u_i, \nu_i)$ will then come from a Reeb chord connecting b to a where a is above b, and now as $u < 0$ increases to 0, a is moving up and yet b is fixed. In local coordinates

$$h_b(x) = u_i < 0, \ \ h_a(x) = -\sum_{i \leq k} x_i^2 + \sum_{j \geq k+1} x_j^2.$$

Then that the Morse index $\mathrm{ind}(D^2 h_{ba})$ where $h_{ba} = h_a - h_b$ will become k instead of $n - k$ (the order of a and b are switched as their heights are switched). Thus if the degree of the original chord is d_i, the degree shifting will be

$$-d(b) - 1 + d(a) - k = -d(b) - 1 + d(a) - \mathrm{ind}(D^2 h_{ba}) = -n + d_i - 2.$$

Proposition 5.3.3. *For $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ a chord generic Legendrian and $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ a sheaf with perfect microstalk F, let $\{(x_i, 0, t_i, 0, u_i, \nu_i)\}_{i \in I}$ be the set*

$$((-\Lambda_q) + \Lambda_r) \cap \{(x, 0, t, 0, u, \nu) | u < 0, \nu < 0\}.$$

Suppose (x_i, t_i, u_i) corresponding to a degree d_i Reeb chord in the bijection defined in Lemma 4.1.4. Then

$$\Gamma_{u \leq u_i}(\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r))_{(x_i, t_i, u_i)} \simeq Hom(F, F)[-n + d_i - 2].$$

5.3.2. Application to the Morse Inequality

Combining the previous propositions, we are able to prove the main theorems 5.0.10 and 5.0.11 using duality exact sequence. The main ingredient for these theorems will be the following Morse inequalities.

Theorem 5.3.4. *For $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ a closed chord generic Legendrian and $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ a microlocal rank r sheaf, let $\mathcal{Q}_j(\Lambda)$ be the set of degree j Reeb chords on Λ. Suppose* $\mathrm{supp}(\mathscr{F})$ *is compact. Then for any $k \in \mathbb{Z}$*

$$r^2 \sum_{j\leq k}(-1)^{k-j}|\mathcal{Q}_j(\Lambda)| \geq \sum_{j\leq k}(-1)^{k-j} \dim H^j Hom_+(\mathscr{F}, \mathscr{F}).$$

In particular, for any $j \in \mathbb{Z}$, $r^2|\mathcal{Q}_j(\Lambda)| \geq \dim H^j Hom_+(\mathscr{F}, \mathscr{F})$.

Theorem 5.3.5. *For $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ a closed chord generic Legendrian and $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ a sheaf with prefect microstalk F, let $\mathcal{Q}_j(\Lambda)$ be the set of degree j Reeb chords on Λ. Suppose* $\mathrm{supp}(\mathscr{F})$ *is compact. Then for any $k \in \mathbb{Z}$*

$$\sum_{j\leq k}(-1)^{k-j}\sum_{i\in\mathbb{Z}} \dim H^i Hom(F,F)|\mathcal{Q}_{j-i}(\Lambda)| \geq \sum_{j\leq k}(-1)^{k-j} \dim H^j Hom_+(\mathscr{F}, \mathscr{F}).$$

In particular, for any $j \in \mathbb{Z}$,

$$\sum_{i\in\mathbb{Z}} \dim H^i Hom(F,F)|\mathcal{Q}_{j-i}(\Lambda)| \geq \dim H^j Hom_+(\mathscr{F}, \mathscr{F}).$$

We now apply the results to relate persistence structure to Reeb chords. We first prove Theorem 5.0.10, 5.0.11 using persistence of $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \mathscr{F})$, and then prove Theorem 5.0.12 using the continuity of persistence of $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \Phi_H^1(\mathscr{F}))$ under Hamiltonian isotopies.

Proof of Theorem 5.0.10 and 5.0.11. Consider the sheaf $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \mathscr{F})$. We know

$$\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \mathscr{F}) = u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r) \simeq \bigoplus_{\alpha\in I} \Bbbk_{(c_\alpha,c'_\alpha]}^{r_\alpha}[n_\alpha].$$

Since $u : M \times \mathbb{R}^2 \to \mathbb{R}$ is proper on $\mathrm{supp}(\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r))$, we know that

$$\Gamma_{u\leq c}(u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r))_c \simeq u_*\Gamma_{u\leq c}(\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r))_{u^{-1}(c)}.$$

On the other hand, given a bar $\Bbbk_{(c,c']}$, we know that

$$\Gamma_{u\leq c}(\Bbbk_{(c,c']})_c \simeq \Bbbk[-1],\ \ \Gamma_{u\leq c'}(\Bbbk_{(c,c']})_{c'} \simeq \Bbbk.$$

Hence by Proposition 5.3.1 we will determine the number of starting point/ending point of bars from the rank of the microstalk.

By Corollary 5.1.3, we know that in degree $j+1$, there are at least $\dim H^j(\Lambda; \Bbbk^{r^2})$ starting points or ending points of bars at $u = 0$. The starting points of such bars should come from bars of the form $\Bbbk_{(0,c_+]}[-j]$ while the ending points of bars should come from bars of the form $\Bbbk_{(c_-,0]}[-j-1]$. By Lemma 4.1.4, the other ending point/starting point of these bars will correspond to signed lengths of Reeb chords in $\mathcal{Q}_\pm(\Lambda)$. By Proposition 5.3.1, we know that for $c_+ > 0$ that corresponds to a degree d_+ Reeb chord, the microstalk

$$\Gamma_{u\leq c_+}(u_*\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r))_{c_+}\simeq \Bbbk^{r^2}[-d_+].$$

Hence the corresponding ending point of a bar $\Bbbk_{(0,c_+]}[-j]$ should be a degree j Reeb chord. Similarly for $c_-<0$ that corresponds to a degree d_- Reeb chord, by Proposition 5.3.3 the microstalk

$$\Gamma_{u\leq c_-}(u_*\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r))_{c_-}\simeq \Bbbk^{r^2}[-n-2+d_-].$$

Hence the corresponding starting point of a bar $\Bbbk_{(c_-,0]}[-j-1]$ should be a degree $n-j$ Reeb chord. Therefore

$$r^2|\mathcal{Q}_j(\Lambda)|+r^2|\mathcal{Q}_{n-j}(\Lambda)|\geq r^2\dim H^j(\Lambda;\Bbbk).$$

This proves Theorem 5.0.10. The proof of Theorem 5.0.11 is similar. □

Finally we prove Theorem 5.0.12, which gives estimates on the Reeb chords between Λ and its Hamiltonian pushoff $\varphi_H^1(\Lambda)$ for a contact Hamiltonian flow φ_H^s $(s\in I)$.

Proof of Theorem 5.0.12. Consider the sheaf $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F},\mathscr{F})$. We know from the previous proof that starting points and ending points of bars at $u=0$ in degree $j+1$ correspond to a basis of $H^j(\Lambda;\Bbbk^{r^2})$. In addition, the corresponding ending point of a bar $\Bbbk_{(0,c_+]}[-j]$ should be a degree j Reeb chord, and the corresponding starting point of a bar $\Bbbk_{(c_-,0]}[-j-1]$ should be a degree $n-j$ Reeb chord. The lengths of these bars at time $s = 0$ will be at least

$$c_j(\Lambda) = c_{n-j}(\Lambda) = \min\{l(\gamma) | \gamma \in \mathcal{Q}_j(\Lambda) \cup \mathcal{Q}_{n-j}(\Lambda)\}.$$

Consider the Hamiltonian $\varphi_H^s\,(s \in I)$. Since

$$\|H\|_{\mathrm{osc}} < c_{j_0}(\Lambda), ..., c_{j_k}(\Lambda),$$

we know by Theorem 5.2.5 that these bars will survive in $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \Phi_H^1(\mathscr{F}))$.

We claim that each bar in $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \Phi_H^1(\mathscr{F}))$ corresponds to a Reeb chord between Λ and $\varphi_H^1(\Lambda)$. Namely the proof is similar to Lemma 4.1.4. Note that $\Lambda_q \cap (\varphi_H^1(\Lambda))_r = \emptyset$, so $(u,\nu) \in SS^\infty(\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \Phi_H^1(\mathscr{F})))$ iff

$$(x, 0, t, 0, u, \nu) \in (-\Lambda_q) + (\varphi_H^1(\Lambda))_r,$$

iff there exists $(x, \xi, t, \tau) \in \Lambda, (x, \xi, t+u, \tau) \in \varphi_H^1(\Lambda)$ (and $\nu = -\tau$). In addition, the computation of microstalks in Proposition 5.3.1 still holds. Hence the endpoints of bars count Reeb chords both from Λ to $\varphi_H^1(\Lambda)$ and from $\varphi_H^1(\Lambda)$ back to Λ, i.e. the chords between Λ and $\varphi_H^1(\Lambda)$. Thus

$$r^2|\mathcal{Q}(\Lambda, \varphi_H^1(\Lambda))| \geq r^2 \sum_{0 \leq i \leq k} \dim H^{j_i}(\Lambda; \Bbbk).$$

This completes the proof of the theorem. $\square$

5.3.3. Horizontal displaceability

As is mentioned in Remark 5.0.7, we show that for all horizontally displaceable closed Legendrians $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ with zero stalk near $M \times \{-\infty\}$ necessarily has compact support. Note that under the assumption that M is noncompact, $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ will always have compact support as the front projection $\pi(\Lambda)$ is compact in $M \times \mathbb{R}$, so we only need to consider the case where M is compact.

Recall that $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ is horizontally displaceable if there is a Hamiltonian flow φ^s_H $(s \in I)$ such that there are no Reeb chords between Λ and $\varphi^1_H(\Lambda)$.

Lemma 5.3.6. *Let $\Lambda, \Lambda' \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be closed Legendrians, and $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R}), \mathscr{F}' \in Sh^b_{\Lambda'}(M \times \mathbb{R})$ such that the stalks near $M \times \{-\infty\}$ are zero. Suppose there are no Reeb chords between Λ and Λ'. Then for any $c \in \mathbb{R}$,*

$$Hom(\mathscr{F}, T_{c,*}\mathscr{F}') \simeq 0.$$

Proof. We know that

$$\Gamma_{u\leq c}(u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{F}'_r))_c \simeq u_*\Gamma_{u\leq c}(\mathscr{H}om(\mathscr{F}_q, \mathscr{F}'_r))_{u^{-1}(c)}.$$

Therefore since there are no Reeb chords between Λ and Λ', by Lemma 4.1.4, we know that $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \mathscr{F}') = u_*\mathscr{H}om(\mathscr{F}_q, \mathscr{F}'_r)$ is a constant sheaf on $\mathbb{R}$.

Consider now $u = -c$ is sufficiently small so that the front $\pi_{M\times\mathbb{R}}(T_{-c,*}(\Lambda'))$ is below $M \times \{-C\}$. Let $i_{u=-c}$ be the inclusion $M \times \mathbb{R} \times \{-c\} \hookrightarrow M \times \mathbb{R}^2$. Then as

Proposition 3.1.7 implies that

$$i_{u=-c}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r') = \mathscr{H}om(\mathscr{F}, T_{-c,*}\mathscr{F}') \simeq \mathrm{D}'\mathscr{F} \otimes T_{-c,*}\mathscr{F}',$$

and the stalk of $\mathscr{F}$ is zero near $\pi_{M\times\mathbb{R}}(\Lambda')$, it is implied that

$$SS^\infty(i_{u=-c}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r')) \subset (-\Lambda) \subset T^{*,\infty}_{\tau<0}(M \times \mathbb{R}).$$

By microlocal Morse lemma we can conclude that

$$\Gamma(M \times \mathbb{R}, i_{u=-c}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r')) \simeq \Gamma(M \times (-\infty, -C), i_{u=-c}^{-1}\mathscr{H}om(\mathscr{F}_q, \mathscr{F}_r')) \simeq 0.$$

Since $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F}, \mathscr{F}')$ is constant this shows the assertion. □

Proposition 5.3.7. *Let M be compact. If $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ is horizontal displaceable, then any $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ that has zero stalk near $M \times \{-\infty\}$ will have compact support.*

Proof. Suppose $\mathrm{supp}(\mathscr{F})$ is noncompact. Then the fact that M is compact and that $\mathscr{F}$ has zero stalk near $M \times \{-\infty\}$ necessarily mean that there for any $T > 0$ sufficiently large, there is $x \in M$ such that $\mathscr{F}_{(x,t)} \neq 0$. Let

$$T > \sup\{t \in \mathbb{R} | \exists\, (x, \xi) \in T^*M,\, (x, \xi, t, 1) \in \Lambda\}.$$

Then $\mathscr{F}$ is locally constant on $M \times [T, +\infty)$ with nonzero stalk.

Since Λ is horizontally displaceable, there is a Hamiltonian flow φ_H^s $(s \in I)$ such that there are no Reeb chords between Λ and $\varphi_H^1(\Lambda)$. Let $\Lambda' = \varphi_H^1(\Lambda)$ and following

Theorem 3.3.1 $\mathscr{F}' = \Phi_H^1(\mathscr{F})$. $\mathscr{F}'$ is also locally constant on $M \times [C, +\infty)$ for sufficiently large $C > 0$ with nonzero stalk. By Lemma 5.3.6,

$$Hom(\mathscr{F}, T_{c,*}\mathscr{F}') \simeq 0.$$

Let $c > 0$ be sufficiently large such that the front projection $\pi_{M\times\mathbb{R}}(T_c(\Lambda'))$ is above $M \times \{C\}$. Then using the formula

$$\mathscr{H}om(\mathscr{F}, T_{c,*}\mathscr{F}') = \mathrm{D}'\mathscr{F} \otimes T_{c,*}\mathscr{F}',$$

near $\pi_{M\times\mathbb{R}}(\Lambda)$ the stalk of $\mathscr{H}om(\mathscr{F}, T_{c,*}\mathscr{F}')$ is zero. Hence

$$SS^\infty(\mathscr{H}om(\mathscr{F}, T_{c,*}\mathscr{F}')) \subset \Lambda' \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R}).$$

By microlocal Morse lemma we can conclude that

$$Hom(\mathscr{F}, T_{c,*}\mathscr{F}') \simeq \Gamma(M \times (C, +\infty), \mathscr{H}om(\mathscr{F}, T_{c,*}\mathscr{F}')) \not\simeq 0,$$

which leads to a contradiction. □

5.4. Application to Non-squeezing into Loose Legendrians

In this section we show that the C^0-limit of a smooth family of Legendrian submanifolds is not going to be stablized or loose when there exists some non-trivial sheaf theoretic invariant. Here is the definition and the theorem.

Definition 5.4.1 (Dimitroglou Rizell-Sullivan)**.** *Let $U \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ be an open subset with $H_n(U;\mathbb{Z}/2\mathbb{Z}) \not\simeq 0$. A Legendrian submanifold $\Lambda \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$ can be squeezed into U if there is a Legendrian isotopy Λ_t with $\Lambda_0 = \Lambda$ and*

$$\Lambda_1 \subset U, \;\; [\Lambda_1] \neq 0 \in H_n(U;\mathbb{Z}/2\mathbb{Z}).$$

Theorem 5.4.1. *Let $\Lambda_{loose} \subset T^{*,\infty}_{\tau>0}(\mathbb{R}^{n+1})$ be a stablized/loose Legendrian, and $\Lambda \subset T^{*,\infty}_{\tau>0}(\mathbb{R}^{n+1})$ be a Legendrian so that there exists $\mathscr{F} \in Sh^b_\Lambda(\mathbb{R}^{n+1})$ whose microstalk has odd dimensional cohomology. Then Λ cannot be squeezed into a tubular neighbourhood of Λ_{loose}.*

The idea is to detect the Legendrian Λ by a fiber $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$ as in Example 5.2.3. First we state a geometric lemma that is needed. This is proved by Dimitroglou Rizell-Sullivan [**47**]. For the concepts including formal Legendrian isotopy, loose Legendrian submanifolds and h-principles, the reader may refer to [**116**].

Lemma 5.4.2 (Dimitroglou Rizell-Sullivan)**.** *For $n \geq 2$, let $\Lambda_{loose} \subset T^{*,\infty}_{\tau>0}(\mathbb{R}^{n+1})$ be any loose Legendrian submanifold. Then for any small $A > \epsilon > 0$, Λ_{loose} is isotopic to Λ'_{loose} that satisfies the following properties:*

(1). there exists $(x_0, t_0) \in \mathbb{R}^{n+1}$ such that there are precisely 2 (transverse) Reeb chords γ_0, γ_1 from Λ'_{loose} to $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$ and

$$l(\gamma_0) - l(\gamma_1) \geq A;$$

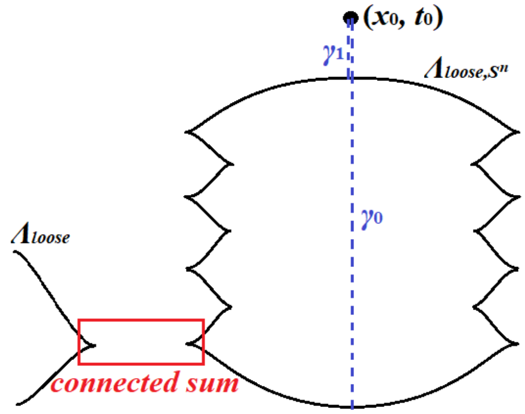

Figure 5.8. On the left there is the loose Legendrian Λ_{loose} and on the right there is a loose Legendrian $\Lambda_{S^n,\text{loose}}$ formally isotopic to the unknotted sphere (the front projection should be spinning around its symmetry axis). In the red region we perform the connected sum construction.

(2). there exists a Hamiltonian H_s $(s \in I)$ with $\|H_s\|_{osc} \leq \epsilon$ such that there are no Reeb chords between $\varphi_H^1(\Lambda'_{loose})$ and $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$.

Proof. We first construct a loose Legendrian sphere $\Lambda_{S^n,\text{loose}}$ that is formally isotopic to the standard unknot sphere $\Lambda_{S^n,\text{st}}$ and satisfies the properties in the lemma. Then we take a connected sum $\Lambda_{\text{loose}}\#\Lambda_{S^n,\text{loose}}$. In fact, since Λ_{loose} is compact, one can find $(x_0,t_0) \in \mathbb{R}^{n+1}$ such that there are no chords between Λ'_{loose} to $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$ (in other words the front projection of Λ'_{loose} is disjoint from the hypersurface $x = x_0$). We choose $\Lambda_{S^n,\text{loose}}$ to be the Legendrian sphere in Figure 5.8, where the number of zigzags is to be determined. There are precisely 2 (transverse) Reeb chords γ_0, γ_1 from Λ'_{loose} to $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$ and

$$l(\gamma_0) - l(\gamma_1) \geq A;$$

It is not hard to see that $\Lambda_{S^n,\mathrm{loose}}$ is formally isotopic to the standard unknot sphere $\Lambda_{S^n,\mathrm{st}}$ and the front projections of Λ_{loose} and $\Lambda_{S^n,\mathrm{loose}}$ are separated by some hypersurface in $\mathbb{R}^{n+1}$. Therefore one can define the connected sum $\Lambda_{\mathrm{loose}}\#\Lambda_{S^n,\mathrm{loose}}$ uniquely up to Legendrian isotopy [**44**, Proposition 4.9].

We show that $\Lambda_{\mathrm{loose}}\#\Lambda_{S^n,\mathrm{loose}}$ is formally isotopic to Λ_{loose}. This is because first $\Lambda_{S^n,\mathrm{loose}}$ is formally isotopic to $\Lambda_{S^n,\mathrm{st}}$ and this isotopy can be chosen to be fixed near the neighbourhood where the connected sum takes place. Second we perform a formal isotopy from $\Lambda_{\mathrm{loose}}\#\Lambda_{S^n,\mathrm{st}}$ to Λ_{loose}. Since locally the connected sum is defined by connecting two cusps [**44**, Section 4.2.2] (see Figure 5.8), one can explicitly see they are isotopic. This proves the claim. Hence by Murphy's h-principle [**116**] $\Lambda_{\mathrm{loose}}\#\Lambda_{S^n,\mathrm{loose}}$ is isotopic to Λ_{loose}.

This constructs $\Lambda'_{\mathrm{loose}} = \Lambda_{\mathrm{loose}}\#\Lambda_{S^n,\mathrm{loose}}$ and by the construction of $\Lambda_{S^n,\mathrm{loose}}$ we know that condition (1) holds.

Now we show condition (2), that one can choose $\Lambda_{S^n,\mathrm{loose}}$ so that $\Lambda_{\mathrm{loose}}\#\Lambda_{S^n,\mathrm{loose}}$ can be displaced from $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$ by a Hamiltonian $H_s\,(s\in I)$ with $\|H_s\|_{\mathrm{osc}}\leq\epsilon$ so that there are no longer Reeb chords between them. This is because we can add sufficiently may zigzags in $\Lambda_{S^n,\mathrm{loose}}$ such that the derivatives of the front

$$\xi_i = \partial t/\partial x_i \in (-\epsilon/2n, \epsilon/2n),\ (1\leq i\leq n)$$

are sufficiently small, i.e. $\Lambda_{S^n,\mathrm{loose}}$ is contained in a neighbourhood of $\mathbb{R}^{n+1}\subset J^1(\mathbb{R}^n)\subset T^{*,\infty}\mathbb{R}^{n+1}$. Then one can easily find a Hamiltonian $H_s\,(s\in I)$ supported in a neighbourhood $U_{\mathbb{R}^{n+1}}$ of $\mathbb{R}^{n+1}$ with $\|H_s\|_{\mathrm{osc}}\leq\epsilon$ that displaces $\Lambda_{S^n,\mathrm{loose}}$ from $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$.

For example, consider a cut-off function $\beta \in C^\infty(\mathbb{R})$ that is equal to 1 in $(-\epsilon/2, \epsilon/2)$ and 0 outside $(-\epsilon, \epsilon)$. Let $H(x, \xi, t) = \beta(|\xi|)\xi_1$, $\|H\|_{\mathrm{osc}} \le \epsilon$,

$$H|_{U_{\mathbb{R}^{n+1}}} = \xi_1,\ X_H|_{U_{\mathbb{R}^{n+1}}} = -\partial/\partial x_1.$$

This will displace $\Lambda_{S^n,\mathrm{loose}}$ from $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$. □

Next we set up the foundation of the persistence module $\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(x_0,t_0)}, \mathscr{F})$ in this case. Note that $\Bbbk_{(x_0,t_0)} \notin Sh^b_{\tau>0}(M\times\mathbb{R})$. However as long as $\mathscr{F} \in Sh^b_{\tau>0}(M \times \mathbb{R})$, all the arguments are still valid. Since Lemma 4.1.4 still holds, one can easily see that all discussions in Section 5.2 on the persistence structure still hold for the sheaf

$$\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(x_0,t_0)}, \mathscr{F}) = u_*\mathscr{H}om((\Bbbk_{(x_0,t_0)})_q, \mathscr{F}_r).$$

Proof of Theorem 5.4.1. First assume that $n \ge 2$. Suppose Λ can be squeezed into a contact neighbourhood U_{loose} of Λ_{loose}. By Lemma 5.4.2, we can apply a contact isotopy so that the contact neighbourhood U_{loose} is mapped to a contact neighbourhood U'_{loose} of $\Lambda'_{\mathrm{loose}}$. Denote by Λ' the image of the original Legendrian submanifold in U'_{loose}. By shrinking the contact neighbourhood U'_{loose} we may assume that for the projection $\pi_{\mathbb{R}^n}\circ\pi_{\mathrm{front}} : U'_{\mathrm{loose}} \to \mathbb{R}^n$, the height of each connected component of U'_{loose} in the fiber of $\pi_{\mathbb{R}^n}\circ\pi_{\mathrm{front}}$ is less than ϵ' where $4\epsilon' < A - \epsilon$.

Lemma 5.4.2 ensures that there exists $(x_0, t_0) \in \mathbb{R}^{n+1}$ such that there are precisely 2 transverse Reeb chords from $\Lambda'_{\mathrm{loose}}$ to $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$, starting from (x_0, t_1) and (x_0, t_2). For $\Lambda' \subset U'_{\mathrm{loose}}$, since the mapping degree $[\Lambda'] \neq 0 \in H_n(\Lambda'_{\mathrm{loose}}; \mathbb{Z}/2\mathbb{Z})$, the preimage of (x_0, t_1) and (x_0, t_2) under the projection $U'_{\text{loose}} \to \Lambda'_{\text{loose}}$ are $p_{1,1}, ..., p_{1,2k+1}$ and $p_{2,1}, ..., p_{2,2k+1}$, and

$$\min_{1 \leq i,j \leq 2k+1} |u(p_{1,i}) - u(p_{2,j})| \geq A - 2\epsilon'.$$

Consider the Hamiltonian $H_s\,(s \in I)$ with $\|H_s\|_{\text{osc}} \leq \epsilon + \epsilon' < A - 2\epsilon'$ and horizontally displaces Λ'_{loose} from the cotangent fiber $T^{*,\infty}_{(x_0,t_0)}\mathbb{R}^{n+1}$ as in Lemma 5.4.2. For a sufficiently small neighbourhood U'_{loose} of Λ'_{loose}, there will be a Hamiltonian $H_s\,(s \in I)$ with $\|H_s\|_{\text{osc}} \leq \epsilon + \epsilon'$ that horizontally displaces U'_{loose}. For $\mathscr{F} \in Sh_{\Lambda'}(\mathbb{R}^{n+1})$ we calculate

$$\mathscr{H}om_{(-\infty,+\infty)}(\Bbbk_{(x_0,t_0)}, \Phi^s_H(\mathscr{F})).$$

By Lemma 4.1.4, $u(p_{1,1}), ..., u(p_{1,2k+1})$ and $u(p_{2,1}), ..., u(p_{2,2k+1})$ correspond to all the starting points and ending points of the bars. In addition, for each point the number of bars $\Bbbk_{(a,b]}$ (either starting or ending there) in the sheaf is at least the rank of the microstalk of $\mathscr{F}$. Denote the rank of the microstalk of $\mathscr{F}$ by $2r + 1$. We argue that there must be a bar starting from $u(p_{1,i})$ and ending at $u(p_{2,j})$. Otherwise all bars start at some $u(p_{1,i})$ will end at some $u(p_{1,j})$ for $i \neq j$. However, there are odd number of points $u(p_{1,1}), ..., u(p_{1,2k+1})$, so there should be $(2r + 1)(2k + 1)/2$ bars connecting them, which leads to a contradiction.

Now that we know there is a bar starting from $u(p_{1,i})$ and ending at $u(p_{2,j})$, it will have length at least $A - 2\epsilon'$. By Theorem 5.2.5, under the Hamiltonian $H_s\,(s \in I)$ with $\|H_s\|_{\text{osc}} \leq \epsilon + \epsilon'$, this bar will persist since $\epsilon + \epsilon' < A - 2\epsilon'$. This leads to a contradiction.

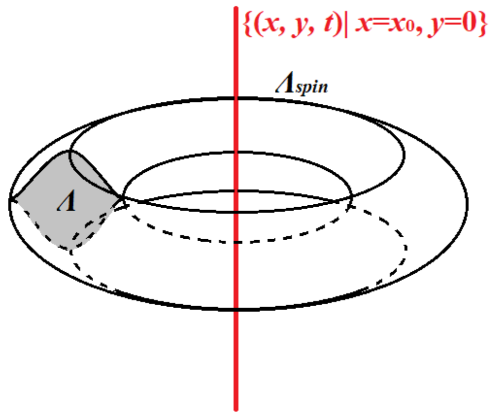

Figure 5.9. The front spinning of a standard unknot.

Finally when $n = 1$, we apply the spinning construction [**55**, Section 4.4] (Figure 5.9) to a stablized Legendrian knot: namely consider a real line $t = t_0$ that is disjoint from the front projection Λ_{loose}, Λ' and spin around the front along the line $x = x_0$. The standard zigzag thus gives a loose chart for the new Legendrian $\Lambda_{\text{loose,spin}}$ and Λ'_{spin} in $T^{*,\infty}_{\tau>0}\mathbb{R}^3$. It is clear from the front projection that, if there is a sheaf with singular support on a knot, then there is also a sheaf with singular support on its spinning. In fact, we consider $\mathbb{R}^3\backslash\{(x,y,t)|x = x_0, y = 0\} \simeq \mathbb{R}^2 \times S^1$ and the projection

$$\pi : \mathbb{R}^3\backslash\{(x,y,t)|x = x_0, y = 0\} \cong \mathbb{R}^2 \times S^1 \hookrightarrow \mathbb{R}^3.$$

Now take the sheaf $\pi^{-1}\mathscr{F}$ then $SS^\infty(\pi^{-1}\mathscr{F}) = \Lambda'_{\text{spin}}$. Note that $\text{supp}(\mathscr{F})$ is compact, so $\pi^{-1}\mathscr{F}$ has zero stalk near the line $\{(x,y,t)|x = x_0, y = 0\}$ and we can easily extend it to a sheaf on $\mathbb{R}^3$. Then applying the argument above will complete the proof. $\square$

CHAPTER 6

Adjoints of Microlocalization and Sperical Adjunction

Spherical adjunctions are introduced by Anno-Logvinenko [**8**] in the dg setting and then by [**49**] in the stable ∞ setting, as a generalization of the notion of spherical objects [**144**]. Like spherical objects, spherical adjunctions provide interesting fiber sequences and autoequivalences of the categories called spherical twists and cotwists.

In algebraic geometry, when we have a smooth variety X with a divisor $i : D \hookrightarrow X$, the push forward functor and pull back functor

$$i_* : Coh(D) \leftrightharpoons Coh(X) : i^*$$

form a spherical adjunction between the dg categories of coherent sheaves, where the spherical twist is $- \otimes \mathcal{O}_X(D)$.

In symplectic geometry, as is suggested by Kontsevich-Katzarkov-Pantev [**98**] and Seidel [**140**], we have another interesting class of spherical adjunctions inspired by long exact sequences in Floer theory [**58, 135, 138, 141**]. For a symplectic Lefschetz fibration $\pi : X \to \mathbb{C}$ with regular fiber $F = \pi^{-1}(\infty)$, let $\mathcal{FS}(X, \pi)$ be the Fukaya-Seidel category associated to Lagrangian thimbles in X and $\mathcal{F}(F)$ the Fukaya category of closed exact Lagrangians in F [**139**]. The cap functor

$$\cap_F : \mathcal{FS}(X, \pi) \to \mathcal{F}(F)$$

defined by intersection of the Lagrangians with F admits a left adjoint $\cup$ called the (left) cup functor [**4**] (see also [**6**, Appendix A]). In an unpublished work, Abouzaid-Ganatra proved that $\cap$ and $\cup$ form a spherical adjunction for general symplectic Landau-Ginzburg models [**4**]. On the other hand, using the formalism of partially wrapped Fukaya categories [**76**, **151**], Sylvan considered the Orlov cup functor

$$\cup_F : \mathcal{W}(F) \to \mathcal{W}(X, F)$$

associated to any Weinstein pair (X, F) and showed that the $\cup$ is a spherical functor[1] as long as the Weinstein stop $F \subset \partial_\infty X$ is a so called swappable stop [**152**]. In this case, the spherical twists/cotwists are the monodromy functors defined by wrapping around the contact boundary.

In microlocal sheaf theory, Nadler has also shown that functors between the pair of microsheaf categories over the symplectic Landau-Ginzburg model $(\mathbb{C}^n, \pi = z_1 \dots z_n)$, aftering (heuristically speaking) adding additional fiberwise stops, form a spherical adjunction. Then, by removing the fiberwise stops, the spherical adjunction for the original pair is also obtained [**122**], but it is unclear how general this argument is in sheaf theory.

Our main result provides a general criterion for the microlocalization functor $m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$ to be spherical. Under the equivalence of Ganatra-Pardon-Shende [**74**], the left adjoint of microlocalization m_Λ^l is equivalent to the Orlov

[1]The data of a spherical functor is equivalent to the data of a spherical adjunction, as will be explained in Section 6.2.1. Here we use spherical functors because the adjoint functor is not explicit from Sylvan's work.

cup functor on wrapped Fukaya categories, while we expect that the microlocalization m_Λ is the cap functor on Fukaya-Seidel categories (see Remark 6.0.2 and 6.0.3). This is based on joint work with C. Kuo.

Let M be a real analytic manifold. Consider a fixed Reeb flow $T_t : T^{*,\infty}M \to T^{*,\infty}M$. Recall that a (time-dependent) contact isotopy $\varphi_t : T^{*,\infty}M \times \mathbb{R} \to T^{*,\infty}M$ is called a positive isotopy if $\alpha(\partial_t \varphi_t) \geq 0$. In the definition, we use the word stop for any compact subanalytic Legendrians (following [**76**, **151**]), meaning that Hamiltonian flows are stopped by the Legendrian.

The geometric notion of a swappable subanalytic Legendrian originates from positive Legendrian loops that avoid the Legendrian at the base point [**42**], and is explicitly introduced by Sylvan [**152**]. Here our definition is slightly different from [**152**].

Definition 6.0.2. *A compact subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$ is called a swappable stop if there exists a compactly supported positive Hamiltonian on $T^{*,\infty}M \backslash \Lambda$ such that the forward flow sends $T_\epsilon(\Lambda)$ to an arbitrary small neighbourhood of $T_{-\epsilon}(\Lambda)$, and the backward flow sends $T_{-\epsilon}(\Lambda)$ to an arbitrary small neighbourhood of $T_\epsilon(\Lambda)$.*

We also introduce the notion of geometric and algebraic full stops, both called full stops for simplicity. We will see in Proposition 6.2.8 that a geometric full stop is always an algebraic full stop.

Definition 6.0.3. *Let M be compact. A compact subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$ is called a geometric full stop if for a collection of generalized linking spheres* *at infinity $\Sigma \subset T^{*,\infty}M$ of Λ, there exists a compactly supported positive Hamiltonian on $T^{*,\infty}M\backslash\Lambda$ such that the Hamiltonian flow sends Σ to an arbitrary small neighbourhood of $T_{-\epsilon}(\Lambda)$.*

More generally, when M is compact, a compact subanalytic Legendrian $\Lambda \subset T^{*,\infty}M$ is called an algebraic full stop if the category of compact objects $Sh^c_\Lambda(M)$ is proper.

Example 6.0.1. *There is a large class of examples of swappable stops and full stops in Section 6.2.3. Here are two cimple classes of examples. (1) For a subanalytic triangulation $\mathcal{S} = \{X_\alpha\}_{\alpha\in I}$, the union of unit conormal bundles $\bigcup_{\alpha\in I}\nu^{*,\infty}_{X_\alpha}M$ is a algebraic full stop (we suspect that it is also a geometric full stop and a swappable stop, but we cannot prove that). (2) For an exact symplectic Landau-Ginzburg model $\pi : T^*M \to \mathbb{C}$, the Lagrangian skeleton $\mathfrak{c}_F$ of a regular fiber at infinity $F = \pi^{-1}(\infty)$ is a swappable stop and when π is a Lefschetz fibration it is a geometric full stop.*

We are able to state our main result, which provides a general criterion for the microlocalization functor m_Λ to be spherical.

Theorem 6.0.3 (Theorem 1.2.1)**.** *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian that is either a full Legendrian stop or a swappable Legendrian stop. Then the microlocalization functor along Λ and its left adjoint*

$$m_\Lambda : Sh_\Lambda(M) \leftrightharpoons \mu Sh_\Lambda(\Lambda) : m_\Lambda^l$$

form a spherical adjunction.

Remark 6.0.2. *According to* [**74**, Proposition 7.24] *there is a commutative diagram between microlocal sheaf categories and wrapped Fukaya categories*

$$\begin{array}{ccc} \mathcal{W}(F) & \xrightarrow{\sim} & \mu Sh^c_{\mathfrak{c}_F}(\mathfrak{c}_F) \\ {\scriptstyle \cup_F}\downarrow & & \downarrow{\scriptstyle m^*_{\mathfrak{c}_F}} \\ \mathcal{W}(T^*M, F) & \xrightarrow{\sim} & Sh^c_{\mathfrak{c}_F}(M). \end{array}$$

Therefore by restricting to the subcategory of compact objects, our theorem recovers the result by Sylvan [**152**] *that*

$$\cup_F : \mathcal{W}(F) \to \mathcal{W}(X, F)$$

is spherical in the case $X = T^*M$. However, different from [**152**]*, we are able to explicitly construct the left and right adjoint functors in the proof.*

Remark 6.0.3. *For a Lefschetz fibration $\pi : T^*M \to \mathbb{C}$ with regular fiber at infinity $F = \pi^{-1}(\infty)$, $\mathcal{W}(T^*M, F)$ is generated by Lagrangian thimbles* [**75**, Corollary 1.14] *and is a proper category* [**74**, Proposition 6.7] *(when $M = T^n$ and F is the Weinstein thickening of the FLTZ skeleton* [**133**]*, this is also proved by* [**105**]*), and hence*

$$Sh^b_{\mathfrak{c}_F}(M) \simeq Sh^c_{\mathfrak{c}_F}(M) \simeq \mathcal{W}(T^*M, F).$$

Since it is expected that $\mathcal{W}(T^*M, F) \simeq \mathcal{FS}(T^*M, \pi)$[2]*, there should be an equivalence $Sh^b_{\mathfrak{c}_F}(M) \simeq \mathcal{FS}(T^*M, \pi)$ (when $M = T^n$, Zhou has sketched a proof in his thesis*

[2] As pointed out in [**74**, Footnote 2], if one takes $\mathcal{W}(T^*M, F)$ as the definition of the Fukaya-Seidel category then this is tautological. However a comparison result between $\mathcal{W}(T^*M, F)$ and the Fukaya-Seidel category defined in [**139**, Part 3] is not yet in the literature.

[**162**]). *Therefore, our theorem should be viewed as a sheaf theory version of the result* [**4**] *that*

$$\cap_F : \mathcal{FS}(T^*M, \pi) \to \mathcal{F}(F)$$

is spherical. However, since we do not know a commutative diagram

$$\begin{array}{ccc} \mathcal{FS}(T^*M, \pi) & \longrightarrow & Sh^b_{\mathfrak{c}_F}(M) \\ \Big\downarrow{\scriptstyle \cap_F} & & \Big\downarrow{\scriptstyle m_{\mathfrak{c}_F}} \\ \mathcal{F}(F) & \longrightarrow & \mu Sh^b_{\mathfrak{c}_F}(\mathfrak{c}_F), \end{array}$$

that result [**4**] *does not directly follow from ours.*

We can write down the spherical twists and cotwists as follows. Previous work of Kuo [**104**] has defined the positive wrapping functor $\mathfrak{W}^+_\Lambda$ (resp. negative wrapping functor $\mathfrak{W}^-_\Lambda$) sending an arbitrary sheaf in $Sh(M)$ to $Sh_\Lambda(M)$ by a colimit (resp. a limit) of positive (resp. negative) wrappings into Λ. The spherical cotwist (resp. the dual cotwist) $Sh_\Lambda(M) \to Sh_\Lambda(M)$ for m_Λ is explicitly as the functor defined by wrapping positively (resp. negatively) around $T^{*,\infty}M$ once along the Reeb flow.

Proposition 6.0.4. *Let* $\Lambda \subset T^{*,\infty}M$ *and* $T_t : T^{*,\infty}M \to T^{*,\infty}M$ *be a Reeb flow. Then the spherical cotwist and dual cotwist are the negative and positive wrap-once functor (where* $\epsilon > 0$ *is sufficiently small)*

$$S^-_\Lambda = \mathfrak{W}^-_\Lambda \circ T_{-\epsilon}, \;\; S^+_\Lambda = \mathfrak{W}^+_\Lambda \circ T_\epsilon.$$

Remark 6.0.4. *By* [**74**, Proposition 7.24]*, we know that the cup functor $\cap_F$ on partially wrapped Fukaya categories is isomorphic to the left adjoint of microlocalization m_Λ^l on sheaf categories. Therefore, we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{W}(T^*M, F) & \xrightarrow{\sim} & Sh^c_{\mathfrak{c}_F}(M) \\ {\scriptstyle \mathcal{S}_F^\pm}\downarrow & & \downarrow{\scriptstyle S^\pm_{\mathfrak{c}_F}} \\ \mathcal{W}(T^*M, F) & \xrightarrow{\sim} & Sh^c_{\mathfrak{c}_F}(M), \end{array}$$

such that our wrap-once functor $S^\pm_{\mathfrak{c}_F}$ is isomorphic to the wrap-once functor of Sylvan [**152**].

We will also write down a formula for spherical twists and dual twists in Section 6.4.1 Corollary 6.4.9, which can be interpreted as the monodromy functors.

On the other hand, from the perspective of Fukaya categories, following a proposal of Kontsevich, Seidel has conjectured [**140**] that for a symplectic Lefschetz fibration, the spherical dual cotwist is the Serre functor

$$\sigma^{-1} : \mathcal{FS}(X, \pi) \to \mathcal{FS}(X, \pi),$$

and proved partial results [**141–143**], while from the perspective of Legendrian contact homology, Ekholm-Etnyre-Sabloff have proved Sabloff duality [**58**, **135**] between linearized homology and cohomology. These results predict a Serre functor, which should be the Poincaré-Lefschetz duality on the category of constructible sheaves with perfect stalks

$$S_\Lambda^- : Sh^b_\Lambda(M) \to Sh^b_\Lambda(M).$$

We show that in the sheaf theoretic setting, the spherical cotwist functor $S_\Lambda^- \otimes \omega_M$ is indeed the Serre functor on $\mathrm{Sh}_\Lambda^b(M)$, which implies a folklore conjecture on Fukaya categories in the case of cotangent bundles.

Proposition 6.0.5 (Proposition 6.3.4)**.** *Let $\Lambda \subset S^*M$ be a full or swappable subanalytic compact Legendrian stop. Then $S_\Lambda^- \otimes \omega_M$ is the Serre functor on $Sh_\Lambda^b(M)_0$ of sheaves microsupported on Λ with perfect stalks and compact supports. In particular, when M is orientable, $S_\Lambda^-[-n]$ is the Serre functor on $Sh_\Lambda^b(M)_0$.*

Remark 6.0.5. *Since our wrap-once functor is isomorphic to the wrap-once functor of Sylvan* [**152**]*, we can prove that the negative wrap-once functor of Sylvan*

$$\mathcal{S}_\Lambda^- : \mathrm{Prop}\,\mathcal{W}(T^*M, F) \to \mathrm{Prop}\,\mathcal{W}(T^*M, F)$$

is the Serre functor when M is orientable. In particular, when $F = \pi^{-1}(\infty)$ is the fiber of a symplectic Lefschetz fibration, then $\mathcal{S}_\Lambda^-$ is the Serré functor on $\mathcal{W}(T^*M, F)$.

Remark 6.0.6. *Spherical adjunctions together with a compatible Serre functor, in the smooth and proper setting, implies existence of a weak relative right Calabi-Yau structure* [**99**]*, but we do not expect the relative Calabi-Yau structures for general Weinstein pairs to be proved this way. See the discussion in Remark 6.3.1.*

6.1. Adjoint Functors of Inclusions Are Wrappings

Following the framework of Ganatra-Pardon-Shende [**76**], for a positive Hamiltonian flow $\varphi : T^{*,\infty}M \times [0,1] \to T^{*,\infty}M$, we will define the canonical map

$$Hom(\mathscr{F}, \mathscr{G}) \to Hom(\mathscr{F}, \Psi^0_{\varphi,1}\mathscr{G})$$

by the multiplication with a canonical element (called the continuation element) $c_\varphi \in H^0 Hom(\mathscr{G}, \mathscr{G}^\varphi)$. In microlocal sheaf theory, the continuation element has been studied by for example Zhou [**163**, Section 2.5] and Kuo [**104**, Section 3.1].

For a positive Hamiltonian flow $\varphi : T^{*,\infty}M \times [0,1] \to T^{*,\infty}M$, recall that Lemma 4.1.1 implies that there are continuation maps induced by canonical maps $\Bbbk_{(-\infty,u)} \to \Bbbk_{(-\infty,u')}$

$$\Psi^0_{\varphi,u}(\mathscr{F}) \to \Psi^0_{\varphi,u'}(\mathscr{F}), \forall\, u \leq u'.$$

For simplicity, we will write $\mathscr{F}^\varphi = \Psi^0_{\varphi,1}(\mathscr{F})$. Then we have continuation maps

$$\mathscr{F} \to \mathscr{F}^\varphi, \forall\, \varphi \geq 0.$$

Proposition 6.1.1 (Kuo [**104**, Definition 3.3 & Proposition 3.4])**.** *Let φ be a positive Hamiltonian on $T^{*,\infty}M$ and $\mathscr{F} \in Sh(M)$. Then there is a canonical continuation element $c_\varphi \in H^0 Hom(\mathscr{F}, \Psi_\varphi(\mathscr{F}))$, such that*

(1) $c_0 = \mathrm{id} \in H^0 Hom(\mathscr{F}, \mathscr{F})$;

(2) $c_{\varphi\circ\psi} = c_\psi \circ c_\varphi \in H^0 Hom(\mathscr{F}, \mathscr{F}^{\varphi\circ\psi}))$;

(3) $c_\varphi = c_{\varphi'} \in H^0 Hom(\mathscr{F}, \mathscr{F}^\varphi)$ *if* φ *and* φ' *are isotopic relative to boundary* $T^{*,\infty}M \times \{0,1\}$.

Consider $X \subset T^{*,\infty}M$ a closed subset. We define the positive wrapping category (which is the diagram category when we take limits and colimits), following [**76**, Section 3.4] and [**104**, Section 3.2].

Definition 6.1.1. *Let* $X \subset T^{*,\infty}M$ *be a closed subset. Then the positive wrapping category* $W^+(T^{*,\infty}M \backslash X)$ *consists of positive Hamiltonians compactly supported away from* Λ*, and the composition of* G *and* H *in the wrapping category is the composition* $G \circ H$ *as Hamiltonian flows.*

One can show that the wrapping category $W^+(T^{*,\infty}M \backslash X)$ is filtered [**76**, Lemma 3.27] [**104**, Proposition 3.17]. In particular we can take colimits with respect to the wrapping category, which leads to the following theorem, that colimit/limit over increasingly positive/negative isotopies provides a description for the tautological inclusion $Sh_X(M) \subseteq Sh(M)$.

Theorem 6.1.2 (Kuo [**104**, Theorem 1.2])**.** *Let* $\iota_{X*} : Sh_X(M) \hookrightarrow Sh(M)$ *denote the tautological inclusion. Then the left and right adjoints are given by the positive/negative colimiting/limiting wrapping*

$$\mathfrak{W}_X^+(\mathscr{F}) = \operatorname*{colim}_{\varphi \in W^+(T^{*,\infty}M \backslash X)} \mathscr{F}^\varphi, \ \mathfrak{W}_X^-(\mathscr{F}) = \lim_{\varphi \in W^+(T^{*,\infty}M \backslash X)} \mathscr{F}^{\varphi^{-1}}.$$

For $\mathscr{G} \in Sh(M)$, it is in general hard to compute $\mathfrak{W}_X^+(\mathscr{G})$ (resp. $\mathfrak{W}_X^-(\mathscr{G})$) since it is given by a colimit (resp. a limit) over a rather large index category. Nevertheless, when $X = \Lambda$ and $SS^\infty(\mathscr{G})$ are both isotropic, the underlying geometry can sometimes provide a cofinal (resp. final) one parameter family $\mathscr{G}_t$ so that $\mathscr{G}_0 = \mathscr{G}$ and $\mathfrak{W}_\Lambda^+(\mathscr{G}) = \operatorname{colim}_{t\to+\infty} \mathscr{G}_t$ (resp. $\mathfrak{W}_\Lambda^-(\mathscr{G}) = \lim_{t\to-\infty} \mathscr{G}_t$). In this case, a natural question is when, for a fixed $F \in Sh(M)$ also with isotropic singular support, the canonical map

$$Hom(\mathscr{F}, \mathscr{G}) \to Hom(\mathscr{F}, \mathfrak{W}_\Lambda^+(\mathscr{G}))$$

is an isomorphism. One such a case which we will encounter is the following:

Lemma 6.1.3. *Let Λ be compact subanalytic Legendrian and $\mathscr{F}, \mathscr{G} \in Sh(M)$ be sheaves with Legendrian singular supports. Assume that $SS^\infty(\mathscr{F}) \cap \Lambda = \varnothing$, and there is a positive isotopy φ_t, $t \in \mathbb{R}$, on $T^{*,\infty}M$ such that for any open neighborhood Ω of Λ, there is $T = T(\Lambda)$ such that $\varphi^t(SS^\infty(\mathscr{G})) \subseteq \Omega$ for $t \geq T$, and $SS^\infty(\mathscr{F}) \cap \varphi^t SS^\infty(\mathscr{G}) = \varnothing$ for all $t \geq 0$, then the canonical map*

$$Hom(\mathscr{F}, \mathscr{G}) \to Hom(\mathscr{F}, \mathfrak{W}_\Lambda^+(\mathscr{G}))$$

is an isomorphism. A similar statement holds for $Hom(\mathscr{G}, \mathscr{F}) \to Hom(\mathfrak{W}_\Lambda^-(\mathscr{G}), \mathscr{F})$ when given a negative isotopy satisfying a similar condition.

Proof. This is essentially [**104**, Theorem 5.15]. The point is that we would like to apply the main theorem about nearby cycle, [**124**, Theorem 4.2]. Although we did not assume compactness on $\operatorname{supp}(\mathscr{F})$ and $\operatorname{supp}(\mathscr{G})$ here as in [**104**, Theorem 5.15], the compactness assumption on Λ will be sufficient to implied the gappedness condition for [**124**, Theorem 4.2]. □

Remark 6.1.1. *In practice, when we can find an increasing sequence of positive Hamiltonian flows* φ_t^k*,* $k \in \mathbb{N}$*, such that for any open neighborhood* Ω *of* Λ*, there is* $K \in \mathbb{N}$ *such that* $\varphi_t^k(SS^\infty(\mathscr{G})) \subseteq \Omega$ *for* $k \geq K$*, then the condition in the lemma holds. Indeed, we can define a time dependent smooth Hamiltonian* φ^t *such that* $\varphi^t(SS^\infty(\mathscr{G})) = \varphi_{t-k}^k(SS^\infty(\mathscr{G}))$ *when* $t \in [k+1-\epsilon, k+1]$*. That satisfies the condition in the lemma.*

Using the description of the adjoint functors ι_Λ^* and $\iota_\Lambda^! : Sh(M) \to Sh_\Lambda(M)$ in terms of positive and negative wrappings, we are able to provide geometric interpretations for both the Sato-Sabloff exact triangle Theorem 4.0.6 and the Guillermou doubling functor Theorem 4.0.7.

Theorem 6.1.2 induces the identification

$$Hom(\mathfrak{W}_\Lambda^- T_{-\epsilon}(\mathscr{F}), \mathscr{G}) = Hom(T_\epsilon(\mathscr{F}), \mathscr{G}) = Hom(\mathscr{F}, T_{-\epsilon}(\mathscr{G}))$$

and the definition of μhom and adjunctions implies that

$$\Gamma(\Lambda, \mu hom(\mathscr{F}, \mathscr{G})) = Hom(m_\Lambda^l m_\Lambda(\mathscr{F}), \mathscr{G}).$$

Then by Sato-Sabloff exact triangle Theorem 4.0.6, we have

$$Hom(\mathfrak{W}_\Lambda^- T_{-\epsilon}(\mathscr{F}), \mathscr{G}) \to Hom(\mathscr{F}, \mathscr{G}) \to Hom(m_\Lambda^l m_\Lambda(\mathscr{F}), \mathscr{G})$$

Equivalently, we have the exact sequence

$$m_\Lambda^l m_\Lambda \to \mathrm{id} \to \mathfrak{W}_\Lambda^+ T_\epsilon \xrightarrow{+1}$$

between endofunctors on $Sh_\Lambda(M)$. A similarly discussion holds for the right adjoints so that we have the exact sequence

$$\mathfrak{W}_\Lambda^- T_{-\epsilon} \to \mathrm{id} \to m_\Lambda^r m_\Lambda \xrightarrow{+1} .$$

Definition 6.1.2. *We define the positive and negative warp-once functor* S_Λ^+ *and* $S_\Lambda^- : Sh_\Lambda(M) \to Sh_\Lambda(M)$ *as the compositions*

$$S_\Lambda^+(\mathscr{F}) := \mathfrak{W}_\Lambda^+ T_\epsilon(\mathscr{F}),\ S_\Lambda^-(\mathscr{F}) := \mathfrak{W}_\Lambda^- T_{-\epsilon}(\mathscr{F}).$$

Corollary 6.1.4. *Let* $\Lambda \subset T^{*,\infty}M$ *be a closed subanalytic Legendrian. Then there are exact triangles of functors*

$$m_\Lambda^l m_\Lambda \to \mathrm{id} \to S_\Lambda^+ \xrightarrow{+1},\ S_\Lambda^- \to \mathrm{id} \to m_\Lambda^r m_\Lambda \xrightarrow{+1} .$$

Then, we recall the adjunction property of the Guillermou doubling functor in Theorem 4.2.10 and 4.2.9. Then Theorem 6.1.2 immediately implies the following result.

Corollary 6.1.5. *Let $\Lambda \subset T^{*,\infty}M$ be a closed subanalytic Legendrian. Then there are equivalences*

$$m_\Lambda^l = \mathfrak{W}_\Lambda^+ \circ w_\Lambda[-1], \;\; m_\Lambda^r = \mathfrak{W}_\Lambda^- \circ w_\Lambda : \mu Sh_\Lambda(\Lambda) \to Sh_\Lambda(M).$$

In particular, the left and the right adjoint m_Λ^l and m_Λ^r can be decomposed to a inclusion followed by a quotient:

$$\mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_\epsilon(\Lambda)\cup T_{-\epsilon}(\Lambda)}(M) \twoheadrightarrow Sh_\Lambda(M).$$

Then one can see that in particular, the exact sequence is compatible in Corollary 6.1.4 is compatible with the exact sequence in Corollary 4.2.8 that

$$T_{-\epsilon} \to T_\epsilon \to w_\Lambda \circ m_\Lambda \xrightarrow{+1}$$

once we apply $\mathfrak{W}_\Lambda^+$ and $\mathfrak{W}_\Lambda^-$ using Theorem 6.1.2.

6.2. Spherical Adjunction, Cotwists and Natural Transformations

With the preparation in the previous sections, we are able to prove Theorem 6.0.3. Our main result in the section is the following theorem.

Theorem 6.2.1. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian full stop or swappable stop. Then the microlocalization functor*

$$m_\Lambda : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$$

is a spherical functor.

Theorem 6.0.3 is going to be a formal consequence of Theorem 6.2.1, as we will discuss in Section 6.2.1. In particular, this will imply that the left adjoint of microlocalization is also a spherical functor.

Corollary 6.2.2. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian full stop or swappable stop. Then the left adjoint of the microlocalization functor*

$$m_\Lambda^l : \mu Sh_\Lambda(\Lambda) \to Sh_\Lambda(M)$$

is a spherical functor.

In Section 6.2.2, we will construct the long exact sequence and natural transformation between the left and right adjoints. Then in Section 6.2.3, we introduce the notion of full stops and swappable stops following [**152**] and explain how these conditions lead to Theorem 6.2.1.

6.2.1. Spherical adjunction and spherical functors

First of all, we recall the definition of spherical adjunctions in Dykerhoff-Kapranov-Schechtman-Soibelman [**49**] in the setting of stable ∞-categories.

Definition 6.2.1 ([**49**, Definition 1.4.8])**.** *Let $\mathscr{A}, \mathscr{B}$ be stable (∞-)categories and*

$$F : \mathscr{A} \leftrightharpoons \mathscr{B} : F^l$$

be an adjunction of ∞-functors. Let T' and S' be the functors that fit into the fiber sequences

$$T' \to \mathrm{id}_{\mathscr{B}} \to F \circ F^l,\ \ F^l \circ F \to \mathrm{id}_{\mathscr{A}} \to S'.$$

Then $F : \mathscr{A} \leftrightharpoons \mathscr{B} : F^l$ is called a spherical adjunction if T' and S' are autoequivalences.

Given a spherical adjunction $F^l \dashv F$, one can in fact show that both F and F^l are spherical functors in the sense of Anno-Logvinenko [**8**]. We recall the definition of spherical functors in the setting of dg categories [**8**] and in the general case [**38**, **49**].

Definition 6.2.2. *Let $\mathscr{A}, \mathscr{B}$ be stable (∞-)categories and $F : \mathscr{A} \to \mathscr{B}$ an (∞-)functor, with left and right adjoints F^l and F^r. Let the spherical twist T, dual twist T', cotwist S and dual cotwist S' be the functors that fit into the fiber sequences*

$$F \circ F^! \to \mathrm{id}_{\mathscr{B}} \to T,\ T' \to \mathrm{id}_{\mathscr{B}} \to F \circ F^*,$$

$$S \to \mathrm{id}_{\mathscr{A}} \to F^! \circ F,\ F^* \circ F \to \mathrm{id}_{\mathscr{A}} \to S'.$$

Then F is a spherical functor if the following conditions hold:

(1) the spherical twist T is an autoequivalence;

(2) the spherical cotwist S is an autoequivalence;

(3) the composition $F^l \circ T[-1] \to F^l \circ F \circ F^r \to F^r$ is an isomorphism;

(4) the composition $F^r \to F^r \circ F \circ F^l \to S \circ F^l[1]$ is an isomorphism.

Proposition 6.2.3 ([**49**, Corollary 2.5.13])**.** *Let $F : \mathscr{A} \leftrightharpoons \mathscr{B} : F^l$ be a spherical adjunction. Then both F and F^l are spherical functors.*

Remark 6.2.1. *Given a spherical adjunction $F^l \dashv F$, let T be the inverse of T' and S the inverse of S'. One can construct the right adjoint of F by setting $F^r = F^l \circ T[-1]$, and the left adjoint of F^l by setting $F_l = F \circ S[1]$. In fact, any spherical functor has iterated left and right adjoints of any order.*

Therefore, to prove a spherical adjunction $F \vdash F^l$, it suffices to show that either of the functors is a spherical functor as in Definition 6.2.2. Moreover, the following theorem shows that it suffices to prove any two out of the four conditions.

Theorem 6.2.4 (Anno-Logvinenko [**8**], Christ [**38**])**.** *Let $\mathscr{A}, \mathscr{B}$ be stable categories, and $F : \mathscr{A} \to \mathscr{B}$ a functor satisfying any two of the four conditions in Definition 6.2.2. Then F is a spherical functor. Moreover, T, T' and S, S' are inverse autoequivalences.*

From the discussion above, we know that in order to prove Theorem 6.0.3, it suffices to prove Theorem 6.2.1 stated at the beginning of the section.

6.2.2. Natural transform between adjoints

Given the adjoint functors and the candidate cotwist in Section 6.1, we will investigate the relation between the left and right adjoints via the algebraically defined natural transformation by the cotwist

$$m_\Lambda^r \to m_\Lambda^r \circ m_\Lambda \circ m_\Lambda^l \to S_\Lambda^- \circ m_\Lambda^l[1].$$

The composition of the natural transformations should induce an equivalence in order for microlocalization to be a spherical functor, as stated in Definition 6.2.2 Condition (4).

For $\mathscr{F} \in Sh_\Lambda(M)$ and $\mathscr{G} \in \mu Sh_\Lambda(\Lambda)$, we thus need to prove that the algebraically defined natural morphism

$$Hom(\mathscr{F}, m_\Lambda^r(\mathscr{G})) \to Hom(\mathscr{F}, m_\Lambda^r m_\Lambda m_\Lambda^l(\mathscr{G})) \to Hom(\mathscr{F}, S_\Lambda^- m_\Lambda^l(\mathscr{G})[1])$$

is an equivalence. On the other hand, Proposition 6.1.2 and 4.1.2 imply that

$$Hom(\mathscr{F}, m_\Lambda^r(\mathscr{G})) = Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})),$$

$$Hom(\mathscr{F}, S_\Lambda^- m_\Lambda^l(\mathscr{G})[1]) = Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})).$$

Positive isotopies will then induce a geometrically defined natural morphism

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \to Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})).$$

Our main result in this section claims that the algebraically defined natural morphism induces an isomorphism if and only if the geometrically defined natural morphism induces an isomorphism.

Proposition 6.2.5. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian. Then for $\mathscr{F} \in Sh_\Lambda(M)$ and $\mathscr{G} \in \mu Sh_\Lambda(\Lambda)$, the natural morphism induced by adjunctions*

$$Hom(\mathscr{F}, m_\Lambda^r(\mathscr{G})) \to Hom(\mathscr{F}, m_\Lambda^r m_\Lambda m_\Lambda^l(\mathscr{G})) \to Hom(\mathscr{F}, S_\Lambda^- m_\Lambda^l(\mathscr{G})[1])$$

is an isomorphism if and only if the natural morphism induced by positive isotopies

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \to Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G}))$$

is an isomorphism.

We need to unpack the algebraic adjunctions between microlocalization and its left and right adjoints using results on the doubling functor.

Firstly, we consider the natural transformation to the cotwist $m_\Lambda^r \circ m_\Lambda \to S_\Lambda^-[1]$. The following lemma follows directly from Corollary 4.2.8 that there is a fiber sequence $T_{-\epsilon} \to T_\epsilon \to w_\Lambda \circ m_\Lambda$.

Lemma 6.2.6. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian. Then for $\mathscr{F} \in Sh_\Lambda(M)$ and $\mathscr{G} \in \mu Sh_\Lambda(\Lambda)$, there is a commutative diagram induced by natural transformations*

$$\begin{array}{ccc} Hom(\mathscr{F}, m_\Lambda^r m_\Lambda m_\Lambda^l(\mathscr{G})) & \longrightarrow & Hom(\mathscr{F}, S_\Lambda^- m_\Lambda^l(\mathscr{G})[1]) \\ \downarrow \wr & & \downarrow \wr \\ Hom(w_\Lambda m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})) & \longrightarrow & Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})). \end{array}$$

Proof. Consider $\mathscr{F} \in Sh_\Lambda(M)$ and $\mathscr{G} \in \mu Sh_\Lambda(\Lambda)$. Theorem 6.1.2 implies that it suffices for us to show the following diagram

$$\begin{array}{ccc} Hom(\mathscr{F}, w_\Lambda m_\Lambda \circ \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})[-1]) & \rightarrow & Hom(\mathscr{F}, T_{-\epsilon}(\mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G}))) \\ \downarrow \wr & & \downarrow \wr \\ Hom(w_\Lambda m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})) & \longrightarrow & Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})). \end{array}$$

Since the two horizontal morphisms are induced by the transformation $w_\Lambda \circ m_\Lambda[-1] \to T_{-\epsilon}$ and respectively $T_\epsilon \to w_\Lambda \circ m_\Lambda$ in Corollary 4.2.8, we can conclude that the diagram commutes. □

Secondly, we need to consider the unit $\mathrm{id} \to m_\Lambda \circ m_\Lambda^l$, which is slightly more difficult. The following lemma relies on Corollary 4.2.8, and the fact that the adjunction in Theorem 4.2.10 factors through the doubling functor by computation in Theorem 4.0.7:

$$\Gamma(\Lambda, \mu hom(m_\Lambda(\mathscr{F}), \mathscr{G})) \xrightarrow{\sim} Hom(w_\Lambda m_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})).$$

Lemma 6.2.7. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian. Then for $\mathscr{F} \in Sh_\Lambda(M)$ and $G \in \mu Sh_\Lambda(\Lambda)$, there is a commutative diagram induced by natural transformations*

$$\begin{array}{ccc} Hom(\mathscr{F}, m_\Lambda^r(\mathscr{G})) & \longrightarrow & Hom(\mathscr{F}, m_\Lambda^r m_\Lambda m_\Lambda^l(\mathscr{G})) \\ \downarrow \wr & & \downarrow \wr \\ Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) & \rightarrow & Hom(w_\Lambda m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})). \end{array}$$

Proof. Consider $\mathscr{F} \in Sh_\Lambda(M)$ and $\mathscr{G} \in \mu Sh_\Lambda(\Lambda)$. By Theorem 6.1.2, it suffices to show that there is a commutative diagram

$$\begin{array}{ccc} Hom(\mathscr{F}, w_\Lambda(\mathscr{G})) & \longrightarrow & Hom(\mathscr{F}, w_\Lambda m_\Lambda \circ \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})[-1]) \\ \downarrow \wr & & \downarrow \wr \\ Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) & \longrightarrow & Hom(w_\Lambda m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})). \end{array}$$

where the morphism on the top is induced by adjunction, and the morphism on the bottom is the composition

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} Hom(w_\Lambda m_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) \to Hom(w_\Lambda m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})).$$

Consider the unit of the adjunction between m_Λ and $m_\Lambda^l = \mathfrak{W}_\Lambda^+ \circ w_\Lambda[-1]$ in Theorem 4.2.10. Then we know that the morphism on the top factors as

$$\begin{array}{ccc} \Gamma(\Lambda, \mu hom(m_\Lambda(\mathscr{F}), \mathscr{G})) & \longrightarrow & \Gamma(\Lambda, \mu hom(m_\Lambda(\mathscr{F}), m_\Lambda \circ \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})[-1])) \\ \downarrow \wr & & \downarrow \wr \\ Hom(\mathscr{F}, w_\Lambda(\mathscr{G})) & \longrightarrow & Hom(\mathscr{F}, w_\Lambda m_\Lambda \circ \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})[-1]). \end{array}$$

Since the top horizontal morphism factors through $Hom(w_\Lambda m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G}))$, it suffices to show that the following composition factors through $\Gamma(\Lambda, \mu hom(m_\Lambda(\mathscr{F}), \mathscr{G}))$

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} Hom(w_\Lambda m_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) \to Hom(w_\Lambda m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ w_\Lambda(\mathscr{G})).$$

Since the adjunction in Theorem 4.2.10 factors through the isomorphism of doubling functor in Theorem 4.0.7

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} Hom(w_\Lambda m_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} \Gamma(\Lambda, \mu hom(m_\Lambda(\mathscr{F}), \mathscr{G})),$$

we can conclude that the diagram above indeed commutes. □

Proof of Proposition 6.2.5. We consider the following commutative diagram, where the horizontal morphisms are induced from the identity $w_\Lambda \circ m_\Lambda = \mathrm{Cone}(T_{-\epsilon} \to T_\epsilon)$, and vertical morphisms are induced by positive isotopies

$$\begin{array}{ccccc} Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) & \longrightarrow & Hom(w_\Lambda \circ m_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) & \longrightarrow & Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \\ \| & & \downarrow & & \downarrow \\ Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) & \to & Hom(w_\Lambda \circ m_\Lambda(\mathscr{F}), \mathfrak{W}_\Lambda^+ \circ w_\Lambda(\mathscr{G})) & \to & Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ \circ w_\Lambda(\mathscr{G})). \end{array}$$

Lemma 6.2.6 and 6.2.7 imply that the algebraically defined natural transformation of functors is an isomorphism if and only if the composition of morphisms in the second row is an isomorphism.

From the computation in Theorem 4.0.7, compared with Theorem 4.2.9 and 4.2.10, we know that horizontal natural morphisms in the first row are isomorphisms

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} Hom(w_\Lambda \circ m_\Lambda(\mathscr{F}), w_\Lambda(\mathscr{G})) \xrightarrow{\sim} Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})).$$

Therefore, the composition of horizontal morphisms in the second row is an isomorphism if and only if the geometrically defined vertical morphism in the last column is an isomorphism. □

6.2.3. Criterion for spherical adjunction

With the presence of the adjunction pairs, fiber sequences and natural transformations in the previous sections, in this section, we will study the spherical cotwist and its dual cotwist, and prove Condition (2) & (4) in Definition 6.2.2 under geometric assumptions. As we will see in the proof, both Condition (2) & (4) in will rely on some full faithfulness of the wrapping functor $\mathfrak{W}_\Lambda^+$.

In this section, we will use the word stop for a closed subanalytic Legendrian in $T^{*,\infty}M$ (meaning that the positive wrappings in $T^{*,\infty}M$ are stopped by the subanalytic Legendrian), which comes from the study of symplectic topology and wrapped Fukaya categories [**76**, **151**].

6.2.3.1. Spherical functor for full stops. We assume that M is compact in this subsection. First, we introduce the notion of an algebraic full stop, which has been frequently used in wrapped Fukaya categories.

Definition 6.2.3. *Let M be compact and $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian. Then Λ is called a full stop if $Sh^c_\Lambda(M)$ is a proper category.*

Remark 6.2.2. *Recall that from Theorem 3.4.3 (*[**121**, Theorem 3.21] *or* [**74**, Corollary 4.23]*), we know that $Sh^b_\Lambda(M) = Sh^{pp}_\Lambda(M)$. From Proposition 3.4.1, we know that in our case $Sh^c_\Lambda(M)$ is smooth, which then implies Corollary 3.4.5 that*

$Sh^b_\Lambda(M) \subseteq Sh^c_\Lambda(M)$. *On the other hand, when* $Sh^c_\Lambda(M)$ *is moreover proper, then we know that* [**74**, Lemma A.8]

$$Sh^c_\Lambda(M) = Sh^b_\Lambda(M).$$

Conversely, when M *and* Λ *are both compact, then if* $Sh^c_\Lambda(M) = Sh^b_\Lambda(M)$, *we can also tell that* $Sh^c_\Lambda(M)$ *is proper using for example Proposition 3.4.4.*

Example 6.2.3. *Let* $\mathcal{S} = \{X_\alpha\}_{\alpha\in I}$ *be a subanalytic triangulation on* M. *Then the union of unit conormal bundles over all strata* $\Lambda = \nu^{*,\infty}_{\mathcal{S}} M = \bigcup_{\alpha\in I} \nu^{*,\infty}_{X_\alpha} M$ *defines a full stop* [**74**, Proposition 4.24].

We recall our notion of a geometric full stop in the introduction. For the definition of generalized linking disks, see for example [**74**, Section 7.1].

Definition 6.2.4. *A closed subanalytic Legendrian* $\Lambda \subset T^{*,\infty}M$ *is called a geometric full stop if for a collection of generalized linking spheres* $\Sigma \subset T^{*,\infty}M$ *of* Λ, *there exists a compactly supported positive Hamiltonian on* $T^{*,\infty}M\backslash\Lambda$ *such that the Hamiltonian flow sends* D *to an arbitrary small neighbourhood of* $T_{-\epsilon}(\Lambda)$.

Following Ganatra-Pardon-Shende [**74**, Proposition 6.7], we prove that a geometric full stop is an algebraic full stop.

Proposition 6.2.8. *Let* $\Lambda \subset T^{*,\infty}M$ *be a geometric full stop. Then* Λ *is also an algebraic full stop.*

To prove the proposition, we recall the following definition by Kuo in [**104**]. Let M be compact, and $\widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(M)$ be the category of constructible sheaves with perfect stalks whose singular support is disjoint from Λ. Let $\mathcal{C}_\Lambda(M)$ be all continuation maps of positive isotopies supported away from Λ. Then the category of wrapped sheaves is [**104**, Definition 4.1]

$$\mathfrak{w}\mathrm{sh}_\Lambda(M) \coloneqq \widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(M)/\mathcal{C}_\Lambda(M).$$

We have $Hom_{\mathfrak{w}\mathrm{sh}_\Lambda(M)}(F, G) = \mathrm{colim}_{\varphi \in W^+(T^{*,\infty}M \backslash \Lambda)} Hom(F, G^\varphi)$, and [**104**, Theorem 1.3]

$$\mathfrak{W}_\Lambda^+ : \mathfrak{w}\mathrm{sh}_\Lambda(M) \xrightarrow{\sim} Sh_\Lambda^c(M).$$

Proof of Proposition 6.2.8. We prove that $\mathfrak{w}\mathrm{sh}_\Lambda(M)$ is a proper category. Namely, for any $\mathscr{F}, \mathscr{G} \in \mathfrak{w}\mathrm{sh}_\Lambda(M)$,

$$Hom_{\mathfrak{w}\mathrm{sh}_\Lambda(M)}(\mathscr{F}, \mathscr{G}) \in \mathrm{Perf}(\Bbbk).$$

Indeed, note that $SS^\infty(\mathscr{G}) \subset T^{*,\infty}M\backslash\Lambda$ is compact. Thus there exists a cofinal wrapping $\varphi_k \in W^+(T^{*,\infty}M\backslash\Lambda)$ such that $\varphi_k^1(SS^\infty(\mathscr{G}))$ is contained in a neighbourhood of Λ, and is in particular away from $SS^\infty(\mathscr{F})$. Therefore

$$\begin{aligned} Hom_{\mathfrak{w}\mathrm{sh}_\Lambda(M)}(\mathscr{F}, \mathscr{G}) &= \operatorname*{colim}_{\varphi \in W^+(T^{*,\infty}M\backslash\Lambda)} Hom(\mathscr{F}, \mathscr{G}^\varphi) \simeq \operatorname*{colim}_{k\to\infty} Hom(\mathscr{F}, \mathscr{G}^{\varphi_k}) \\ &\simeq \operatorname*{colim}_{k\to\infty} Hom(\mathscr{F}, \mathscr{G}) = Hom(\mathscr{F}, \mathscr{G}) \in \mathrm{Perf}(\Bbbk), \end{aligned}$$

which completes the proof for the first case of a geometric full stop. $\square$

Example 6.2.4. *Let* $\pi : T^*M \to \mathbb{C}$ *be an exact symplectic Lefschetz fibration (whose existence is ensured by Giroux-Pardon* [**82**]*), and* $F = \pi^{-1}(\infty)$ *a regular Weinstein fiber at infinity. Then the Lagrangian skeleton of the Weinstein manifold* $\Lambda = \mathfrak{c}_F \subset T^{*,\infty}M$ *defines a full stop* [**75**, Corollary 1.14] *&* [**74**, Proposition 6.7] *under the equivalence between wrapped Fukaya categories and microlocal sheaf categories* [**74**]*.*

Let $\Lambda = \Lambda_\Sigma^\infty \subset S^*T^n$ *be the FLTZ skeleton associated to the toric fan* Σ [**69**, **70**, **133**]*. Gammage-Shende (under an extra assumption)* [**73**] *and Zhou (without extra assumptions)* [**164**] *show that it is indeed the Lagrangian skeleton of a regular fiber of a symplectic fibration* $\pi : T^*T^n \to \mathbb{C}$*, which is expected to be a Lefschetz fibration when the mirror toric stack* $\mathcal{X}_\Sigma$ *is smooth. The fact that* Λ_Σ^∞ *is a full stop (when* $\mathcal{X}_\Sigma$ *is smooth) is also independently proved by Kuwagaki using mirror symmetry* [**105**]*.*

When $\Lambda \subset T^{*,\infty}M$ is a full Legendrian stop, we know that $Sh_\Lambda(M) = \mathrm{Ind}(Sh_\Lambda^b(M))$. Therefore we only focus on results on the small category $Sh_\Lambda^b(M)$.

To show Condition (2) that S_Λ^+ and S_Λ^- are equivalences, we appeal to Section 4.1.3 where S_Λ^- is shown to abide Serre duality (up to a twist) on $Sh_\Lambda^b(M)$.

Proposition 6.2.9. *Let* $\Lambda \subset T^{*,\infty}M$ *be a compact full Legendrian stop where* M *is a closed manifold. Then there is a pair of inverse autoequivalences*

$$S_\Lambda^+ : Sh_\Lambda^b(M) \rightleftharpoons Sh_\Lambda^b(M) : S_\Lambda^-.$$

Proof. For $\mathscr{F}, \mathscr{G} \in Sh^b_\Lambda(M)$, we claim that

$$Hom(\mathscr{F}, \mathscr{G}) \to Hom_{\mathfrak{wsh}_\Lambda(M)}(T_\epsilon(\mathscr{F}), T_\epsilon(\mathscr{G})).$$

Indeed, consider a sequence of descending open neighbourhoods $\{\Omega_k\}_{k\in\mathbb{N}}$ of $\Lambda \subset T^{*,\infty}M$ such that $\Omega_{k+1} \subseteq \overline{\Omega_k}$ and $\bigcap_{k\in\mathbb{N}} \Omega_k = \Lambda$. Let the sequence cofinal wrapping be the Reeb flow $T_{1/t}$ on $T^{*,\infty}M \backslash \Omega_k$ and identity on Ω_{k+1}. Then

$$\begin{aligned} Hom_{\mathfrak{wsh}_\Lambda(M)}(T_\epsilon(\mathscr{F}), T_\epsilon(\mathscr{G})) &= \operatorname*{colim}_{\delta\to 0^+} Hom(T_\delta(\mathscr{F}), T_\epsilon(\mathscr{G})) \\ &= \operatorname*{colim}_{\delta\to 0^+} Hom(\mathscr{F}, T_{\epsilon-\delta}(\mathscr{G})). \end{aligned}$$

Then the right hand side is a constant since $Hom(\mathscr{F}, T_{\epsilon-\delta}(\mathscr{G})) \xleftarrow{\sim} Hom(\mathscr{F}, \mathscr{G})$ by the perturbation trick Proposition 4.1.2.

On the other hand, we also know that $\mathfrak{W}^+_\Lambda : \mathfrak{w}\mathrm{sh}_\Lambda(M) \to Sh^b_\Lambda(M)$ is an equivalence [**104**, Theorem 1.3]. Therefore

$$Hom_w(T_\epsilon(\mathscr{F}), T_\epsilon(\mathscr{G})) \simeq Hom(\mathfrak{W}^+_\Lambda \circ T_\epsilon(\mathscr{F}), \mathfrak{W}^+_\Lambda \circ T_\epsilon(\mathscr{G})) = Hom(S^+_\Lambda(\mathscr{F}), S^+_\Lambda(\mathscr{G})).$$

Since S^-_Λ is the right adjoint of S^+_Λ, we know that $S^-_\Lambda \circ S^+_\Lambda = \mathrm{id}_{Sh^b_\Lambda(M)}$.

Then consider $\mathscr{F}, \mathscr{G} \in Sh^b_\Lambda(M)$. Sabloff-Serre duality Proposition 4.1.6 implies that

$$Hom(S^-_\Lambda(\mathscr{F}), S^-_\Lambda(\mathscr{G})) = Hom(\mathscr{G} \otimes \omega_M^{-1}, S^-_\Lambda(\mathscr{F}))^\vee = Hom(\mathscr{F}, \mathscr{G}).$$

Then since S^-_Λ is the right adjoint of S^+_Λ, we know that $S^-_\Lambda \circ S^+_\Lambda = \mathrm{id}_{Sh^b_\Lambda(M)}$. $\square$

Next, we show Condition (4) that there is a natural isomorphism of functors $m^r_\Lambda \xrightarrow{\sim} S^-_\Lambda \circ m^l_\Lambda[1]$, which again requires Serre duality in Section 4.1.3.

Proposition 6.2.10. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic full Legendrian stop. Then for any $\mathscr{F} \in Sh^b_\Lambda(M)$ and $\mathscr{G} \in \mu Sh^c_\Lambda(\Lambda)$ there is an isomorphism*

$$Hom(\mathscr{F}, m^r_\Lambda(\mathscr{G})) \to Hom(w_\Lambda \circ m_\Lambda(\mathscr{F}), m^l_\Lambda(\mathscr{G})) \to Hom(T_\epsilon(\mathscr{F}), m^l_\Lambda(\mathscr{G}))$$

Proof. Let $\mu_i \in \mu Sh^c_\Lambda(\Lambda)$ be the corepresentatives of microstalks at the point p_i on the smooth stratum $\Lambda_i \subset \Lambda$, which (split) generate the category $\mu Sh^c_\Lambda(\Lambda)$ following Proposition 3.4.2. By Proposition 6.2.5, it suffices to show that for any $\mathscr{F} \in Sh^b_\Lambda(M)$,

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mu_i)) \simeq Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}^+_\Lambda \circ w_\Lambda(\mu_i)).$$

Note that $m^l_\Lambda(\mu_i) \in Sh^b_\Lambda(M)$. By Sabloff-Serre duality Proposition 4.1.6, we know that the right hand side is

$$Hom(T_\epsilon(\mathscr{F}), m^l_\Lambda(\mu_i)[1])^\vee = Hom(\mathscr{F}, T_{-\epsilon} \circ m^l_\Lambda(\mu_i)[1])^\vee = Hom(m^l_\Lambda(\mu_i)[1], \mathscr{F} \otimes \omega_M).$$

On the other hand, since $\mathscr{F}$ is cohomologically constructible, by Theorem 4.2.9 and Proposition 3.1.7, the left hand side is

$$\begin{aligned} Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mu_i))^\vee &= p_*(\mathrm{D}'_M\mathscr{F} \otimes w_\Lambda(\mu_i))^\vee = Hom(\mathrm{D}'_M\mathscr{F} \otimes w_\Lambda(\mu_i), p^!\Bbbk) \\ &= Hom(w_\Lambda(\mu_i), \mathrm{D}_M \circ \mathrm{D}'_M\mathscr{F}) = Hom(w_\Lambda(\mu_i), \mathscr{F} \otimes \omega_M). \end{aligned}$$

Moreover, the continuation map is exactly induced by the continuation map $w_\Lambda(\mu_i) \to \mathfrak{W}_\Lambda^+ \circ w_\Lambda(\mu_i)$. Therefore, the result immediately follows from Proposition 6.1.2. $\square$

By Proposition 6.2.9 and 6.2.10, we can immediately finish the proof of Theorem 6.2.1 and hence the full stop part in Theorem 6.0.3.

6.2.3.2. Spherical functor for swappable stops. Next, we define the notion of a swappable stop, which is introduced by Sylvan [**152**], but is a priori weaker than his terminology.

Definition 6.2.5. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian. Then Λ is called a swappable Legendrian stop if there exists a positive wrapping fixing Λ that sends $T_\epsilon(\Lambda)$ to an arbitrarily small neighbourhood of $T_{-\epsilon}(\Lambda)$.*

Example 6.2.5. *The Legendrian stops in Example 6.2.4 are swappable, and we conjecture that Legendrian stops in Example 6.2.3 are swappable as well. More generally, when $F \subset T^{*,\infty}M$ is a Weinstein page of a contact open book decomposition for $T^{*,\infty}M$* [**80**, **91**]*, then it is a swappable stop.*

However, swappable stops are not necessarily full stops. For instance, for the Landau-Ginzburg model $\pi : T^*M \to \mathbb{C}$ that are not Lefschetz fibrations, $F \subset T^{*,\infty}M$ will in general not be a full stop (one can consider $\pi : \mathbb{C}^n \to \mathbb{C}; \quad \pi = z_1 z_2 \dots z_n$ [**2**, **122**]*). Sylvan* [**152**, Example 1.4] *also explained that one can take any monodromy invariant subset for the Lagrangian skeleton of fiber of a Lefschetz fibration and get a swappable stop.*

Example 6.2.6. *There is another way to get new swappable stops from old ones*[3]*. Consider* $\Lambda \subset \partial_\infty X$ *to be a swappable stop in* $\partial_\infty X$ *the contact boundary of some Liouville domain, and* $\partial_\infty X'$ *the contact boundary of some other Liouville domain. When we take the Liouville connected sum of* X *and* X' *along some subcritical Weinstein hypersurface* $F \hookrightarrow \partial_\infty X$ *and* $F \hookrightarrow \partial_\infty X'$ [**15**]*, as the skeleton* $\mathfrak{c}_F$ *is of subcritical dimension, the positive loop of* Λ *will generically avoid* $\mathfrak{c}_F$*. Therefore,* Λ *is also swappable in* $\partial_\infty(X \#_F X')$*.*

In particular, let $X = \mathbb{C}^n$ *and* $F = \mathbb{C}^{n-1}$*, then* $X \#_F X'$ *is the 1-handle connected sum of* $\mathbb{C}^n$ *and* X'*, which is Liouville homotopy equivalent to* X'*. In particular, for any swappable stop* $\Lambda \subset D^{2n-1} \subset S^{2n-1}$ *(for example, the skeleton of any page of contact open book decomposition), putting it in a Darboux ball in* $T^{*,\infty}M$*, we will get a swappable stop in* $T^{*,\infty}M$*.*

When $\Lambda \subset T^{*,\infty}M$ is a swappable Legendrian stop, then there exists a cofinal positive (resp. negative) wrapping that sends $T_\epsilon(\Lambda)$ to an arbitrary small neighbourhood of $T_{-\epsilon}(\Lambda)$ (resp. sends $T_{-\epsilon}(\Lambda)$ to an arbitrary small neighbourhood of $T_\epsilon(\Lambda)$). So there exists a (cofinal sequence of) positive contact flow φ_k^t, $k \in \mathbb{N}$, supported away from Λ such that $\varphi_k^1(T_\epsilon(\Lambda))$ is contained in a small neighbourhood of $T_{-\epsilon}(\Lambda)$ for $k \gg 0$. We will fix the cofinal sequence of positive flow and check condition (2) and (4) of Definition 6.2.2 by considering this particular wrapping.

Proposition 6.2.11. *Assume* Λ *is swappable. The functors* S_Λ^+ *and* S_Λ^- *are equivalences.*

[3]The authors would like to thank Emmy Murphy who explains to us this construction.

Proof. It's sufficient to check that they are fully-faithful since $S_\Lambda^+ \vdash S_\Lambda^-$ is an adjunction pair. The computation is symmetric so we check that $Hom(S_\Lambda^+(\mathscr{F}), S_\Lambda^+(\mathscr{G})) = Hom(\mathscr{F}, \mathscr{G})$, or equivalently, the canonical map

$$Hom(F, G) = Hom(T_\epsilon(\mathscr{F}), T_\epsilon(\mathscr{G})) \to Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ \circ T_\epsilon(\mathscr{G}))$$

is an isomorphism. First apply Proposition 4.1.2, so that the map factorizes as

$$Hom(T_\epsilon(\mathscr{F}), T_\epsilon(\mathscr{G})) \xrightarrow{\sim} Hom(T_\epsilon(\mathscr{F}), T_{\epsilon+\delta}(\mathscr{G})) \to Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ \circ T_\epsilon(\mathscr{G}))$$

for some $0 < \delta \ll \epsilon$. Then by the swappable assumption, for a sequence of descending open neighbourhoods $\{\Omega_k\}_{k\in\mathbb{N}}$ of $\Lambda \subset T^{*,\infty}M$ such that $\Omega_{k+1} \subseteq \overline{\Omega_k}$ and $\bigcap_{k\in\mathbb{N}} \Omega_k = \Lambda$, there exist (an increasing sequence of) positive Hamiltonian flows φ_k^t, $k \in \mathbb{N}$, supported away from Λ such that

$$\varphi_k^1(T_\epsilon(\Lambda)) \subseteq T_{-1/k}(\Omega_k)$$

for $k \gg 0$. Thus we are in the situation of Lemma 6.1.3. □

Proposition 6.2.12. *When Λ is swappable. The canonical map $m_\Lambda^r \to m_\Lambda^r \circ m_\Lambda \circ m_\Lambda^l \to S_\Lambda^- \circ m_\Lambda^l[1]$ is an isomorphism.*

Proof. By Proposition 6.2.5, it's sufficient to show that the map

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) \to Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ \circ w_\Lambda(\mathscr{G}))$$

is an isomorphism. Since $SS^\infty(w_\Lambda(\mathscr{G})) \subseteq T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)$, we can again flow it forward by the Reeb flow T_δ for some $0 < \delta \ll \epsilon$ by Proposition 4.1.2 such that the above map factorizes as

$$Hom(T_\epsilon(\mathscr{F}), w_\Lambda(\mathscr{G})) = Hom(T_\epsilon(\mathscr{F}), T_\delta \circ w_\Lambda(\mathscr{G})) \to Hom(T_\epsilon(\mathscr{F}), \mathfrak{W}_\Lambda^+ \circ w_\Lambda(\mathscr{G})).$$

But then the same proof from the last proposition applies. By the swappable assumption, for a sequence of descending open neighbourhoods $\{\Omega_k\}_{k\in\mathbb{N}}$ of $\Lambda \subset T^{*,\infty}M$ such that $\Omega_{k+1} \subseteq \overline{\Omega_k}$ and $\bigcap_{k\in\mathbb{N}} \Omega_k = \Lambda$, there exist (an increasing sequence of) positive Hamiltonian flows φ_k^t, $k \in \mathbb{N}$, supported away from Λ such that

$$\varphi_k^t(T_\epsilon(\Lambda) \cup T_{-\epsilon}(\Lambda)) \subseteq T_{-1/k}(\Omega_k) \cup T_{-1/2k}(\Lambda)$$

for $k \gg 0$. We can again apply Lemma 6.1.3 to conclude that the map is an isomorphism. □

By Proposition 6.2.11 and 6.2.12, we can immediately finish the proof of Theorem 6.2.1 and hence the swappable stop part of Theorem 6.0.3.

6.3. Spherical Adjunction and Serre Duality on Subcategories

Spherical adjunction on the category of all sheaves and microsheaves will also tell us about information on the corresponding subcategories. Restricting attention to the pair of sheaf categories of compact objects, and the corresponding pair of sheaf categories of proper objects when the manifold is compact, which are the sheaf theoretic models of suitable versions of Fukaya categories, we can show the following corollary.

Corollary 6.3.1. *Let* $\Lambda \subset T^{*,\infty}M$ *be a closed subanalytic Legendrian. Suppose* Λ *is either a swappable stop or a geometric full stop. Then the microlocalization functor along* Λ *on the sheaf category of objects with perfect stalks*

$$m_\Lambda : Sh^b_\Lambda(M) \to \mu Sh^b_\Lambda(\Lambda)$$

is a spherical functor. Respectively, the left adjoint of the microlocalization functor on the sheaf category of compact objects

$$m^l_\Lambda : \mu Sh^c_\Lambda(\Lambda) \to Sh^c_\Lambda(M)$$

is also a spherical functor.

We also prove that S^-_Λ is in fact the Serre functor on $Sh^b_\Lambda(M)$, when Λ is a full stop or swappable stop. Thus we will prove the folklore conjecture on partially wrapped Fukaya categories associated to Lefschetz fibrations formulated by Seidel [**140**] (who attributes the conjecture to Kontsevich) with partial results in [**141–143**], in the case of partially wrapped Fukaya categories on cotangent bundles.

6.3.1. Spherical adjunction on subcategories

In this section, we will restrict to the subcategories of proper objects and compact objects of sheaves. Over the category of proper objects of sheaves (equivalently, sheaves with perfect stalks), we will show that the microlocalization

$$m_\Lambda : Sh^b_\Lambda(M) \to \mu Sh^b_\Lambda(\Lambda)$$

is a spherical functor, and over the category of compact objects of sheaves, we will show that the left adjoint of the microlocalization

$$m^l_\Lambda : \mu Sh^c_\Lambda(M) \to Sh^c_\Lambda(M)$$

is a spherical functor. We know that autoequivalences coming from twists and cotwists immediately restrict to these corresponding subcategories. As a result, once we know the corresponding functors restrict, they will be spherical.

First, we consider the subcategories of compact objects. For the spherical adjunction $m^l_\Lambda \dashv m_\Lambda$. We know that the left adjoint m^l_Λ preserves compact objects, i.e. we have

$$m^l_\Lambda : \mu Sh^c_\Lambda(\Lambda) \to Sh^c_\Lambda(M).$$

However, it is not clear whether microlocalization m_Λ also preserves these objects.

Lemma 6.3.2. *Let $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian stop. When m^r_Λ admits a right adjoint, the essential image of the microlocalization functor*

$$m_\Lambda : Sh^c_\Lambda(M) \to \mu Sh_\Lambda(\Lambda)$$

is contained in $\mu Sh^c_\Lambda(\Lambda)$

Proof. We know that m_Λ^r preserves colimits as it admits a right adjoint. Now since the right adjoint of m_Λ preserves colimits, we can conclude that m_Λ preserves compact objects. □

Whenever $m_\Lambda \vdash m_\Lambda^l$ is a spherical adjunction, we know by Remark 6.2.1 that m_Λ^r admits a right adjoint. Therefore spherical adjunction can always be restricted to the subcategories of compact objects, as we have claimed in Corollary 6.3.1.

Then we consider the subcategories of proper objects. We know that the microlocalization functor preserve proper objects (or equivalently objects with perfect stalks), i.e. we have

$$m_\Lambda : Sh_\Lambda^b(M) \to \mu Sh_\Lambda^b(\Lambda).$$

However, it is not clear whether the left adjoint m_Λ^l and right adjoint m_Λ^r preserves these objects.

Lemma 6.3.3. *Let $\Lambda \subset T^{*,\infty}M$ be a subanalytic Legendrian stop. When m_Λ^r admits a right adjoint, the essential image of the left adjoint of microlocalization functor*

$$m_\Lambda^r : \mu Sh_\Lambda^b(\Lambda) \to Sh_\Lambda(M)$$

is also contained in $Sh_\Lambda^b(M)$.

Proof. We recall Theorem 3.4.3 that

$$Sh_\Lambda^b(M) = \mathrm{Fun}^{ex}(Sh_\Lambda^c(M)^{op}, \mathrm{Perf}(\Bbbk)), \ \ \mu sh_\Lambda^b(\Lambda) = \mathrm{Fun}^{ex}(\mu Sh_\Lambda^c(\Lambda)^{op}, \mathrm{Perf}(\Bbbk)),$$

where the isomorphism is given by the $Hom(-,-)$ pairing on $Sh^c_\Lambda(M)^{op} \times Sh^b_\Lambda(M)$ and respectively on $\mu Sh^c_\Lambda(\Lambda)^{op} \times \mu Sh^b_\Lambda(\Lambda)$. Then since we know that microlocalization preserves compact objects

$$m_\Lambda : Sh^c_\Lambda(M) \to \mu Sh^c_\Lambda(\Lambda),$$

the right adjoint m^r_Λ clearly preserves proper objects. □

Whenever $m_\Lambda \vdash m^l_\Lambda$ is a spherical adjunction, we know by Remark 6.2.1 that m^r_Λ admits a right adjoint, so the adjunction $m^r_\Lambda \vdash m_\Lambda$ can be restricted to the subcategories of proper objects.

Therefore by Remark 6.2.1 the spherical adjunction $m_\Lambda \vdash m^l_\Lambda$ can always be restricted to the subcategories of proper objects, as we have claimed in Corollary 6.3.1.

As we have seen, the candidate right adjoint of m^l_Λ will simply be the microlocalization functor

$$m_\Lambda : \, Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda).$$

The candidate left adjoint of m^l_Λ, by Remark 6.2.1, is the functor

$$m_\Lambda \circ S^-_\Lambda[1] : \, Sh_\Lambda(M) \to \mu sh_\Lambda(\Lambda).$$

The readers may be confused about the non-symmetry as m^l_Λ is supposed to be the cup functor on wrapped Fukaya categories and there is no non-symmetry there. We can provide an explanation as follows. Consider the category of wrapped sheaves in [**104**], we have a preferred equivalence $\mathfrak{W}_\Lambda^+ : \mathfrak{w}\mathrm{sh}_\Lambda(M) \to Sh^c_\Lambda(M)$. If we try to instead replace the domain $Sh^c_\Lambda(M)$ by $\mathfrak{w}\mathrm{sh}_\Lambda(M)$, then m^l_Λ can be replaced by the doubling functor $w_\Lambda[-1]$, and one can easily see that

$$m^r_\Lambda = m_\Lambda \circ \mathfrak{W}^+_\Lambda,\ \ m^l_\Lambda = m_\Lambda \circ \mathfrak{W}^-_\Lambda[1].$$

Then m^r_Λ (resp. m^l_Λ) is indeed the cap functor by wrapping positively (resp. negatively) into the Legendrian Λ and then take microlocalization, i.e. the sheaf theoretic restriction.

6.3.2. Serre functor on proper subcategory

In this section, we finally prove that S^-_Λ is in fact the Serre functor on $Sh^b_\Lambda(M)$, when Λ is a full stop or swappable stop. Thus we will prove the folklore conjecture on partially wrapped Fukaya categories associated to Lefschetz fibrations in the case of partially wrapped Fukaya categories on cotangent bundles.

Let $\mathscr{A}$ be a stable category over $\Bbbk$. Recall that $\mathscr{A}$ is proper category if for any $X, Y \in \mathscr{A}$,

$$Hom_{\mathscr{A}}(X, Y) \in \mathrm{Perf}(\Bbbk).$$

When $\mathscr{A}$ is a proper category, by the above lemma, we are always able to define the (right) dualizing bi-module $\mathscr{A}^*$.

Definition 6.3.1. *For a proper stable category $\mathscr{A}$, the (right) dualizing bi-module $\mathscr{A}^*$ is defined by*

$$\mathscr{A}^*(X,Y) = Hom_{\mathscr{A}}(X,Y)^\vee = Hom_{\Bbbk}(Hom_{\mathscr{A}}(X,Y),\Bbbk).$$

Definition 6.3.2. *For a proper stable category $\mathscr{A}$, a Serre functor $S_{\mathscr{A}}$ is the functor that represents the right dualizing bimodule $\mathscr{A}^*$, i.e.*

$$Hom_{\mathscr{A}}(-,-)^\vee \simeq Hom_{\mathscr{A}}(-, S_{\mathscr{A}}(-)).$$

Proposition 6.3.4. *Let $\Lambda \subset T^{*,\infty}M$ be a full or swappable compact subanalytic Legendrian stop. Then $S_\Lambda^- \otimes \omega_M$ is the Serre functor on $Sh_\Lambda^b(M)_0$ of sheaves with perfect stalks and compact supports. In particular, when M is orientable, $S_\Lambda^-[-n]$ is the Serre functor on $Sh_\Lambda^b(M)_0$.*

Proof. First, by Lemma 6.3.3, we know that $S_\Lambda^- : Sh_\Lambda^b(M) \to Sh_\Lambda^b(M)$ preserves perfect stalks. Moreover, when M is noncompact and Λ is a swappable stop, we argue that S_Λ^- also preserves compact supports. By the swappable assumption, for a sequence of descending open neighbourhoods $\{\Omega_k\}_{k\in\mathbb{N}}$ of $\Lambda \subset T^{*,\infty}M$ such that $\Omega_{k+1} \subseteq \overline{\Omega_k}$ and $\bigcap_{k\in\mathbb{N}} \Omega_k = \Lambda$, there exist (an increasing sequence of) positive Hamiltonian flows φ_k^t, $k \in \mathbb{N}$, supported away from Λ such that

$$\varphi_k^{-1}(T_{-\epsilon}(\Lambda)) \subset T_{1/k}(\Omega_k)$$

for $k \gg 0$. Without loss of generality, we can even assume that φ_k^t are all supported in some common compact subset. Since M is noncompact, consider any unbounded region U such that $\Lambda \cap S^*U = \varnothing$. Then there exists an open subset $U' \subset U \subset M$, $\varphi_k^t(T_\epsilon(\Lambda)) \cap S^*U' = \varnothing$ for $k \gg 0$. Then

$$\Gamma(U', S_\Lambda^-(F)) = \Gamma\Big(U', \lim_{k\to\infty} K(\varphi_k^{-1}) \circ T_{-\epsilon}(F)\Big) = 0.$$

Since $SS^\infty(S_\Lambda^-(F)) \subseteq \Lambda$, we get $\Gamma(U, S_\Lambda^-(F)) = 0$, which implies that $S_\Lambda^-(F)$ has compact support.

Since we have concluded that $S_\Lambda^- : Sh_\Lambda^b(M)_0 \to Sh_\Lambda^b(M)_0$ preserves perfect stalks and compact supports, the proposition then immediately follows from Theorem 6.1.2 and 4.1.6. $\square$

We should remark that, even though Proposition 4.1.6 is true in general, the above statement is not without the assumption on $\Lambda \subset T^{*,\infty}M$. For example, in Section 6.5 we will see an example where S_Λ^- fails to be an equivalence on $Sh_\Lambda^b(M)$.

Finally, we explain the implication of the above result in partially wrapped Fukaya categories. Ganatra-Pardon-Shende [**74**, Proposition 7.24] have proved that there is a commuative diagram intertwining the cup functor and the left adjoint of microlocalization functor

$$\begin{array}{ccc} \mathcal{W}(F) & \xrightarrow{\sim} & \mu Sh^c_{\mathfrak{c}_F}(\mathfrak{c}_F) \\ \Big\downarrow{\scriptstyle \cup_F} & & \Big\downarrow{\scriptstyle m^*_{\mathfrak{c}_F}} \\ \mathcal{W}(T^*M, F) & \xrightarrow{\sim} & Sh^c_{\mathfrak{c}_F}(M). \end{array}$$

Sylvan has shown that the spherical twist associated to the cup functor is the wrap-once functor [**152**], and hence it intertwines with the wrap-once functor in sheaf categories. Consequently, we have proven that the negative wrap-once functor

$$\mathcal{S}_\Lambda^- : \mathrm{Prop}\,\mathcal{W}(T^*M, F) \to \mathrm{Prop}\,\mathcal{W}(T^*M, F)$$

is indeed the Serre functor on $\mathrm{Prop}\,\mathcal{W}(T^*M, F)$.

In particular, let $\pi : T^*M \to \mathbb{C}$ be a symplectic Lefschetz fibration and $F = \pi^{-1}(\infty)$ be the Weinstein fiber. Let $\mathfrak{c}_F$ be the Lagrangian skeleton of F. Then by Ganatra-Pardon-Shende [**74, 75**] we know that

$$\mathrm{Perf}\,\mathcal{W}(T^*M, F) = \mathrm{Prop}\,\mathcal{W}(T^*M, F)$$

is a proper subcategory. Therefore, S_Λ^- is the Serre functor on partially wrapped Fukaya category associated to Lefschetz fibrations.

Remark 6.3.1. *Finally, we remark that according to the result of Katzarkov-Pandit-Spaide* [**99**]*, existence of spherical adjunction in the next section together with a compatible Serre functor will imply existence of the weak relative proper Calabi-Yau structure introduced in* [**21**] *of the pair*

$$m_\Lambda : Sh^b_\Lambda(M) \to \mu Sh^b_\Lambda(\Lambda)$$

(even though we do not explicitly show compatibility of the Serre functor, we believe that it is basically clear from the definition in [**99**]*).*

However, we will see in the last section that spherical adjunction does not hold in all these pairs that are expected to be relative Calabi-Yau (on the contrary, as explained in Sylvan [**152**] *or Remark 4.2.6, when we consider microlocalization along a single component of a Legendrian stop with multiple components*

$$m_{\Lambda_i} : Sh_\Lambda(M) \to \mu Sh_\Lambda(\Lambda) \to \mu Sh_{\Lambda_i}(\Lambda_i)$$

we will still get spherical adjunctions, but the pair is unlikely to be relative Calabi-Yau). We will investigate relative Calabi-Yau structures separately (and hopefully, in full generality) in future works.

6.4. Spherical Pairs and Perverse Schöbers

The description of spherical pairs comes from the relation between spherical functors and perverse sheaves of categories (called perverse schöbers) on a disk with one singularity [**96**]. For a perverse schöber on $\mathbb{D}^2$ with singularity at 0 associated to the spherical functor

$$F : \mathscr{A} \to \mathscr{B}$$

consider a single cut $[0, 1] \subset \mathbb{D}^2$. Then the nearby category at 0 is $\mathscr{A}$ while the vanishing category on $(0, 1]$ is $\mathscr{B}$, and the spherical twist is determined by monodromy around $\mathbb{D}^2\backslash\{0\}$. Kapranov-Schechtman realized a symmetric description of the perverse schöber determined by the diagram

$$\mathscr{B}_- \xleftarrow{F_-} \mathscr{C} \xrightarrow{F_+} \mathscr{B}_+,$$

by considering a double cut on the disk $[-1,1] \subset \mathbb{D}^2$. The nearby category at 0 is $\mathscr{C}$ while the vanishing category on $[-1,0)$ (resp. $(0,1]$) is $\mathscr{B}_-$ (resp. $\mathscr{B}_+$). The nearby category $\mathscr{C}$ will carry a 4-periodic semi-orthogonal decomposition. Such a viewpoint will provide new information of the microlocal sheaf categories.

Given our formalism of spherical adjunctions, we will prove as a corollary spherical pairs which give rise to non-trivial equivalences of microlocal sheaf categories over different Lagrangian skeleta that do not a priori require non-characteristic deformations (while in known examples of such equivalences, [**48**, **165**] the Lagrangian skelata are related by non-characteristic deformations).

Definition 6.4.1. *Let $\Lambda_\pm \subset S^*M$ be two disjoint closed subanalytic Legendrian stops. Suppose there exists both a positive and a negative compactly supported Hamiltonian flow that sends Λ_+ to an arbitrary small neighbourhood of Λ_-, whose backward flows send Λ_- to an arbitrary small neighbourhood of Λ_+. Then (Λ_-, Λ_+) is called a swappable pair.*

Remark 6.4.1. *When $\Lambda_\pm \subset S^*M$ are Lagrangian skeleta of Weinstein hypersurfaces $F_\pm \subset S^*M$, we do not know whether $(X, F_\pm)$ are in fact Weinstein homotopic, though in some examples we will mention we suspect that they are. Moreover, it is in general a hard question when a singular Lagrangian will arise as the skeleton of a Weinstein manifold* [**61**, Problem 1.1 & Remark 1.2].

We will show that a swappable pair of Legendrian stops produces a spherical pair.

Theorem 6.4.1 (Theorem 6.4.10)**.** *Let $\Lambda_\pm \subset S^*M$ be a swappable pair of closed Legendrian stops. Then $Sh_{\Lambda_-}(M) \simeq Sh_{\Lambda_+}(M), \mu Sh_{\Lambda_-}(\Lambda_-) \simeq \mu Sh_{\Lambda_+}(\Lambda_+)$, and there is a spherical pair*

$$Sh_{\Lambda_-}(M) \rightleftharpoons Sh_{\Lambda_+ \cup \Lambda_-}(M) \leftrightharpoons Sh_{\Lambda_+}(M).$$

Remark 6.4.2. *As in the previous result, we can show that the spherical pair can be restricted to the subcategories of compact objects of sheaf categories, which therefore leads to a result on partially wrapped Fukaya categories.*

Remark 6.4.3. *For symplectic topologists, this result may seem boring since one may suspect that the corresponding Weinstein pairs turn out to be Weinstein homotopic. However, when considering Fukaya-Seidel categories given a Landau-Ginzburg potential, the Weinstein hypersurfaces $F_\pm$ can be fibers of different potential functions. Therefore, studying the behaviour of their Lagrangian skeleta provides a way to compare the categories directly.*

In this section, we discuss how the adjunctions give rise to spherical pairs and semi-orthogonal decompositions, and prove Proposition 6.4.8 and Theorem 6.4.10. Using Proposition 6.4.8, we will also give an explicit characterization of the spherical twists and dual twists.

We will provide precise definitions of the terminologies in this section and then show the results in the introduction. Note that none of the arguments in this section essentially relies on microlocal sheaf theory, and therefore they can all be rewritten using Lagrangian Floer theory.

6.4.1. Semi-orthogonal decomposition

Firstly, we explain how spherical adjunctions give rise to 4-periodic semi-orthogonal decompositions and spherical twists are given by iterated mutations Halpern–Laistner-Shipman [**90**] in the case of dg categories and Dykerhoff-Kapranov-Schechtman-Soibelman [**49**] in general (this is how Sylvan proved that the Orlov cup functor is spherical [**152**]).

Theorem 6.4.2 (Halpern–Laistner-Shipman [**90**], [**49**]). *Let $F : \mathscr{A} \to \mathscr{B}$ be an ∞-functor, and $\mathscr{C}$ be the semi-orthogonal gluing of $\mathscr{A}$ and $\mathscr{B}$ along the graphical bi-module $\Gamma(F)$. Then F is spherical if and only if $\mathscr{C}$ fits into a 4-periodic semi-orthogonal decomposition such that $\mathscr{A}^{\perp\perp\perp\perp} = \mathscr{A}$. The dual twist is the iterated mutation $T_{\mathscr{A}} = R_{\mathscr{A}} \circ R_{\mathscr{A}^{\perp\perp}}$, and the dual cotwist is $S_{\mathscr{B}} = L_{\mathscr{A}} \circ L_{\mathscr{A}^{\perp\perp}}$.*

Remark 6.4.4. *Given a pair of semi-orthogonal decompositions $\mathscr{C} = \langle \mathscr{A}_+, \mathscr{B} \rangle \simeq \langle \mathscr{B}, \mathscr{A}_- \rangle$, the right mutation functor is the equivalence $R_{\mathscr{A}} : \mathscr{A}_+ \to \mathscr{A}_-$ defined by the composition of embedding and projection.*

Therefore, restricting to our setting of sheaf categories, our main theorem is equivalent to the following statement.

Proposition 6.4.3. *Let $\Lambda \subset T^{*,\infty}M$ be a compact subanalytic Legendrian stop. Then under the fully faithful embedding $w_\Lambda : \mu Sh_\Lambda(\Lambda) \to Sh_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)}(M)$, there is* *a semi-orthogonal decomposition*

$$Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) \simeq \langle \mu sh_\Lambda(\Lambda), Sh_{T_\epsilon(\Lambda)}(M)\rangle.$$

Proof. We can check that $Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M)$ is the semi-orthogonal gluing (i.e. the Grothendieck construction) of $\mu Sh_\Lambda(\Lambda)$ and $Sh_\Lambda(M)$ along the graphical bi-module $\Gamma(m_\Lambda)$. First, we show that

$$\langle \mu Sh_\Lambda(\Lambda), Sh_{T_\epsilon(\Lambda)}(M)\rangle \hookrightarrow Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M),$$

By Theorem 4.2.10, Remark 4.2.4 and 4.2.5, we know that

$$Hom(w_\Lambda(\mathscr{F}), T_\epsilon(\mathscr{G})) \simeq 0, \;\; Hom(T_\epsilon(\mathscr{G}), w_\Lambda(\mathscr{F})) \simeq Hom(m_\Lambda(\mathscr{G}), \mathscr{F}).$$

This proves full faithfulness.

For essential surjectivity, consider $\mathscr{F} \in Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M)$. Then Theorem 4.2.9 and Remark 4.2.6 implies the following fiber sequence

$$\mathfrak{W}^+_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)} w_{T_{-\epsilon}(\Lambda)} m_{T_\epsilon(\Lambda)}(\mathscr{F}) \to \mathscr{F} \to \mathfrak{W}^+_{T_\epsilon(\Lambda)}(\mathscr{F}).$$

Since $\mathfrak{W}^+_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)} w_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)}(\mathscr{F}) = w_\Lambda m_{T_{-\epsilon}(\Lambda)}(\mathscr{F})$, we can conclude that

$$w_\Lambda m_{T_{-\epsilon}(\Lambda)}(\mathscr{F}) \to \mathscr{F} \to \mathfrak{W}^+_{T_\epsilon(\Lambda)}(\mathscr{F}),$$

where $\mathfrak{W}^+_{T_{-\epsilon}(\Lambda)}(\mathscr{F}) \in Sh_{T_\epsilon(\Lambda)}(M)$ and $m_{T_{-\epsilon}(\Lambda)}(\mathscr{F}) \in \mu Sh_\Lambda(\Lambda)$. This shows the essential surjectivity and thus shows the semi-orthogonal decomposition. $\square$

Corollary 6.4.4. *The semi-orthogonal decomposition can be restricted to compact objects, namely there is a semi-orthogonal decomposition*

$$Sh^c_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) \simeq \langle Sh^c_{T_\epsilon(\Lambda)}(M), \mu Sh^c_\Lambda(\Lambda)\rangle.$$

Proof. Consider the fiber sequence of categories

$$\mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) \twoheadrightarrow Sh_{T_\epsilon(\Lambda)}(M).$$

For $\mathscr{F} \in Sh^c_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M)$, we know that $\mathfrak{W}^+_{T_\epsilon(\Lambda)}(\mathscr{F}) \in Sh^c_{T_\epsilon(\Lambda)}(M)$ since by $\mathfrak{W}^+_{T_\epsilon(\Lambda)}$ is the stop removal functor as explained in Remark 3.4.2 and Proposition 3.4.2.

Moreover, by Proposition 3.4.2 we know that the fiber of the stop removal functor is compactly generated by the corepresentatives of microstalks at $T_{-\epsilon}(\Lambda)$ in the category $Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M)$, which by definition are

$$m^l_{T_{-\epsilon}(\Lambda)}(\mu_i) = \mathfrak{W}^+_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)} w_{T_{-\epsilon}(\Lambda)}(\mu_i) = w_\Lambda(\mu_i),$$

where $\mu_i \in \mu Sh_\Lambda(\Lambda)$ are corepresentatives of the microstalks in the category $\mu Sh_\Lambda(\Lambda)$. Then when restricting to compact objects, the fiber is $\mu Sh^c_\Lambda(\Lambda)$ (split) generated by $\mu_i \in \mu Sh^c_\Lambda(\Lambda)$. □

Remark 6.4.5. *We explain how this is related to Sylvan's proof of the spherical adjunction* [**152**, Section 4]. *Sylvan considered a sectorial gluing of the original Weinstein sector* (X, F) *and the* A_2*-sector* $F\langle 2\rangle = (\mathbb{C}, \{e^{2\pi\sqrt{-1}j/3}\infty\}_{0\le j\le 2}) \times F$, *and* *showed semi-orthogonal decompositions for the ambient sector* $(X, F) \cup_F F\langle 2\rangle$*. However, the sector* $(X, F) \cup_F F\langle 2\rangle$ *is exactly* $(X, T_{-\epsilon}(F) \cup T_\epsilon(F))$*.*

Example 6.4.6. *Let* $\Lambda \subset J^1(M) \cong S^*_{\tau>0}(M \times \mathbb{R})$ *be a smooth Legendrian with no Reeb chords (i.e.* Λ *is the Legendrian lift of an embedded Lagrangian). Then by* [**84**, Proposition 24.1] *we know that the compactly supported sheaves* $Sh_\Lambda(M \times \mathbb{R})_0 \simeq 0$*, and therefore the compactly supported sheaves with singular support on the double copied Legendrian is*

$$Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M \times \mathbb{R})_0 \simeq \mu Sh_\Lambda(\Lambda).$$

It is interesting to consider the case when Λ *is a singular Legendrian with no Reeb chords, which should relate to the framework of Nadler-Shende* [**124**, **146**] *where they embedded the Lagrangian skeleton of a Weinstein sector into unit cotangent bundles via the h-principle.*

Example 6.4.7. *Let* $\Lambda_{loose} \subset J^1(M) \cong S^*_{\tau>0}(M \times \mathbb{R})$ *be a stabilized or (not necessarily smooth) loose Legendrian* [**40**, Chapter 7; **116**; **117**]*. Then by* [**148**, Proposition 5.8] *we know that the compactly supported sheaves* $Sh_{\Lambda_{loose}}(M \times \mathbb{R})_0 \simeq 0$*, and therefore the compactly supported sheaves with singular support on the double copied Legendrian is*

$$Sh_{T_{-\epsilon}(\Lambda_{loose})\cup T_\epsilon(\Lambda_{loose})}(M \times \mathbb{R})_0 \simeq \mu Sh_{\Lambda_{loose}}(\Lambda_{loose}).$$

Moreover, from the point of view of K-theory, the semi-orthogonal gluing (i.e. the Gronthendieck construction) fits into a (diagram of) simplicial ∞-category being the relative Waldhausen S-construction. Following Dyckerhoff-Kapranov-Schechtman-Soilbelman [**49**], we state the following conjecture.

Conjecture 6.4.5. *The relative Waldhausen S-construction of the left adjoint of microlocalization functor is the simplicial ∞-category*

$$S_n(m_\Lambda^l) = Sh_{\bigcup_{j=0}^n T_{j\epsilon}(\Lambda)}(M).$$

Remark 6.4.8. *Consider a sectorial gluing of the original Weinstein sector (X, F) and the A_{n+1}-sector $F\langle n+1\rangle = (\mathbb{C}, \{e^{2\pi\sqrt{-1}j/(n+2)}\infty\}_{0\leq j\leq n+1}) \times F$. Then our geometric model for Waldhausen S-construction is*

$$(X, F) \cup_F F\langle n+1\rangle = \Big(X, \bigcup_{0\leq j\leq n} T_{j\epsilon}(F)\Big).$$

Going back to semi-orthogonal decompositions and spherical adjunctions, combining Theorem 6.4.2 and Proposition 6.4.3, we immediately get the following corollary from the spherical adjunction we have proved.

Corollary 6.4.6. *Let $\Lambda \subset T^{*,\infty}M$ be a swappable Legendrian stop or full Legendrian stop. Then under the fully faithful functor $w_\Lambda : \mu Sh_\Lambda(\Lambda) \to Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M)$, there are semi-orthogonal decompositions*

$$Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) \simeq \langle \mu Sh_\Lambda(\Lambda), Sh_{T_\epsilon(\Lambda)}(M)\rangle \simeq \langle Sh_{T_{-\epsilon}(\Lambda)}(M), \mu Sh_\Lambda(\Lambda)\rangle.$$

Conjecture 6.4.7. *When $\Lambda \subset T^{*,\infty}M$ be a swappable Legendrian stop or full Legendrian stop, we expect that the simplicial ∞-category*

$$S_n(m_\Lambda^l) = Sh_{\bigcup_{j=0}^n T_{j\epsilon}(\Lambda)}(M)$$

can be lifted to a paracyclic ∞-category.

In fact, the semi-orthogonal decompositions

$$Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) = \langle \mu Sh_\Lambda(\Lambda), Sh_{T_\epsilon(\Lambda)}(M)\rangle = \langle Sh_{T_{-\epsilon}(\Lambda)}(M), \mu Sh_\Lambda(\Lambda)\rangle$$

also provide (trivial) examples of spherical pairs, which we now introduce.

For a diagram of ∞-functors over stable ∞-categories

$$\mathscr{B}_- \xleftarrow{F_-} \mathscr{C} \xrightarrow{F_+} \mathscr{B}_+$$

where $F_\pm$ admit fully faithful left and adjoints $F_\pm^{l,r}$, we can write

$$\mathscr{A}_- = \ker(F_+) =^\perp (F_+^r \mathscr{B}_+) = (F_+^l \mathscr{B}_+)^\perp, \ \ \mathscr{A}_+ = \ker(F_-) =^\perp (F_-^r \mathscr{B}_-) = (F_-^l \mathscr{B}_-)^\perp,$$

and write $\iota_\pm : \mathscr{A}_\pm \to \mathscr{C}$ (which admits left and right adjoints $\iota_\pm^*$ and $\iota_\pm^!$). The following definition is essentially a reinterpretation of the conditions in Theorem 6.4.2.

Definition 6.4.2. *A diagram of ∞-functors over stable ∞-categories*

$$\mathscr{B}_- \xleftarrow{F_-} \mathscr{C} \xrightarrow{F_+} \mathscr{B}_+$$

is called a spherical pair if $F_\pm$ admit fully faithful left and right adjoints $F_\pm^{l,r}$ such that

(1) the compositions $F_+^l \circ F_- : \mathscr{B}_+ \to \mathscr{B}_-$, $F_-^l \circ F_+ : \mathscr{B}_- \to \mathscr{B}_+$ are equivalences;

(2) the compositions $\iota_+^! \circ \iota_- : \mathscr{A}_+ \to \mathscr{A}_-$, $\iota_-^! \circ \iota_+ : \mathscr{A}_- \to \mathscr{A}_+$ are equivalences.

Then Corollary 6.4.6 immediately implies the following proposition.

Proposition 6.4.8. *Let $\Lambda \subset T^{*,\infty}M$ be a swappable Legendrian stop or full Legendrian stop. Then there exists spherical pairs of the form*

$$\mu sh_{T_{-\epsilon}(\Lambda)}(T_{-\epsilon}(\Lambda)) \leftarrow Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) \to \mu sh_{T_\epsilon(\Lambda)}(T_{-\epsilon}(\Lambda)).$$

Meanwhile, there is also a spherical pair

$$Sh_{T_{-\epsilon}(\Lambda)}(M) \to Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) \leftarrow Sh_{T_\epsilon(\Lambda)}(M).$$

Remark 6.4.9. *We can restrict the spherical pairs to the subcategories of compact or proper objects as explained in Section 6.3.1 and Corollary 6.4.4.*

Moreover, from the description in Theorem 6.4.2, we can show that the spherical twists (resp. dual twists) are simply the positive (resp. negative) monodromy functor, under the inclusion by the doubling functor.

Corollary 6.4.9. *Let $\Lambda \subset T^{*,\infty}M$ be a swappable Legendrian stop or full Legendrian stop. Under the inclusion $w_\Lambda : \mu Sh_\Lambda(\Lambda) \hookrightarrow Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M)$, the spherical* *dual twist is computed by*

$$S^-_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)} \circ S^-_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}|_{w_\Lambda(\mu Sh_\Lambda(\Lambda))}[2].$$

Similarly, the spherical dual cotwist is computed by

$$S^+_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)} \circ S^+_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}|_{Sh_{T_\epsilon(\Lambda)}(M)} = S^+_{T_\epsilon(\Lambda)}.$$

Proof. By Proposition 6.4.8 and Theorem 6.4.2, it suffices to show that the right mutation functor

$$R_{\mu Sh_\Lambda(\Lambda)} : w_\Lambda(\mu Sh_\Lambda(\Lambda)) \hookrightarrow Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) \to w_\Lambda(\mu sh_\Lambda(\Lambda))$$

is the functor $S^-_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}|_{w_\Lambda(\mu Sh_\Lambda(\Lambda))}[1]$. Consider the pair of semi-orthogonal decompositions

$$Sh_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}(M) = \left\langle \mu Sh_\Lambda(\Lambda), Sh_{T_\epsilon(\Lambda)}(M)\right\rangle = \left\langle Sh_{T_{-\epsilon}(\Lambda)}(M), \mu Sh_\Lambda(\Lambda)\right\rangle.$$

The first semi-orthogonal decomposition is realized in Proposition 6.4.3 by

$$m^l_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)}(\mathscr{F}) \to \mathscr{F} \to \mathfrak{W}^+_{T_\epsilon(\Lambda)}\mathscr{F}.$$

Following a similar argument, the second semi-orthogonal decomposition is realized by the fiber sequence

$$\mathfrak{W}^-_{T_{-\epsilon}(\Lambda)}\mathscr{F} \to \mathscr{F} \to m^r_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)}(\mathscr{F}).$$

Therefore, one can show that the right mutation functor associated to the pair of semi-orthogonal decompositions using Theorem 4.2.9 and Remark 4.2.6

$$\begin{aligned} R_{\mu Sh_\Lambda(\Lambda)}(w_\Lambda m_{T_{-\epsilon}(\Lambda)}(\mathscr{F})) &= m^r_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)} \circ m^l_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)}(\mathscr{F})[1] \\ &= m^r_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)} \circ m^l_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)}(\mathscr{F})[1] = m^r_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)}(\mathscr{F})[1] \\ &= \mathfrak{W}^-_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)} w_{T_{-\epsilon}(\Lambda)} m_{T_{-\epsilon}(\Lambda)}(\mathscr{F})[1] = S^-_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)} w_\Lambda m_{T_{-\epsilon}(\Lambda)}(\mathscr{F})[1]. \end{aligned}$$

One can also compute $R_{\mu Sh_\Lambda(\Lambda)^{\perp\perp}}$ in the same way, which implies the result on spherical twists.

For spherical cotwists, it suffices to show that the left mutation functor is

$$S^+_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)}|_{Sh_{T_\epsilon(\Lambda)}(M)}.$$

Then using the above semi-orthogonal decompositions, we have

$$L_{Sh_{T_\epsilon(\Lambda)}(M)}(\mathfrak{W}^-_{T_\epsilon(\Lambda)} \mathscr{F}) = \mathfrak{W}^+_{T_{-\epsilon}(\Lambda)} \circ \mathfrak{W}^-_{T_\epsilon(\Lambda)} \mathscr{F} = S^+_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)} \circ \mathfrak{W}^-_{T_\epsilon(\Lambda)} \mathscr{F}.$$

One also can compute $L_{Sh_{T_{-\epsilon}(\Lambda)}(\Lambda)}$ in the same way. This implies the result on spherical dual cotwists. Finally, we note that from the computation

$$S^+_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)} \circ S^+_{T_{-\epsilon}(\Lambda) \cup T_\epsilon(\Lambda)}(\mathfrak{W}^-_{T_\epsilon(\Lambda)} \mathscr{F}) = \mathfrak{W}^+_{T_{-\epsilon}(\Lambda)} \circ \mathfrak{W}^+_{T_\epsilon(\Lambda)}(\mathfrak{W}^-_{T_\epsilon(\Lambda)} \mathscr{F}) = S^+_{T_\epsilon(\Lambda)}(\mathfrak{W}^-_{T_\epsilon(\Lambda)} \mathscr{F}),$$

which confirms the last assertion. □

Remark 6.4.10. *Note that the functor* $S^+_{T_{-\epsilon}(\Lambda)\cup T_\epsilon(\Lambda)}$ *intertwines* $T_{-\epsilon}(\Lambda)$ *and* $T_\epsilon(\Lambda)$ *by a positive isotopy. Hence applying the functor twice has the effect of the monodromy functor.*

These seem to be trivial examples of spherical pairs. In the following section, we will provide some examples that are less trivial, namely pairs of different subanalytic Legendrians.

6.4.2. Spherical pairs from variation of skeleta

In general, there are subanalytic Legendrian stops that give rise to spherical pairs which are not necessarily homeomorphic. For example, Donovan-Kuwagaki [**48**] have considered two specific examples from homological mirror symmetry of toric stacks. They presented equivalences between sheaf categories

$$Sh^c_{\Lambda_+}(T^n) \xrightarrow{\sim} Sh^c_{\Lambda_-}(T^n)$$

that are mirror to certain flop-flop equivalences of the mirror toric stacks defined by GIT quotients [**18**] (more generally, it is discussed in [**165**] how different GIT quotients are related by semi-orthogonal decompositions).

Remark 6.4.11. *Unlike in algebraic geometry, where the equivalence is between derived categories of varieties related by flops that are only birational equivalent, in symplectic geometry, we expect that the Weinstein sectors* (T^*T^n, Λ_-) *and* (T^*T^n, Λ_+) *are Weinstein homotopic (see* [**40**, **61**] *for the definition), even though the Lagrangian* *skeleta and associated Landau-Ginzburg potentials* [**73**, **133**, **164**] *are different. This reflects the flexibility on the symplectic side.*

Here we provide a general criterion for this type of equivalences between microlocal sheaf categories. In known examples of such equivalences between microlocal sheaf categories, the Legendrian stops are required to be related by non-characteristic deformations [**48**, **165**]. However, we provide a criterion that does not a priori require existence of non-characteristic deformations thanks to the equivalences from spherical adjunctions (though it often turns out that the Legendrian stops are related by such deformation).

Note that any two Legendrian stops are generically disjoint after a small contact perturbation.

Definition 6.4.3. *Let $\Lambda_\pm \subset T^{*,\infty}M$ be two disjoint closed subanalytic Legendrian stops. Suppose there exists both a positive and a negative Hamiltonian flow that sends Λ_+ to an arbitrary small neighbourhood of Λ_-, and there also exists a positive and a negative Hamiltonian flow that sends Λ_- to an arbitrary small neighbourhood of Λ_+. Then (Λ_-, Λ_+) is called a swappable pair.*

Both of the following examples are considered in [**48**, **165**], though from a different perspective.

Example 6.4.12. *We can consider the mirror to the flops associated to* $X_0 = \mathbb{C}^2/\mathbb{Z}_2$. *On the one hand, consider the Deligne-Mumford quotient stack*

$$X_- = [\mathbb{C}^2/\mathbb{Z}_2].$$

On the other hand, consider the minimal resolution

$$X_+ = \widetilde{\mathbb{C}^2/\mathbb{Z}_2} = \mathrm{Tot}(\mathcal{O}_{\mathbb{CP}^1}(-2)).$$

Under homological mirror symmetry (or coherent-constructible correspondence) of toric stacks [**105**], *the mirrors are Weinstein sectors* $(T^*T^2, \Lambda_\pm)$ *as shown in Figure 6.1. One can show that after a small Reeb perturbation,* $(T_{-\epsilon}(\Lambda_-), T_\epsilon(\Lambda_+))$ *and* $(T_\epsilon(\Lambda_-), T_{-\epsilon}(\Lambda_+))$ *are both swappable pairs of Legendrian stops, as shown in Figure 6.1.*

We can also consider the Atiyah flops associated to $X_0 = \{(x, y, z, w) | zy - zw = 0\} \subset \mathbb{C}^4$ *and their mirror. There are two crepant resolutions along two exceptional rational curves*

$$X_\pm = \mathrm{Tot}(\mathcal{O}_{E_\pm}(-1)^{\oplus 2}).$$

Under homological mirror symmetry (or coherent-constructible correspondence) of toric stacks [**105**], *the mirrors are Weinstein sectors* $(T^*T^3, \Lambda_\pm)$. *One can similarly show that* $(T_{-\epsilon}(\Lambda_-), T_\epsilon(\Lambda_+))$ *and* $(T_\epsilon(\Lambda_-), T_{-\epsilon}(\Lambda_+))$ *are both swappable pairs of Legendrian stops.*

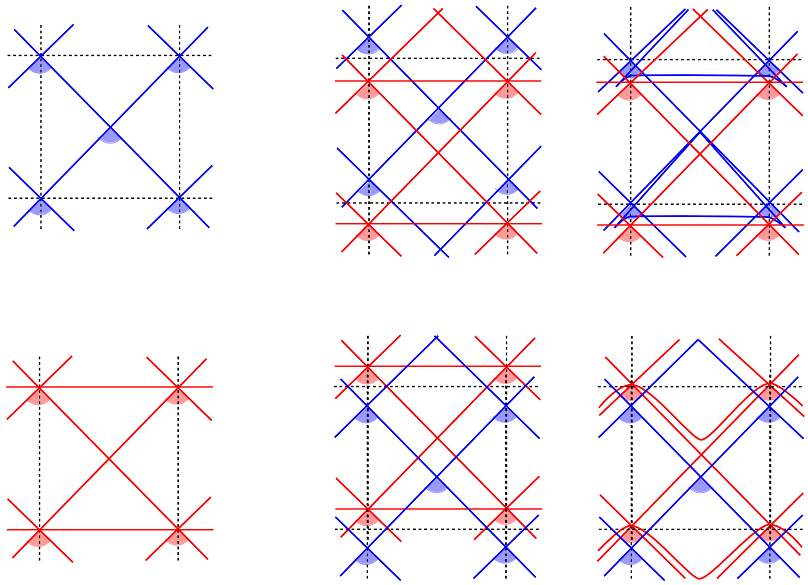

Figure 6.1. The figure on the left illustrates the swappable pair $\Lambda_\pm \subset S^*T^2$ mirror to the flops associated to $X_0 = \mathbb{C}^2/\mathbb{Z}_2$, where all the covectors are pointing downward. The figure on the right illustrates a cofinal wrapping that sends $T_{-\epsilon}(\Lambda)$ to a neighbourhood of $T_\epsilon(\Lambda)$ and one that sends $T_\epsilon(\Lambda_+)$ to a neighbourhood of $T_{-\epsilon}(\Lambda)$.

The main theorem of the section is the following statement, that swappable pairs of Legendrian stops induce spherical pairs of sheaf categories.

Theorem 6.4.10. *Let $\Lambda_\pm \subset T^{*,\infty}M$ be a swappable pair of closed Legendrian stops. Then $Sh_{\Lambda_-}(M) \simeq Sh_{\Lambda_+}(M), \mu sh_{\Lambda_-}(\Lambda_-) \simeq \mu sh_{\Lambda_+}(\Lambda_+)$, and there is a spherical pair*

$$Sh_{\Lambda_-}(M) \leftarrow Sh_{\Lambda_+ \cup \Lambda_-}(M) \to Sh_{\Lambda_+}(M).$$

Proof. We notice that since (Λ_-, Λ_+) is a swappable pair, this implies that $\Lambda_\pm$ are independently swappable in $T^{*,\infty}M$: in fact, we can wrap Λ_- into a small neighbourhood of Λ_+, and then follow the wrapping which sends the neighbourhood of

Λ_+ back into a small neighbourhood of Λ_-. Therefore, the left adjoints of microlocalization

$$m^l_{\Lambda_\pm} : \mu sh_{\Lambda_\pm}(\Lambda_\pm) \to Sh_{\Lambda_\pm}(M)$$

are spherical functors whose spherical twists are $S^+_{\Lambda_\pm}$. On the other hand, for any $\mathscr{F} \in Sh_{\Lambda_-}(M)$ and $\mathscr{G} \in Sh_{\Lambda_+}(M)$, we can define the swapping functors

$$R^+_{\Lambda_-,\Lambda_+}(\mathscr{F}) = S^+_{\Lambda_-\cup\Lambda_+}\mathscr{F} = \mathfrak{W}^+_{\Lambda_+}\mathscr{F}, \;\; R^+_{\Lambda_+,\Lambda_-}(\mathscr{G}) = S^+_{\Lambda_-\cup\Lambda_+}\mathscr{G} = \mathfrak{W}^+_{\Lambda_-}\mathscr{G}.$$

From Corollary 6.4.9, the compositions of swapping functor give the spherical twists

$$S^+_{\Lambda_-} = R^+_{\Lambda_+,\Lambda_-} \circ R^+_{\Lambda_-,\Lambda_+}, \; S^+_{\Lambda_+} = R^+_{\Lambda_-,\Lambda_+} \circ R^+_{\Lambda_+,\Lambda_-}.$$

As a result, we know that $R^+_{\Lambda_-,\Lambda_+}, R^+_{\Lambda_+,\Lambda_-}$ are equivalences. Similarly we can also consider spherical cotwists and show that the corresponding co-swapping functors are equivalences. This implies that

$$Sh_{\Lambda_-}(M) \simeq Sh_{\Lambda_+}(M), \;\; \mu Sh_{\Lambda_-}(\Lambda_-) \simeq \mu Sh_{\Lambda_+}(\Lambda_+).$$

Then consider two new pairs of Legendrian stops $(\Lambda_-, T_\epsilon(\Lambda_-))$ and $(T_{-\epsilon}(\Lambda_+), \Lambda_+)$, obtained by a sufficiently small Reeb push-off. We show that there are equivalences

$$Sh_{\Lambda_-\cup T_\epsilon(\Lambda_-)}(M) \simeq Sh_{\Lambda_-\cup\Lambda_+}(M) \simeq Sh_{T_{-\epsilon}(\Lambda_+)\cup\Lambda_+}(M),$$

such that the corresponding restrictions are the swapping functor

$$R^+_{\Lambda_-,\Lambda_+} : Sh_{\Lambda_-}(M) \to Sh_{T_{-\epsilon}(\Lambda_+)}(M), \;\; Sh_{T_\epsilon(\Lambda_-)}(M) \to Sh_{\Lambda_+}(M)$$

Then since $Sh_{\Lambda_- \cup T_\epsilon(\Lambda_-)}(M)$ and $Sh_{T_{-\epsilon}(\Lambda_+) \cup \Lambda_+}(M)$ are both endowed with semi-orthogonal decompositions by Proposition 6.4.3, this will complete the proof.

Consider the non-nagative wrapping that fixes Λ_- while sending $T_\epsilon(\Lambda_-)$ into Λ_+, and another non-negative wrapping that fixes Λ_- while sending Λ_+ into $T_\epsilon(\Lambda_-)$. Viewing $\Lambda_- \cup T_\epsilon(\Lambda_-)$ and $\Lambda_- \cup \Lambda_+$ independently as two Legendrian stops, the above observation shows that they form a swappable pair as well. Hence we have

$$R^+_{\Lambda_- \cup T_\epsilon(\Lambda_-), \Lambda_- \cup \Lambda_+} : Sh_{\Lambda_- \cup T_\epsilon(\Lambda_-)}(M) \to Sh_{\Lambda_- \cup \Lambda_+}(M),$$

$$R^-_{\Lambda_- \cup \Lambda_+, \Lambda_- \cup T_\epsilon(\Lambda_-)} : Sh_{\Lambda_- \cup \Lambda_+}(M) \to Sh_{\Lambda_- \cup T_\epsilon(\Lambda_-)}(M),$$

where $R^+_{\Lambda_- \cup T_\epsilon(\Lambda_-), \Lambda_- \cup \Lambda_+}$ and $R^-_{\Lambda_- \cup \Lambda_+, \Lambda_- \cup T_\epsilon(\Lambda_-)}$ are inverse equivalences by the definition above. Hence we have shown the equivalence

$$Sh_{\Lambda_- \cup \Lambda_+}(M) \simeq Sh_{\Lambda_- \cup T_\epsilon(\Lambda_-)}(M)$$

which realizes $Sh_{\Lambda_- \cup \Lambda_+}(M)$ as semi-orthogonal decompositions.

Finally, to identify the projection functors with the stop removal functors, i.e. positive wrapping functors, we only need to notice that the following diagram commutes

$$
\begin{array}{ccccccc}
Sh_{\Lambda_-}(M) & \xleftarrow{\mathfrak{w}^+_{\Lambda_-}} & Sh_{\Lambda_-\cup\Lambda_+}(M) & = & Sh_{\Lambda_-\cup\Lambda_+}(M) & \xrightarrow{\mathfrak{w}^+_{\Lambda_+}} & Sh_{\Lambda_+}(M) \\
\Vert & & \mathfrak{w}^+_{\Lambda_-\cup\Lambda_+} \uparrow \wr & & \wr \downarrow \mathfrak{w}^+_{T_{-\epsilon}(\Lambda_+)\cup\Lambda_+} & & \Vert \\
Sh_{\Lambda_-}(M) & \xleftarrow[\mathfrak{w}^+_{\Lambda_-}]{} & Sh_{\Lambda_-\cup T_\epsilon(\Lambda_-)}(M) & \xrightarrow{\sim} & Sh_{T_{-\epsilon}(\Lambda_+)\cup\Lambda_+}(M) & \xrightarrow[\mathfrak{w}^+_{\Lambda_+}]{} & Sh_{\Lambda_+}(M).
\end{array}
$$

This completes the proof of the theorem. □

We now explain how the above result gives rise to more general equivalences and autoequivalences of microlocal sheaf categories other than the spherical twists and cotwists defined by wrapping around. In particular, we will discuss the relation to the work of Donovan-Kuwagaki [**48**], who provide autoequivalences that are mirror to the flop-flop equivalence in two specific examples.

Throughout this subsection, we will work with the subcategory of compact objects $Sh^c_\Lambda(M)$ instead of the large stable category $Sh_\Lambda(M)$.

Let $\Lambda_\pm = \Lambda_0 \cup H_\pm \subset T^{*,\infty}M$ be subanalytic Legendrian stops such that there is a positive Hamiltonian isotopy fixed on Λ_0 that sends H_- to an arbitrary small neighbourhood of Λ_+, and a positive isotopy fixed on Λ_0 that sends H_+ to an arbitrary small neighbourhood of Λ_-.

Then by a small perturbation, we know that the pair

$$(T_{-\epsilon}(\Lambda_-), T_\epsilon(\Lambda_+)), \quad (T_{-\epsilon}(\Lambda_+), T_\epsilon(\Lambda_-))$$

form a swappable pair. By results in the previous section, the following compositions are equivalences

$$Sh^c_{\Lambda_-}(M) \to Sh^c_{T_{-\epsilon}(\Lambda_-)\cup T_\epsilon(\Lambda_+)}(M) \to Sh^c_{\Lambda_+}(M),$$

$$Sh^c_{\Lambda_+}(M) \to Sh^c_{T_{-\epsilon}(\Lambda_+)\cup T_\epsilon(\Lambda_-)}(M) \to Sh^c_{\Lambda_-}(M).$$

Example 6.4.13. *Weinstein pairs in Example 6.4.12 will give examples. Suppose* $\Lambda_0 = \Lambda_- \cap \Lambda_+$. *We can then write* $\Lambda_\pm = \Lambda_0 \cup H_\pm$, *where* $H_\pm$ *are Legendrian disks with boundary* $\partial H_\pm \subset \Lambda_0$ *as in Figure 6.2. In fact, we expect that the Weinstein thickenings of* $\Lambda_\pm$ *are Weinstein homotopic, and the Weinstein thickening of* $\Lambda_-\cup\Lambda_+$ *is a Weinstein stacking* [**108**] *of the two Weinstein hypersurfaces.*

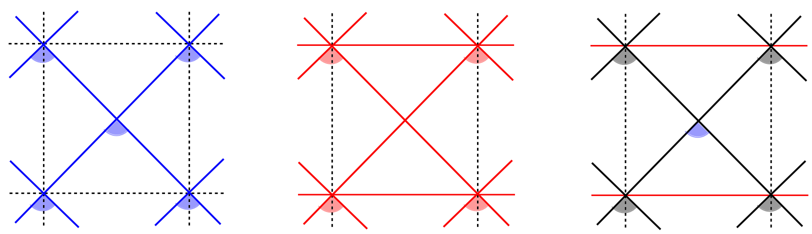

Figure 6.2. The figure on the left and in the middle are the Legendrian pairs $\Lambda_\pm \subset S^*T^2$, where all conormal directions are pointing downward. The figure on the right is $\Lambda_-\cup\Lambda_+ = \Lambda_0\cup H_-\cup H_1 \subset S^*T^2$ where $H_\pm$ have the corresponding colors as $\Lambda_\pm$.

In the above equivalences, the autoequivalence obtained by compositions

$$Sh^c_{\Lambda_-}(M) \xrightarrow{\sim} Sh^c_{\Lambda_+}(M) \xrightarrow{\sim} Sh^c_{\Lambda_-}(M)$$

is no longer the spherical twist by wrapping around the contact boundary as Theorem 6.4.10. Yet we expect that under appropriate assumptions, these autoequivalences are spherical twists of a spherical pair

$$Sh^c_{\Lambda_-}(M) \leftarrow Sh^c_{\Lambda_- \cup \Lambda_+}(M) \rightarrow Sh^c_{\Lambda_+}(M),$$

and are mirror to the spherical pair associated to the flop-flop equivalence from Bozenta-Bondal [**18**]. Donovan-Kuwagaki [**48**] have proved this for Example 6.4.13.

However, currently, using microlocal sheaf theoretic methods, almost the only way to get semi-orthogonal decompositions or reversed inclusions

$$Sh^c_{\Lambda_-}(M) \hookrightarrow Sh^c_{\Lambda_- \cup \Lambda_+}(M)$$

is to assume that $\Lambda_\pm$ are full stops, i.e. $Sh^c_{\Lambda_\pm}(M) = Sh^b_{\Lambda_\pm}(M)$ [**41**, Section 6.3]. Using homological mirror symmetry or coherent-constructible correspondence, it is possible to prove semi-orthogonal decompositions for some limited cases of non-full stops like Example 6.4.13 [**48**, **164**], but to our knowledge, for the moment there is no general statement.

6.5. Example That Wrap-once Is Not Equivalence

When introducing the notion of a swappable stop, Sylvan has already noticed the strong constraint that swappability puts on the stop [**152**]. Now we have proved that full stops also implies sphericality. However, it is not know whether being a full or swappable stop is a necessary condition, or even whether any condition is really needed to show spherical adjunction.

Here we provide an example where $S_\Lambda^+ : Sh_\Lambda^c(M) \to Sh_\Lambda^c(M)$ is not an equivalence. Moreover, our computation implies that in this case, m_Λ does not preserve compact objects and m_Λ^l does not preserve proper modules (or equivalently, sheaves with perfect stalks).

In order to compute the wrap-once functor, we need the following geometric criterion for cofinal wrappings.

Lemma 6.5.1 ([**76**, Lemma 3.29]). *Let $\Sigma \subset T^{*,\infty}M\backslash\Lambda$ be a subanalytic Legendrian, and φ_k be an increasing sequence of contact flows on $T^{*,\infty}M\backslash\Lambda$. Suppose there exists a contact form α on $T^{*,\infty}M\backslash\Lambda$ such that*

$$\lim_{k\to\infty}\int_0^1 \min_{\varphi_k^t(\Sigma)} \alpha\big(\partial_t\varphi_k|_{\varphi_k^t(\Sigma)}\big)\,dt = \infty.$$

Then $\{\varphi_k\}_{k\in\mathbb{N}}$ is a cofinal sequence of wrappings in the category of positive wrappings of Σ in $T^{*,\infty}M\backslash\Lambda$.

Proposition 6.5.2. *Let $M = T^n = \mathbb{R}^n/\mathbb{Z}^n$, $\Lambda = T_0^{*,\infty}T^n \subset T^{*,\infty}T^n$ $(n \geq 2)$, and $\overline{B_\epsilon(0)}$ be a closed ball of radius ϵ around 0. Then $S_\Lambda^-(1_0) = \mathfrak{W}_\Lambda^- \circ \Bbbk_{\overline{B_\epsilon(0)}} \notin Sh_\Lambda^b(T^n)$. In particular, S_Λ^- does not induce an equivalence on the proper subcategory $Sh_\Lambda^b(T^n)$.*

Proof. Let $\epsilon > 0$ be a small positive number and $\eta_k : T^n \to [0,1]$ be a smooth cut-off function such that for small neighbourhoods $\overline{B_{\epsilon/2k^2}(0)} \subset \overline{B_{\epsilon/k^2}(0)}$ around 0 we have

$$\eta_k|_{\overline{B_{\epsilon/2k^2}(0)}} = 0, \;\; \eta|_{T^n\backslash\overline{B_{\epsilon/k^2}(0)}} = 1.$$

Let the decreasing sequence of negative Hamiltonians be $H_k(x,\xi) = -k\,\eta_k(x)|\xi|^2$ and the contact flow be φ_k. Rescale the contact form on $T^*T^n\backslash T_0^*T^n$ by a function $\delta(x) \to \infty$, $x \to 0$. One can easily check by the above lemma that φ_k is a cofinal sequence of positive wrappings on $T^{*,\infty}T^n\backslash\Lambda$. We will show that $\lim_{k\to\infty} K(\varphi_k) \circ \Bbbk_{\overline{B_\epsilon(0)}}$ does not have perfect stalk at $T^n\backslash\{0\}$.

For simplicity, we now assume that $n = 2$. Consider the universal cover $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong T^n$. Let the lifting of the Hamiltonian be $\overline{H}_k(x,\xi) = -k\,\eta_k(\pi(x))|\xi|^2$ and the lifting of the flow be $\overline{\varphi}_k$. Then

$$K(\varphi_k) \circ \Bbbk_{\overline{B_\epsilon(0)}} = \pi_*\big(K(\overline{\varphi}_k) \circ \Bbbk_{\overline{B_\epsilon(0)}}\big).$$

We then show that in each region of the form $\square_m = [m+\epsilon, m+1-\epsilon] \times [\epsilon, 1-\epsilon]$ where $m \geq 0$, when $k \geq m+1$ we have

$$\Bbbk \hookrightarrow \Gamma\big(\square_m, K(\varphi_k) \circ \Bbbk_{\overline{B_\epsilon(0)}}\big).$$

In fact, for the outward unit conormal bundle of $\overline{B_\epsilon(0)}$, under the Hamiltonian H_k, the boundary arc of the sector S_k in between the rays $\theta = \arcsin(\epsilon/k^2)$ and $\theta = \arctan(1/k) - \arcsin(\epsilon/k^2(1+k^2)^{1/2})$ will follow the inverse geodesic flow $\overline{H}_k = -k|\xi|^2$ determined by $\partial_t\varphi_k = k\partial/\partial r$. Therefore there is an injection

$$\Bbbk \hookrightarrow \Gamma\big(S_k, K(\varphi_k) \circ \Bbbk_{\overline{B_\epsilon(0)}}\big) \hookrightarrow \Gamma\big(\square_m, K(\overline{\varphi}_k) \circ \Bbbk_{\overline{B_\epsilon(0)}}\big).$$

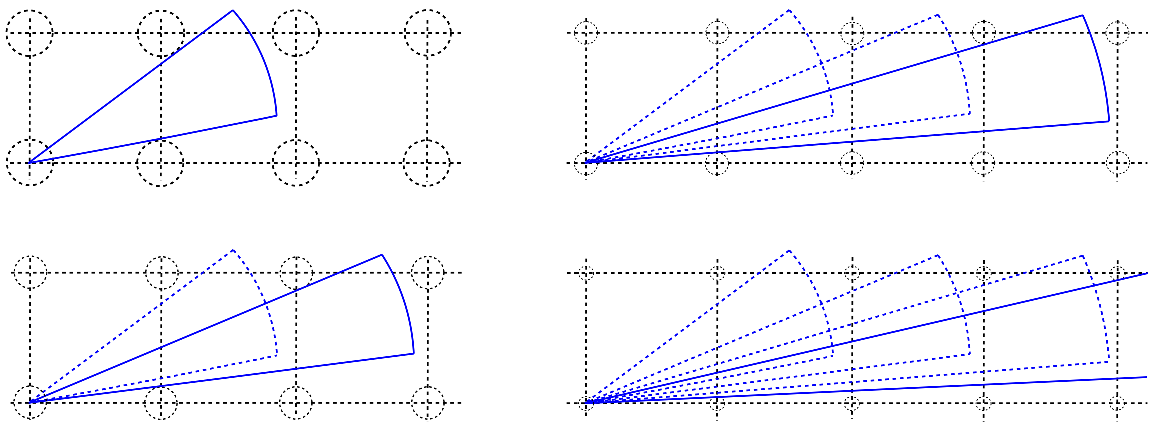

Figure 6.3. The black circles are the boundary of the regions where the Hamiltonian H_k are cut off by the function η_k. The blue sectors are the sectors which completely follow the inverse geodesic flow as they do not intersect the black circles. Since radii of the the black circles decreases (and converges to 0), the slope of the lower edge of the sectors that follow the geodesic flow also decreases (and converges to 0). Even though the slope of the upper edge of the sectors are decreasing as more and more black circles appear on the top right pat of the plane, the sequence of sectors will not shrink to nothing and can go arbitrary far away.

Then under the projection map $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong T^n$, write $\square = [\epsilon, 1-\epsilon]\times[\epsilon, 1-\epsilon] \subset T^n$. We know that when $k \geq m+1$,

$$\Bbbk^{\oplus m} \hookrightarrow \Gamma\big(\square, K(\varphi_k) \circ \Bbbk_{\overline{B_\epsilon(0)}}\big).$$

Therefore, $\Gamma\big(\square, \lim_{k\to\infty} K(\varphi_k)\circ\Bbbk_{\overline{B_\epsilon(0)}}\big) \notin \mathrm{Perf}(\Bbbk)$. Since $\lim_{k\to\infty} K(\varphi_k)\circ\Bbbk_{\overline{B_\epsilon(0)}} \in Sh_{T_0^{*,\infty}T^n}(T^n)$ is constructible, we can conclude that the stalk at $x' \neq x$ is isomorphic to the sections on $\square$, and hence is not perfect. $\square$

Corollary 6.5.3. *Let $M = T^n$ and $\Lambda = T_0^{*,\infty}T^n \subset T^{*,\infty}T^n$ $(n \geq 2)$. S_Λ^+ is not an equivalence on $Sh_\Lambda^c(T^n)$.*

Proof. Assume that S^+_Λ is an equivalence on $Sh^c_\Lambda(T^n)$. Consider the equivalence $Sh^b_\Lambda(T^n) \simeq \mathrm{Fun}^{\mathrm{ex}}(Sh^c_\Lambda(T^n)^{\mathrm{op}}, \mathrm{Perf}(\Bbbk))$ given by the homomorphism pairing as stated in Theorem 3.4.3. For $\mathscr{F} \in Sh^c_\Lambda(T^n), \mathscr{G} \in Sh^b_\Lambda(T^n)$, since

$$Hom(S^+_\Lambda(\mathscr{F}), \mathscr{G}) = Hom(\mathscr{F}, S^-_\Lambda(\mathscr{G})),$$

we know that S^-_Λ has to be an equivalence on $Sh^b_\Lambda(T^n)$. This contradicts the proposition. □

Corollary 6.5.4. *Let $M = T^n$ and $\Lambda = T^{*,\infty}_0 T^n \subset T^{*,\infty}T^n\,(n \geq 2)$. Then m_Λ does not preserve compact objects.*

Proof. This follows immediately from the fiber sequence $m^l_\Lambda \circ m_\Lambda \to \mathrm{id}_{Sh_\Lambda(T^n)} \to S^+_\Lambda$. □

Remark 6.5.1. *We believe one can also show that m^l_Λ does not perserve proper modules (or objects with perfect stalks) using a similar argument.*

We can then deduce the following geometric result which shows that the Weinstein stop is not a swappable stop.

Corollary 6.5.5. *Let $M = T^n\,(n \geq 2)$ and $\Lambda = T^{*,\infty}_0 T^n$. Then the Weinstein hypersurface F_Λ defined as the ribbon of $\Lambda \subset T^{*,\infty}T^n$ is not a swappable hypersurface.*

We can compare our result with the following result of Dahinden [**42**].

Theorem 6.5.6 (Dahinden [**42**])**.** *Let M be a connected manifold with* $\dim M \geq 2$. *Suppose there exists a positive Legendrian isotopy $\Lambda_t \subset T^{*,\infty}M$ such that $\Lambda_0 =$*

$\Lambda_1 = \Lambda = T^{*,\infty}_x M$ *and* $\Lambda_t \cap \Lambda = \varnothing \, (t \in (0,1))$. *Then* M *is simply connected or* $M = \mathbb{RP}^n$.

Dahinden's theorem does not imply that the Weinstein ribbon F_Λ of $\Lambda \subset T^{*,\infty}M$ is a swappable hypersurface, because firstly, it is in general unknown whether the exact symplectomorphism F_Λ defined by the positive loop sends the zero section to itself (note that this may be closely related to the nearby Lagrangian conjecture). Therefore, our corollary is at least a priori stronger than the theorem.

CHAPTER 7

Functorial Specialization and Lagrangian Cobordisms

Consider a Liouville domain X and a Liouville subdomain X' following Section 2.4, Abouzaid-Seidel [**5**] first constructed a partially defined Viterbo restriction functor between their wrapped Fukaya categories.

$$\mathcal{W}(X) \dashrightarrow \mathcal{W}(X')$$

Recently, using the framework of Liouville sectors, Sylvan [**152**] managed to define the Viterbo restriction functor for Liouville domain X and a Liouville subdomain X' (see also [**75**, Section 8.2])

$$\mathcal{W}(X) \longrightarrow \mathcal{W}(X').$$

which is a homological epimorphism (i.e. the right adjoint is fully faithful) when X, X' are Weinstein. This can be generalized to Liouville subsector embeddings (that send sectorial boundary to sectorial boundary) following Section 2.4. On the other hand, from the prespective of microlocal sheaves, using the technique of Nadler-Shende [**124**], we also get the specialization functor whose left adjoint (after restricting to compact objects) is the Viterbo restriction

$$\mu Sh_{\mathfrak{c}_{X'}}(\mathfrak{c}_{X'}) \hookrightarrow \mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X).$$

However, the constructions of Viterbo restrictions are highly nontrivial, and to the author's knowledge, neither of the works explained how to show that Viterbo restriction is functorial, in the sense that compositions of embeddings induce compositions of functors. We show that this is indeed the case.

Theorem 7.0.7 (Theorem 1.4.1)**.** *Let X_0, X_1, and X_2 be Weinstein sectors with Lagrangian skeleta $\mathfrak{c}_{X_0}$, $\mathfrak{c}_{X_1}$, and $\mathfrak{c}_{X_2}$ equipped with Maslov data, such that $i_{01} : X_0 \hookrightarrow X_1$ and $i_{12} : X_1 \hookrightarrow X_2$ are Liouville subsector embeddings sending sectorial boundary to sectorial boundary. Denote by $\Phi_{ij} : \mu Sh_{\mathfrak{c}_{X_i}}(\mathfrak{c}_{X_i}) \hookrightarrow \mu Sh_{\mathfrak{c}_{X_j}}(\mathfrak{c}_{X_j})$ the embeddings of microsheaf categories. Then*

$$\Phi_{12} \circ \Phi_{01} \simeq \Phi_{02} : \mu Sh_{\mathfrak{c}_{X_0}}(\mathfrak{c}_{X_0}) \hookrightarrow \mu Sh_{\mathfrak{c}_{X_2}}(\mathfrak{c}_{X_2}).$$

Our main application of the Viterbo (co)restriction is Lagrangian cobordisms between Legendrian submanifolds. Consider the symplectization $(Y \times \mathbb{R}_r, d(e^r\alpha))$ of the contact manifold $(Y, \ker\alpha)$. Following [**63**, Section 2.8], Chantraine [**27**] and Ekholm [**50**], for instance, considered the category of Lagrangian cobordisms.

Under certain conditions on $(Y, \ker\alpha)$ (for example, when Y has no closed Reeb orbits or when it has an exact symplectic filling) previous works in this field considered a dg algebra called Legendrian contact homology/Chekanov-Eliashberg dg algebra $\mathcal{A}(\Lambda)$ associated to a Legendrian submanifold Λ generated by Reeb trajectories starting and ending on Λ [**36**, **57**]. We consider the version that is a dg algebra over the dg algebra $C_{-*}(\Omega_*\Lambda)$ where $\Omega_*\Lambda$ is the based loop space of Λ [**60**]. Following [**50**, **59**], a Lagrangian cobordism L from Λ_- to Λ_+ is expected to induce a homomorphism

$$\Phi_L^* : \mathcal{A}(\Lambda_+) \to \mathcal{A}(\Lambda_-) \otimes_{C_{-*}(\Omega_*\Lambda_-)} C_{-*}(\Omega_* L).$$

The representations of $\mathcal{A}(\Lambda)$ over $\Bbbk$ are called augmentations. Given an augmentation $\epsilon_- : \mathcal{A}(\Lambda_-) \to \Bbbk$, its restriction

$$\epsilon_-|_{C_{-*}(\Omega_*\Lambda_-)} : C_{-*}(\Omega_*\Lambda_-) \to \Bbbk$$

defines a rank 1 local system $\delta_{\Lambda_-} \in Hom(C_0(\Omega_*\Lambda_-);\Bbbk) \cong H^1(\Lambda_-;\Bbbk^\times)$. For any rank 1 local system $\delta_L \in \mathrm{Hom}(C_0(\Omega_* L);\Bbbk) \cong H^1(L;\Bbbk^\times)$ that restricts to δ_{Λ_-} on Λ_-, we are able to define

$$\epsilon_+ = \Phi_L(\epsilon_-, \delta_L) : \mathcal{A}(\Lambda_+) \xrightarrow{\Phi_L^*} \mathcal{A}(\Lambda_-) \otimes_{C_{-*}(\Omega_*\Lambda_-)} C_{-*}(\Omega_* L) \xrightarrow{(\epsilon_-,\delta_L)} \Bbbk$$

(see [**129**] for the case of Legendrian knots).

For augmentations of $\mathcal{A}(\Lambda)$, Bourgeois-Chantraine [**19**] defined a non-unital $\mathcal{A}_\infty$-category $\mathcal{A}ug_-(\Lambda)$, while Ng-Rutherford-Sivek-Shende-Zaslow [**127**] defined a (strictly) unital $\mathcal{A}_\infty$-category $\mathcal{A}ug_+(\Lambda)$ for Legendrian knots in $\mathbb{R}^3_{\mathrm{std}}$[1]. A Lagrangian cobordism L from Λ_- to Λ_+ is expected to induce a functor between the corresponding augmentation categories

$$\Phi_L : \mathcal{A}ug_\pm(\Lambda_-) \times_{Loc^1(\Lambda_-)} Loc^1(L) \to \mathcal{A}ug_\pm(\Lambda_+),$$

[1]The $\pm$ signs come from the fact that $\mathcal{A}ug_-(\Lambda)$ can be defined using small negative Reeb pushoffs of Λ, while $\mathcal{A}ug_+(\Lambda)$ is defined using positive pushoffs of Λ.

where $Loc^1(-)$ stands for rank 1 local systems.

Let Y be the ideal contact boundary of a Weinstein manifold X. Consider a Lagrangian cobordism $L \subset Y \times \mathbb{R}$ between Legendrian submanifolds from Λ_- to $\Lambda_+ \subset Y$. View (X, Λ_-), (X, Λ_+) and T^*L as a Weinstein sector following Section 2.4, then there is a subsector embedding (after cutting off cylindrical ends)

$$(X, \Lambda_-) \cup_{T^*(\Lambda_- \times [-1,1])} T^*L \hookrightarrow (X, \Lambda_+).$$

Considering their partially wrapped Fukaya categories, by gluing formula, we can write the Viterbo restriction functor as

$$\mathcal{W}(X, \Lambda_+) \to \mathcal{W}(X, \Lambda_-) \otimes_{\mathcal{W}(T^*(\Lambda_- \times [-1,1]))} \mathcal{W}(T^*L).$$

By the Legendrian surgery formula [**14**, **20**, **52**, **60**], we know that there is an equivalence between Legendrian contact homology (with loop space coefficients) and partially wrapped Fukaya categories

$$\mathrm{Perf}\, \mathcal{W}(X, \Lambda) \simeq \mathrm{Perf}\, \mathcal{A}(\Lambda).$$

Hence the Viterbo restriction functor indeed realizes the Lagrangian cobordism map between the dg algebras.

Therefore, we construct a Lagrangian cobordism functor between microlocal sheaf categories using the result of Nadler-Shende [**124**]. Our construction is independent of Floer theory and symplectic field theory.

Theorem 7.0.8 (Theorem 1.4.2)**.** *Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, $\Lambda_-, \Lambda_+ \subset \partial_\infty X$ be Legendrian submanifolds, and $L \subset \partial_\infty X \times \mathbb{R}$ an exact Lagrangian cobordism from Λ_- to Λ_+. There is a fully faithful Lagrangian cobordism functor between microlocal sheaf categories*

$$\Phi_L : \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L) \hookrightarrow \mu Sh_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}),$$

such that concatenations of Lagrangian cobordisms give rise to compositions of cobordism functors.

In particular, when $X = T^*M$, there is a fully faithful cobordism functor between sheaf categories

$$\Phi_L : Sh_{\Lambda_-}(M) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L) \hookrightarrow Sh_{\Lambda_+}(M).$$

Remark 7.0.2. *The fiber product of categories*

$$\mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L)$$

is defined as the homotopy pull back of the following diagram

$$\mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \longrightarrow \mu Sh_{\Lambda_-}(\Lambda_-) \longleftarrow \mu Sh_L(L)$$

where the arrows are the restriction functors in Section 3.4. In particular, when $X = T^*M$ the restriction functor

$$Sh_{\Lambda_-}(M) \longrightarrow \mu Sh_{\Lambda_-}(\Lambda_-)$$

is the microlocalization functor.

Remark 7.0.3. *When the Legendrian* Λ *is has vanishing Maslov class and relative second Stiefel-Whitney class, we know by Theorem 3.2.4 that*

$$\mu Sh_\Lambda(\Lambda) \simeq Loc(\Lambda).$$

The category of local systems is derived Morita equivalent to the chains on based loop space , i.e. $Loc(\Lambda) \simeq \mathrm{Mod}\, C_{-*}(\Omega_*\Lambda)$*; the category of compact local systems* $Loc^c(\Lambda) \simeq \mathrm{Perf}\, C_{-*}(\Omega_*\Lambda)$.

Remark 7.0.4. *Our result also works in the singular setting, including immersed exact Lagrangian cobordisms with vanishing action self intersection points (which lifts to immersed Legendrians with no Reeb chords), and even subanalytic Lagrangian cobordisms between subanalytic Legendrians satisfying the condition above (see Remark 7.2.2).*

On the other hand, in Section 4.4, we explained that an exact Lagrangian cobordism $L \subset J^1(M) \times \mathbb{R}$ can be lifted to a Legendrian cobordism $\widetilde{L} \subset J^1(M \times \mathbb{R}_{>0})$ with conical ends. Following Pan-Rutherford, the dg algebra map can be viewed as a bimodule [**130**]. By enhancing with loop space coefficients, we expect

$$\mathcal{A}(\Lambda_-) \otimes_{C_{-*}(\Omega_*\Lambda_-)} C_{-*}(\Omega_*L) \to \mathcal{A}(\widetilde{L}) \leftarrow \mathcal{A}(\Lambda_+)$$

Indeed, we peoved a sheaf quantization result Theorem 4.0.9 which realizes the cobordism functor by extending the sheaf on the negative end together with local systems by the sheaf quantization functor and then restricting the sheaf to the positive end. We prove that these two approaches are equivalent.

Theorem 7.0.9 (Theorem 1.4.3)**.** *Let $L \subset J^1(M) \times \mathbb{R}_{>0}$ be an exact Lagrangian cobordism from Λ_- to $\Lambda_+ \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, and $\widetilde{L} \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R} \times \mathbb{R}_{>0})$ be the conical Legendrian lifting. Then there is a commutative diagram*

$$\begin{array}{ccc} & Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0}) & \\ {\scriptstyle (i_-^{-1}, m_L)} \swarrow & & \searrow {\scriptstyle i_+^{-1}} \\ Sh_{\Lambda_-}(M \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L) & \xrightarrow{\quad \Phi_L \quad} & Sh_{\Lambda_+}(M \times \mathbb{R}) \end{array}$$

where $i_- : M \times \mathbb{R} \times s_- \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$ for $s_- > 0$ sufficiently small and $i_+ : M \times \mathbb{R} \times s_+ \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$ for $s_+ > 0$ sufficiently large.

Full faithfulness of the Lagrangian cobordism functor induces a number of strong implications, including a number of exact triangles that are analogous to the ones in Legendrian contact homology deduced by Chantraine-Dimitroglou Rizell-Ghiggini-Golovko using the Cthulhu complex [**31**].

In particular, combining with the systematic approaches to compute the number of microlocal rank 1 sheaves over $\mathbb{F}_q$ for certain Legendrian surfaces using flag moduli developed by Treumann-Zaslow [**157**] and Casals-Zaslow [**26**] have developed, we will be able to get new obstructions to Lagrangian cobordisms for these Legendrian surfaces.

The following examples of Legendrian surfaces $\Lambda_{g,k}$ are considered in [**43**] and [**137**] ($\Lambda_{g,0}$ are the unknotted Legendrian surfaces and $\Lambda_{g,g}$ are Clifford Legendrian surfaces). Dimitroglou Rizell showed that those $\Lambda_{g,k}$'s admit $\mathbb{Z}/2\mathbb{Z}$-coefficient augmentations and generating families only when $k = 0$, and hence it may not be easy to study Lagrangian cobordisms between them when $k \geq 1$. However, using the Legendrian weave description, we are able to show the following.

Theorem 7.0.10 (Theorem 1.5.3)**.** *Let $\Gamma_{Unknot}, \Gamma_{Cliff}$ be the 2-graphs in S^2 shown in Figure 7.1, and $\Lambda_{Unknot}, \Lambda_{Cliff}$ the corresponding Legendrian weaves in $J^1(S^2) \subset T^{*,\infty}\mathbb{R}^3$. Let $\Lambda_{g,k}$ be the Legendrian surface with genus g by taking k copies of Λ_{Cliff} and $g - k$ copies of Λ_{Unknot}. Then*

(1) for any $g' \leq g$, there are Lagrangian cobordisms from $\Lambda_{g,k}$ to $\Lambda_{g',k}$ and also from $\Lambda_{g',k}$ to $\Lambda_{g,k}$;

(2) (Dimitroglou Rizell) for any $k \geq 1$, there are no Lagrangian cobordisms with vanishing Maslov class from $\Lambda_{g,0}$ to $\Lambda_{g',k}$;

(3) for any $k \geq 1, k' \geq 0$, there are Lagrangian cobordisms L from $\Lambda_{g,k}$ to $\Lambda_{g,k'}$ such that $\dim \operatorname{coker}(H^1(L) \to H^1(\Lambda_{g,k})) \geq 2$;

(4) for any $k < k'$, there are no Lagrangian cobordisms L with vanishing Maslov class from $\Lambda_{g,k}$ to $\Lambda_{g,k'}$ such that $H^1(L) \twoheadrightarrow H^1(\Lambda_{g,k})$; in particular there are no such Lagrangian concordances.

Remark 7.0.5. *Part (2) is a direct corollary of either* [**43**] *or* [**157**].

Roughly speaking, the Legendrian $\Lambda_{g,k}$ is closer to being Lagrangian fillable when k is smaller (in particular $\Lambda_{g,0}$ are the only Lagrangian fillable ones). We would expect that it is difficult to have a Lagrangian cobordism from $\Lambda_{g,k}$ to $\Lambda_{g,k'}$ if $k > k'$.

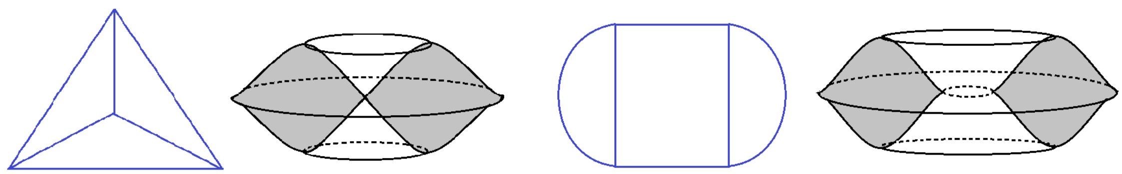

Figure 7.1. On the left is the Clifford Legendrian torus and its corresponding 2-graph, and on the right is the unknotted Legendrian torus and its corresponding 2-graph.

Our theorem shows that, for $k > k'$, there are indeed obstructions for Lagrangian cobordisms to exist from $\Lambda_{g,k}$ to $\Lambda_{g,k'}$ assuming either (2) $k = 0$ or (4) $H^1(L) \to H^1(\Lambda_{g,k})$ is surjective. On the contrary, as long as we assume (3) $k \geq 1$ and $H^1(L) \to H^1(\Lambda_{g,k})$ is not surjective, then we enter the world of flexibility and there are no obstructions for Lagrangian cobordisms (and $\dim \operatorname{coker}(H^1(L) \to H^1(\Lambda_{g,k}))$ can even be very small).

7.1. Functorial Specialization of Weinstein Subsector Embeddings

In this section, we review the construction of Nadler-Shende [**124**] realizing embedded exact Lagrangians as objects in the microlocal sheaf category of the Lagrangian skeleton. We explain how their result essentially gives a specialization functor for any Liouville subsector embeddings. Finally, we prove the functoriality Theorem 7.0.7.

7.1.1. Specialization of Lagrangians in Weinstein Sectors

We state the series of results by Nadler-Shende [**124**], and explain why they induce specializations functor for any Weinstein subsector embeddings.

Basically, Nadler-Shende are able to embed the Weinstein manifold $(X, d\lambda)$ into the contact boundary of some cotangent bundle and thus construct a microlocal sheaf category $\mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X)$ from the Lagrangian skeleton $\mathfrak{c}_X$ of X. Moreover, they are able to construct functors with respect to Liouville subsector embeddings and homotopies that are fully faithful.

Firstly, let us recall their construction of the microlocal sheaf category $\mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X)$ for any Weinstein manifold X with subanalytic skeleton $\mathfrak{c}_X$ ([**124**, Section 8]).

Remark 7.1.1. *It is explained in* [**74**, Section 7.7] *how to make the Lagrangian skeleton* $\mathfrak{c}_X$ *of a Weinstein manifold* X *subanalytic. Namely any Weinstein manifold admits some Weinstein structure with a subanalytic skeleton.*

Gromov's h-principle [**65**, Theorem 12.3.1] enables us to embed the contactization of the Weinstein manifold $(X \times \mathbb{R}, \ker(dt - \lambda))$ into the contact boundary of a higher dimensional cotangent bundle $T^{*,\infty}N$, as long as (1) $\dim T^*N \geq \dim X + 2$ and (2) there is a bundle map $\Psi_s : TX \times T\mathbb{R} \to T(T^{*,\infty}N)$ covering a smooth embedding $f : X \times \mathbb{R} \hookrightarrow T^{*,\infty}N$ such that $\Psi_0 = df$ and $\Psi_1|_{TX\times\mathbb{R}}$ is a symplectic bundle map into the contact distribution $\xi_{T^{*,\infty}N}$. The second condition is purely algebraic topological. For example, $N = \mathbb{R}^m$ for sufficiently large m, this is satisfied as long as X is stably polarizable [**146**].

Consider the symplectic normal bundle $\nu_{X\times\mathbb{R}}(T^{*,\infty}N)$ of $X \times \mathbb{R} \hookrightarrow T^{*,\infty}N$. Assume that by choosing $\dim T^*N > 0$ to be sufficiently large, we can find a Lagrangian subbundle $(X \times \mathbb{R})_\sigma \subset \nu_{X\times\mathbb{R}}(T^{*,\infty}N)$ by choosing a section σ of the Lagrangian Grassmannian of the normal bundle $\nu_{X\times\mathbb{R}}(T^{*,\infty}N)$, as in [**124**, Lemma 9.1]. This is a null homotopy of

$$X \times \mathbb{R} \to BU \to BLGr$$

(where $BLGr$ is the classifying space of the stable Lagrangian Grassmannian). Let the Legendrian thickening of $\mathfrak{c}_X$ be

$$\mathfrak{c}_{X,\sigma} = (X \times \mathbb{R})_\sigma|_{\mathfrak{c}_X \times \{0\}}.$$

Definition 7.1.1. *The microlocal sheaf category on a Weinstein manifold X, with a chosen section σ in the stable Lagrangian Grassmannian, is defined by*

$$\mu Sh_{\mathfrak{c}_X} = \mu Sh_{\mathfrak{c}_{X,\sigma}}|_{\mathfrak{c}_X \times \{0\}}.$$

Remark 7.1.2. *Nadler-Shende showed that this microlocal sheaf category is independent of the Lagrangian skeleton and the contact embedding we choose. It does depend on the thickening because that is determined by the section in Lagrangian Grassmannian.*

Remark 7.1.3. *More generally, the existence of a section in the stable Lagrangian Grassmannian can be relaxed to simply the existence of a section σ : $X \times \mathbb{R} \to B\mathrm{Pic}(\Bbbk)|_{X \times \mathbb{R}}$, which is classified by Maslov data* [**124**, Definition 10.6], *i.e. a null homotopy*

$$X \times \mathbb{R} \to B^2\mathrm{Pic}(\Bbbk),$$

and the microlocal sheaf category can be defined by $\sigma^{-1}\mu Sh_{B\mathrm{Pic}(\Bbbk)|_{\mathfrak{c}_X}}$. *The Maslov data for ring spectrum coefficients are carefully studied by Jin* [**93**] *and* [**124**, Section 11]. *When* $\Bbbk$ *is a ring, this is ensured as long as* $2c_1(X) = 0$.

Therefore, from now on we will always assume the existence of a section in the Lagrangian Grassmannian of the stable normal bundle without loss of generality.

Given a Weinstein subdomain $X' \subset X$ equipped with Maslov data, let $\lambda' = \lambda - df$ be the Liouville form on X such that the Liouville flow $Z_{\lambda'}$ is transverse to $\partial_\infty X'$, and $\mathfrak{c}_{X'}$ the skeleton of X' under the Liouville flow $Z_{\lambda'}$. Then the primitive $f|_U : U \to \mathbb{R}$ determines the Legendrian lift of the skeleton $\mathfrak{c}_{X'}$ in $X \times \mathbb{R}$ being $\widetilde{\mathfrak{c}}_{X'} = \{(x, f(x)) | x \in \mathfrak{c}_{X'}\}$. Define

$$\mu Sh_{\widetilde{\mathfrak{c}}_{X'}} = \mu Sh_{\widetilde{\mathfrak{c}}_{X',\sigma}}|_{\widetilde{\mathfrak{c}}_{X'}}.$$

In particular, when $X' = T^*L$ is a Weinstein subdomain, we write $\widetilde{L}$ for the Legendrian lift of L and consider $\mu Sh_{\widetilde{L}}$. It will be natural to construct an embedding functor

$$\mu Sh_{\widetilde{\mathfrak{c}}_{X'}}(\widetilde{\mathfrak{c}}_{X'}) \longrightarrow \mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X).$$

Nadler-Shende's main result is about constructing such an embedding functor and proving its full faithfulness. When $X' = T^*L$, this realizes exact Lagrangian submanifolds $L \subset X$ as objects in the microlocal sheaf category.

Definition 7.1.2 (Nadler-Shende [**124**, Definition 2.9])**.** *Let* $\Lambda_\zeta, \Lambda'_\zeta \subset Y$ $(\zeta \in \mathbb{R})$ *be two families of subsets in a contact manifold.* $\Lambda_\zeta, \Lambda'_\zeta$ *are gapped if there exists*

$\epsilon > 0$, so that for any $\zeta \in \mathbb{R}$ there are no Reeb chords connecting Λ_ζ with Λ'_ζ with length shorter than ϵ.

Theorem 7.1.1 (Nadler-Shende [**124**, Theorem 8.3 & 9.7])**.** *Consider a subanalytic Legendrian $\Lambda_1 \subset X \times \mathbb{R}$, which is either compact or locally closed, relatively compact with cylindrical ends. Let $\varphi_H^\zeta : X \times \mathbb{R} \to X \times \mathbb{R}$ be a contact isotopy for $\zeta \in (0,1]$ conical near the cylindrical ends. Let $\Lambda_H \subset X \times \mathbb{R} \times (0,1]$ be the Legendrian movie of φ_H^ζ and $\overline{\Lambda}_H$ be the closure of Λ_H in $X \times \mathbb{R} \times [0,1]$. Let $\Lambda_0 = \overline{\Lambda}_H \cap (X \times \mathbb{R} \times \{0\}) \subset X \times \mathbb{R}$ be the set of limit points of $\varphi_H^\zeta(\Lambda_1)$ as $\zeta \to 0$.*

Assume that for some contact form on $X \times \mathbb{R}$, the family $\varphi_H^\zeta(\Lambda_1)\,(\zeta \in (0,1])$ is self gapped. Then there is a fully faithful functor

$$\mu Sh_{\Lambda_1}(\Lambda_1) \hookrightarrow \mu Sh_{\Lambda_0}(\Lambda_0).$$

In particular, when $X' \subset X$ is a Weinstein subdomain (with Liouville complement), consider the Liouville vector field Z_λ on $(X, d\lambda)$ defined by

$$\iota(Z_\lambda) d\lambda = \lambda.$$

The Liouville flow of Z_λ for negative time will compress $\mathfrak{c}_{X'}$ onto $\mathfrak{c}_X$ as $z \to -\infty$. The Liouville flow on X extends to a contact flow φ_Z^z in $X \times \mathbb{R}$ with

$$d\varphi_Z^z / dz = t\partial/\partial t + Z_\lambda,$$

and thus we can consider the Legendrian movie of $\widetilde{\mathfrak{c}}_{X'}$ under the flow. The theorem then gives a fully faithful embedding of microlocal sheaves on $\widetilde{\mathfrak{c}}_{X'}$ to sheaves on $\lim_{z\to-\infty}\varphi_Z^z(\widetilde{\mathfrak{c}}_U) \subset \mathfrak{c}_X \times \{0\}$. Write $\phi_Z^\zeta = \varphi_Z^{\ln z}$. Applying the flow $\varphi_Z^z\,(z \in (-\infty,0])$ or $\phi_Z^\zeta\,(\zeta \in (0,1])$, we have [**124**, Section 8.2]

$$\begin{aligned}\mu Sh_{\mathfrak{c}_{X'}}(\mathfrak{c}_{X'}) &\hookrightarrow \mu Sh_{\lim_{z\to-\infty}\varphi_Z^z(\widetilde{\mathfrak{c}}_{X'})}(\lim{}_{z\to-\infty}\varphi_Z^z(\widetilde{\mathfrak{c}}_{X'}))\\ &\xrightarrow{\sim} \mu Sh_{\lim_{\zeta\to0}\phi_Z^\zeta(\widetilde{\mathfrak{c}}_{X'})}(\lim{}_{\zeta\to0}\,\phi_Z^\zeta(\widetilde{\mathfrak{c}}_{X'})) \hookrightarrow \mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X)\end{aligned}$$

For the proof of the theorem, consider a contact embedding $X \times \mathbb{R} \hookrightarrow T^{*,\infty}N$ and choose a Lagrangian section $(X\times\mathbb{R})_\sigma \subset \nu_{X\times\mathbb{R}}(T^{*,\infty}N)$. One can pull back the contact form and the contact isotopy via the projection $\nu_{X\times\mathbb{R}}(T^{*,\infty}N) \to X\times\mathbb{R}$. Then $\varphi_H^\zeta(\Lambda_{1,\sigma})\,(\zeta \in (0,1])$ is self gapped iff $\varphi_H^\zeta(\Lambda_1)\,(\zeta \in (0,1])$ is. Hence one can replace $X\times\mathbb{R}$ in the theorem by $T^{*,\infty}N$.

The proof consists of two steps. First, we need to construct a fully faithful embedding from $\mu Sh_\Lambda(\Lambda)$ back to $Sh(N)$ where we have Grothendieck's six functors; second, we need to construct a fully faithful functor between subcategories of $Sh(N)$.

Here is the first step, the antimicrolocalization construction, which we have discussed in Section 4.2; see [**75**; **84**, Section 8; **152**] for related constructions. Unlike our approach, the approach of Nadler-Shende is highly nonexplicit (and in particular it is hard to deduce adjunction and exact triangles). However, their construction is done for Legendrians relative to collar ends, which we only sketched an approach in Section 4.2.1.

Definition 7.1.3. *Let $\Lambda \subset T^{*,\infty}N$ be a subanalytic Legendrian with cylindrical end $\partial\Lambda$, i.e. a contact embedding*

$$(T^*(U \times (-1,1)) \times \mathbb{R}, \partial\Lambda \times [0,1)) \hookrightarrow (T^{*,\infty}N, \Lambda).$$

Let φ_s $(s \in \mathbb{R})$ be some Reeb flow on $T^{*,\infty}N$. For $\partial\Lambda_{\pm s} \times [0,1) \subset T^*(U \times (-1,1)) \times \mathbb{R}$, connect the ends $\partial\Lambda_{\pm s}$ by a family of standard cusps $\partial\Lambda \times \prec$. Then

$$(\Lambda, \partial\Lambda)_s^{\prec} = \Lambda_{-s} \cup \Lambda_s \cup (\partial\Lambda \times \prec).$$

Theorem 7.1.2 (Nadler-Shende [**124**, Theorem 7.28])**.** *Let $\Lambda \subset T^{*,\infty}N$ be a subanalytic Legendrian, which is either compact or locally closed, relatively compact with cylindrical ends. Let c be the shortest length of Reeb orbits starting and ending on Λ. For $\epsilon < c/2$, the microlocalization functor*

$$Sh_{(\Lambda,\partial\Lambda)_\epsilon^{\prec}}(N)_0 \to \mu Sh_{\Lambda_{-\epsilon}}(\Lambda_{-\epsilon})$$

admits a right inverse. Here the subscript 0 means the subcategory of objects with 0 stalk away from a compact set.

By applying the antimicrolocalization functor, we now only need to construct a functor in $Sh(N)$. Namely we consider the nearby cycle functor and show that it is fully faithful in our setting. This full faithfulness criterion is proposed by Nadler [**120**] and proved for families of Legendrians by Zhou [**163**, Proposition 3.2].

Definition 7.1.4. *For a fibration $\pi_{\mathbb{R}} : N \times \mathbb{R} \to \mathbb{R}$, let the projection of the cotangent bundle to the fiber be $\Pi : T^*(N \times \mathbb{R}) \to T^*(N \times \mathbb{R})/\pi_{\mathbb{R}}^* T^*\mathbb{R}$. For $\mathscr{F} \in Sh(N \times \mathbb{R})$, the singular support relative to $\pi_{\mathbb{R}}$ is*

$$SS_\pi(\mathscr{F}) = \overline{\Pi(SS(\mathscr{F}))}.$$

Theorem 7.1.3 (Nadler-Shende [**124**, Theorem 5.1])**.** *Let $\mathscr{F}, \mathscr{G}$ be weakly constructible sheaves on $N \times \mathbb{R} \times \mathbb{R} \backslash \{0\}$. Write $j : N \times \mathbb{R} \times \mathbb{R}\backslash\{0\} \to N \times \mathbb{R}^2$ and $i : N \times \mathbb{R} \times \{0\} \to N \times \mathbb{R}^2$. Suppose*

(1) $SS^\infty(\mathscr{F}), SS^\infty(\mathscr{G}) \cap \pi_{\mathbb{R}}^* T^*(\mathbb{R}\backslash\{0\}) = \emptyset$;

(2) The family of pairs $SS^\infty_\pi(\mathscr{F}), SS^\infty_\pi(\mathscr{G})$ are gapped for some contact form.

Then we have a natural isomorphism

$$\Gamma(i^{-1}\mathscr{H}om(j_*\mathscr{F}, j_*\mathscr{G})) \xrightarrow{\sim} Hom(i^{-1}j_*\mathscr{F}, i^{-1}j_*\mathscr{G}).$$

Finally, instead of considering the whole category $Sh(N)$, we need to restrict to the subcategories $Sh_{(\Lambda_1, \partial\Lambda_1)^\prec_\epsilon}(N)$ and $Sh_{(\Lambda_0, \partial\Lambda_0)^\prec_\epsilon}(N)$. Therefore we need the following estimate, which follows from Proposition 3.1.9 and 3.1.10 [**97**, Theorem 6.3.1 & Corollary 6.4.4].

Lemma 7.1.4 ([**124**, Lemma 3.16])**.** *For $\mathscr{F} \in Sh(N \times \mathbb{R}_{>0})$, denoting $j : N \times \mathbb{R}_{>0} \to N \times \mathbb{R}_{\geq 0}$ and $i : N \times \{0\} \to N \times \mathbb{R}_{\geq 0}$,*

$$SS(i^{-1}j_*\mathscr{F}) \subset \overline{\Pi(SS(\mathscr{F}))} \cap T^*(N \times \{0\}).$$

Note that by Theorem 3.3.2, since Λ_H is the Legendrian movie of Λ_1 under the flow φ_H^ζ ($\zeta \in (0,1]$), we have a quasi-equivalence of categories

$$\mu Sh_{\Lambda_1}(\Lambda_1) \simeq \mu Sh_{\Lambda_H}(\Lambda_H).$$

Using Theorem 7.1.2, 7.1.3 together with Lemma 7.1.4, Theorem 7.1.1 now immediately follows.

7.1.2. Functoriality of the Specialization Functors

Having explained the construction of Nadler-Shende, we prove the functoriality of the specialization functors Theorem 7.0.7. Our strategy is as follows.

Let X_0, X_1, X_2 be Weinstein sectors with Lagrangian skeleta $\mathfrak{c}_{X_0}, \mathfrak{c}_{X_1}, \mathfrak{c}_{X_2}$ equipped with Maslov data, such that $i_{01} : X_0 \hookrightarrow X_1$ and $i_{12} : X_1 \hookrightarrow X_2$ are Liouville embeddings sending sectorial boundaries to sectorial boundaries. Denote by $\Phi_{ij} : \mu Sh_{\mathfrak{c}_{X_i}}(\mathfrak{c}_{X_i}) \hookrightarrow \mu Sh_{\mathfrak{c}_{X_j}}(\mathfrak{c}_{X_j})$ the embedding of microlocal sheaf categories. Φ_{02} is defined by using the Liouville flow to compress $\mathfrak{c}_{X_0}$ to the ambient skeleton $\mathfrak{c}_{X_2}$ directly, and $\Phi_{12} \circ \Phi_{01}$ is defined by first compressing $\mathfrak{c}_{X_0}$ to the skeleton $\mathfrak{c}_{X_1}$, and next compressing $\mathfrak{c}_{X_1}$ to the ambient skeleton $\mathfrak{c}_{X_2}$. We will try to define a 2-parametric family of contact flow that interpolates between them. Then following the construction, Φ_{01} and $\Phi_{12} \circ \Phi_{01}$ are two different compositions of nearby cycles, and the theorem is reduced to commutativity of the nearby cycle functors.

Therefore, we need the commutativity criterion of nearby cycle functors in for example [**123**] or [**101, 113**]. In order to keep the proof self contained, we extract the main technical lemma as follows, which is a base change formula that does not hold in general.

Write the projection maps

$$\pi_i : N \times [0,1] \times [0,1] \to [0,1],\ (x, t_1, t_2) \mapsto t_i,\ (i = 1, 2)$$

and $\pi = \pi_1 \times \pi_2 : N \times [0,1] \times [0,1] \to [0,1] \times [0,1]$. Write the inclusions

$$\begin{array}{ccc} N \times \{0\} \times (0,1] & \xrightarrow{i} & N \times [0,1] \times (0,1] \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \bar{j}} \\ N \times \{0\} \times [0,1] & \xrightarrow{\bar{j}} & N \times [0,1] \times [0,1]. \end{array}$$

Proposition 7.1.5. *Let $\mathscr{F} \in Sh(N \times [0,1] \times (0,1])$ be a sheaf such that*

(1) $i^{\#} SS^{\infty}(\mathscr{F}) \cap \pi_2^* T^{*,\infty}(0,1] = \varnothing$,

(2) $SS^{\infty}(\mathscr{F}) \cap \pi^* T^{*,\infty}((0,1] \times (0,1]) = \varnothing$,

(3) $\overline{SS^{\infty}_{\pi}(\mathscr{F})} \cap T^{*,\infty} N \times \{(0,0)\}$ is a subanalytic Legendrian.

Then there is a natural isomorphism of sheaves

$$\bar{i}^{-1} \bar{j}_* \mathscr{F} \simeq j_* i^{-1} \mathscr{F}.$$

Remark 7.1.4. *For the applications, $\mathscr{F}$ will always be the push forward of a sheaf $\mathscr{F}_0 \in Sh(N \times (0,1] \times (0,1])$, in which case Condition (1) can be easily checked. We choose to state a more general result without assuming that.*

Remark 7.1.5. *We remark the importance of Condition (3). The following example is due to an anonymous referee. Let* $N = \mathbb{R}$*,* $S = \{(x, t_1, t_2) | t_1 = xt_2\} \subset N \times [0, 1] \times (0, 1]$ *and* $\mathscr{F} = \Bbbk_S$*. Then Condition (3) does not hold and one can check that the base change formula does not hold.*

We have a natural morphism $\bar{i}^{-1}\bar{j}_*\mathscr{F} \to j_* i^{-1}\mathscr{F}$ by adjunction. Since the natural morphism induces quasi-isomorphisms on stalks on $N \times 0 \times (0, 1]$, it suffices to show that the it also induces quasi-isomorphisms on stalks on $N \times \{(0, 0)\}$.

First we compute the stalks of $\bar{i}^{-1}\bar{j}_*\mathscr{F}$ at $(x, 0, 0)$. The following lemma is basically [**124**, Corollary 4.4]. Let U_x be a sufficiently small open ball around $x \in N$, $D_{(0,0)}(\epsilon) = [0, \epsilon) \times [0, \epsilon)$ for $\epsilon > 0$ sufficiently small, and $D^\circ_{(0,0)}(\epsilon) = [0, \epsilon) \times (\delta, \epsilon)$ for $\delta \ll \epsilon$.

Lemma 7.1.6. *Let* $\mathscr{F} \in Sh(N \times [0, 1] \times (0, 1])$ *be a sheaf so that* $i^\# SS^\infty(\mathscr{F}) \cap \pi_2^* T^{*,\infty}(0, 1] = \varnothing$*,* $SS^\infty(\mathscr{F}) \cap \pi^* T^{*,\infty}((0, 1] \times (0, 1]) = \varnothing$*, and* $\overline{SS^\infty_\pi(\mathscr{F})} \cap T^{*,\infty}N \times \{(0, 0)\}$ *is a subanalytic Legendrian. Then for* $x \in N$*,* $U_x \subset N$ *a sufficiently small open neighbourhood and* $\epsilon > 0$ *sufficiently small,*

$$\bar{j}_*\mathscr{F}_{(x,0,0)} \simeq \Gamma\big(\overline{U}_x \times \overline{D}^\circ_{(0,0)}(\epsilon), \mathscr{F}\big).$$

Proof. Since $\overline{SS^\infty_\pi(\mathscr{F})} \cap T^{*,\infty}N \times \{(0, 0)\}$ is a subanalytic Legendrian, for a small neighbourhood $U_x \times D_{(0,0)}(\epsilon)$ of $(x, 0, 0) \in N \times [0, 1] \times [0, 1]$, we have

$$\overline{SS^\infty_\pi(\mathscr{F})} \cap \nu^{*,\infty}_{U_x,\pm} N \times \{(0, 0)\} = \varnothing$$

for sufficiently small neighbourhoods U_x by general position argument.

Consider $N \times (0,1] \times [0,1]$. Since $SS^\infty(\mathscr{F}) \cap \pi^* T^{*,\infty}((0,1] \times (0,1]) = \varnothing$, we can get an injective projection to the relative singular support in the relative cotangent bundle $SS^\infty(\mathscr{F}) \hookrightarrow SS^\infty_\pi(\mathscr{F})$ on $N \times (0,1] \times (0,1]$. Hence there is an injective projection

$$SS^\infty(\mathscr{F}) \cap \nu^{*,\infty}_{U_x \times D_{(0,0)}(\epsilon),\pm}(N \times (0,1] \times (0,1]) \hookrightarrow SS^\infty_\pi(\mathscr{F}) \cap \nu^{*,\infty}_{U_x,\pm} N \times D_{(0,0)}(\epsilon).$$

Then consider $N \times \{0\} \times (0,1]$. Since $i^\# SS^\infty(\mathscr{F}) \cap \pi_2^* T^{*,\infty}(0,1] = \varnothing$ and $\nu^{*,\infty}_{U_x \times \{0\} \times D_0(\epsilon),\pm}(N \times \{0\} \times (0,1])$ only consists of covectors tangent to $N \times \{0\} \times (0,1]$, there is also an injection

$$SS^\infty(\mathscr{F}) \cap \nu^{*,\infty}_{U_x \times \{0\} \times D_0(\epsilon),\pm}(N \times \{0\} \times (0,1]) \hookrightarrow i^\# SS^\infty(\mathscr{F}) \cap \nu^{*,\infty}_{U_x \times \{0\} \times D_0(\epsilon),\pm}(N \times \{0\} \times (0,1])$$

Then since $i^\# SS^\infty(\mathscr{F}) \cap \pi_2^* T^{*,\infty}(0,1] = \varnothing$, we have an injective projection

$$i^\# SS^\infty(\mathscr{F}) \cap \nu^{*,\infty}_{U_x \times \{0\} \times D_0(\epsilon),\pm}(N \times \{0\} \times (0,1]) \hookrightarrow i^\# SS^\infty_{\pi_2}(\mathscr{F}) \cap \nu^{*,\infty}_{U_x,\pm} N \times \{0\} \times D_0(\epsilon).$$

However, as $\epsilon \to 0$ the limit points in the above relative singular support are contained in $\overline{SS^\infty_\pi(\mathscr{F})} \cap \nu^{*,\infty}_{U_x,\pm} N \times \{(0,0)\} = \varnothing$. Therefore, the set of the limit points in the relative singular support is empty. Hence we can conclude that for sufficiently small $\epsilon > 0$,

$$SS^\infty(\mathscr{F}) \cap \nu^{*,\infty}_{U_x \times D_{(0,0)}(\epsilon),\pm}(N \times [0,1] \times (0,1]) = \varnothing.$$

Consequently, by non-characteristic deformation lemma Proposition 3.1.2 applied to the family $U_x \times D_{(0,0)}(\epsilon)$ and $U_x \times D^\circ_{(0,0)}(\epsilon)$ for sufficiently small $\epsilon > 0$ and $\delta \ll \epsilon$, we can conclude that

$$\begin{aligned}\bar{j}_*\mathscr{F}_{(x,0,0)} &\simeq \Gamma\big(U_x \times D_{(0,0)}(\epsilon), \bar{j}_*\mathscr{F}\big) \simeq \Gamma\big(\overline{U}_x \times \overline{D}_{(0,0)}(\epsilon), \bar{j}_*\mathscr{F}\big)\\ &\simeq \Gamma\big(U_x \times D^\circ_{(0,0)}(\epsilon), \mathscr{F}\big) \simeq \Gamma\big(\overline{U}_x \times \overline{D}^\circ_{(0,0)}(\epsilon), \mathscr{F}\big).\end{aligned} \qquad \square$$

Then we compute the stalks of $j_*i^{-1}\mathscr{F}$ at $(x,0)$. Let U_x be a sufficiently small open ball around $x \in N$, $D_0 = [0,\epsilon)$ for $\epsilon > 0$ sufficiently small and $D^\circ_0 = (\delta,\epsilon)$ for $\delta \ll \epsilon$.

Lemma 7.1.7. *Let $\mathscr{G} \in Sh(N \times (0,1])$ be a sheaf such that $SS^\infty(\mathscr{G}) \cap \pi^*T^{*,\infty}(0,1] = \varnothing$, and $\overline{SS^\infty_\pi(\mathscr{G})} \cap T^{*,\infty}N \times \{0\}$ is subanalytic Legendrian. Then for any $x \in N$, $U_x \subset N$ a sufficiently small open neighbourhood and $\epsilon > 0$ sufficiently small,*

$$j_*\mathscr{G}_{(x,0)} \simeq \Gamma\big(\overline{U}_x \times \overline{D}^\circ_0(\epsilon), \mathscr{G}\big).$$

Proof. Since $\overline{SS^\infty_\pi(\mathscr{G})} \cap T^{*,\infty}N \times \{0\}$ is a subanalytic Legendrian, for a small neighbourhood $U_x \times D_0(\epsilon)$ of $(x,0) \in N \times [0,1]$, we have

$$\overline{SS^\infty_\pi(\mathscr{G})} \cap \nu^{*,\infty}_{U_x,\pm}N \times \{0\} = \varnothing$$

by general position argument. Since $SS^\infty(\mathscr{G}) \cap \pi^*T^{*,\infty}(0,1] = \varnothing$, we have an injective projection to the relative singular support in the relative cotangent bundle

$SS^\infty(\mathscr{G}) \hookrightarrow SS^\infty_\pi(\mathscr{G})$. Hence there is an injective projection

$$SS^\infty(\mathscr{G}) \cap \nu^{*,\infty}_{U_x \times D_0(\epsilon),\pm}(N \times (0,1]) \hookrightarrow SS^\infty_\pi(\mathscr{G}) \cap \nu^{*,\infty}_{U_x,\pm}N \times D_0(\epsilon).$$

as $\epsilon \to 0$ the limit points in the above relative singular support are contained in $\overline{SS^\infty_\pi(\mathscr{G})} \cap \nu^{*,\infty}_{U_x,\pm}N \times 0 = \varnothing$. Hence we can conclude that when $\epsilon > 0$ is sufficiently small, the intersection between relative singular support and $\nu^{*,\infty}_{U_x,\pm}N \times D_0(\epsilon)$ is empty. Therefore, by non-characteristic deformation lemma Proposition 3.1.2 applied to the family $U_x \times D_0(\epsilon)$ and $U_x \times D_0^\circ(\epsilon)$, we have

$$\begin{aligned} j_*\mathscr{G}_{(x,0)} &\simeq \Gamma\big(U_x \times D_0(\epsilon), j_*\mathscr{G}\big) \simeq \Gamma\big(\overline{U}_x \times \overline{D}_0(\epsilon), j_*\mathscr{G}\big) \\ &\simeq \Gamma\big(U_x \times D_0^\circ(\epsilon), \mathscr{G}\big) \simeq \Gamma\big(\overline{U}_x \times \overline{D}_0^\circ(\epsilon), \mathscr{G}\big). \end{aligned}$$

□

Remark 7.1.6. *The above lemmas will also follow from the weak constructibility of $\mathscr{F}$* [**123**, Section 2]*. For the applications, we believe that in fact both conditions hold.*

Proof of Proposition 7.1.5. We apply Lemma 7.1.6 to $\mathscr{F}$ and apply Lemma 7.1.7 and Theorem 3.1.10 to $i^{-1}\mathscr{F}$. Then it suffices to show that

$$\Gamma\big(\overline{U}_x \times \overline{D}^\circ_{(0,0)}(\epsilon), \mathscr{F}\big) \simeq \Gamma\big(\overline{U}_x \times \overline{D}_0^\circ(\epsilon), \mathscr{F}\big).$$

Since $\overline{SS^\infty_\pi(\mathscr{F})} \cap T^{*,\infty}N \times \{(0,0)\}$ is a subanalytic Legendrian, for a small neighbourhood $U_x \times D_{(0,0)}(\epsilon)$ of $(x,0,0) \in N \times [0,1] \times [0,1]$, we have

$$\overline{SS^\infty_\pi(\mathscr{F})} \cap \nu^{*,\infty}_{U_x,\pm}N \times \{(0,0)\} = \varnothing$$

for sufficiently small neighbourhoods U_x by general position argument. Write $D^\circ_{(0,0)}(\epsilon,\epsilon') = [0,\epsilon') \times (\delta,\epsilon)$ for $0 \leq \epsilon' \leq \epsilon$. Since $SS^\infty(\mathscr{F}) \cap \pi^*T^{*,\infty}((0,1] \times (0,1]) = \varnothing$, we know that there is an injective projection

$$SS^\infty(\mathscr{F}) \cap \nu^{*,\infty}_{U_x \times D^\circ_{(0,0)}(\epsilon,\epsilon'),\pm}(N \times (0,1] \times (0,1]) \hookrightarrow SS^\infty_\pi(\mathscr{F}) \cap \nu^{*,\infty}_{U_x,\pm}N \times \overline{D}^\circ_{(0,0)}(\epsilon,\epsilon').$$

However, as $\epsilon,\epsilon' \to 0$, the limit points of the relative singular support are contained in $\overline{SS_\pi(\mathscr{F})} \cap \nu^{*,\infty}_{U_x,\pm}N \times \{(0,0)\} = \varnothing$. Hence we can conclude that when $\epsilon,\epsilon' > 0$ are sufficiently small, the intersection of the relative singular support and $\nu^{*,\infty}_{U_x,\pm}N \times D^\circ_{(0,0)}(\epsilon,\epsilon')$ is empty. By non-characteristic deformation lemma Proposition 3.1.2 applied to the family $D^\circ_{(0,0)}(\epsilon,\epsilon')$, we can conclude that

$$\Gamma\big(\overline{U}_x \times \overline{D}^\circ_{(0,0)}(\epsilon), \mathscr{F}\big) \simeq \Gamma\big(\overline{U}_x \times \overline{D}^\circ_0(\epsilon,\epsilon'), \mathscr{F}\big) \simeq \Gamma\big(\overline{U}_x \times \overline{D}^\circ_0(\epsilon), \mathscr{F}\big).$$

This completes the proof. □

Remark 7.1.7. *When applying non-characteristic deformation lemma, one should notice that $\partial(\overline{U}_x \times \overline{D}^\circ_{(0,t)})$ is piecewise smooth. Therefore, we need to use the condition that $SS^\infty(\mathscr{F}) \cap \pi^*T^{*,\infty}((0,1] \times (0,1]) = \varnothing$ rather than only considering the intersection with $\pi_1^*T^{*,\infty}(0,1]$ and $\pi_2^*T^{*,\infty}(0,1]$. For the same reason, we need to use* *the estimate on* $\overline{SS^\infty_\pi(\mathscr{F})} \cap T^{*,\infty}N \times 0 \times 0$ *rather than the estimates on* $\overline{SS^\infty_{\pi_1}(\mathscr{F})}$ *and* $\overline{SS^\infty_{\pi_2}(\mathscr{F})} \cap T^{*,\infty}N \times 0 \times 0$. *The author is grateful to an anonymous referee for pointing out the mistake in the proposition.*

We can start the proof of the theorem. Let λ_i be the Liouville form, Z_i the Liouville vector field, and $\varphi^z_{Z_i}$ the Liouville flow on the Weinstein sector X_i. Consider the lifting of the flow $\varphi^z_{Z_i}$ in $T^{*,\infty}N$ that satisfies

$$d\varphi^z_{Z_i}/dz = t\partial/\partial t + Z_{\lambda_i}$$

on $X_i \times \mathbb{R}$. Then we know that

$$\lim_{z\to-\infty} \varphi^z_{Z_1}(\mathfrak{c}_{X_0}) \subset \mathfrak{c}_{X_1},\ \lim_{z\to-\infty} \varphi^z_{Z_2}(\mathfrak{c}_{X_0}),\ \lim_{z\to-\infty} \varphi^z_{Z_2}(\mathfrak{c}_{X_1}) \subset \mathfrak{c}_{X_2}.$$

Write $\phi^\zeta_{Z_i} = \varphi^{\ln\zeta}_{Z_i}$. Now consider the 2-parameter family of contact Hamiltonian $\phi^{\zeta,\eta}_{\overline{Z}} = \phi^\zeta_{Z_2} \circ \phi^{\eta-\zeta}_{Z_1}$. Then $\phi^{\zeta,\zeta}_{\overline{Z}} = \varphi^\zeta_{Z_2},\ \phi^{1,\eta}_{\overline{Z}} = \varphi^\eta_{Z_1}$. In particular, the limits satisfy

$$\lim_{\zeta\to 0} \phi^{\zeta,\eta}_{\overline{Z}}(-) = \lim_{\zeta\to 0} \phi^\zeta_{Z_2}(-) = \lim_{z\to-\infty} \varphi^z_{Z_2}(-),$$
$$\lim_{\eta\to 0} \phi^{\zeta,\eta}_{\overline{Z}}(-) = \phi^\zeta_{Z_2}\Big(\lim_{\eta\to 0} \phi^\eta_{Z_1}(-)\Big) = \phi^\zeta_{Z_2}\Big(\lim_{y\to-\infty} \varphi^y_{Z_1}(-)\Big).$$

Write $\Delta = \{(\zeta,\eta) | 0 < \eta \leq \zeta \leq 1\}$, $\overline{\Delta} = \{(\zeta,\eta) | 0 \leq \eta \leq \zeta \leq 1\}$ and $\overline{\Delta}_0 = \overline{\Delta}\backslash\{(0,0)\}$.

Proof of Theorem 7.0.7. Consider the 2-parameter family of contact flows $\phi^{\zeta,\eta}_{\overline{Z}}$ $((\zeta,\eta) \in \Delta)$. By Theorem 3.3.2 Remark 3.3.2, for $\mathscr{F} \in \mu Sh_{\mathfrak{c}_{X_0}}(\mathfrak{c}_{X_0})$, we can get a sheaf

$$\Psi^{\zeta,\eta}_{\overline{Z}}(\mathscr{F}) \in \mu Sh_{(\mathfrak{c}_{X_0})_{\overline{Z}}}\big((\mathfrak{c}_{X_0})_{\overline{Z}}\big),$$

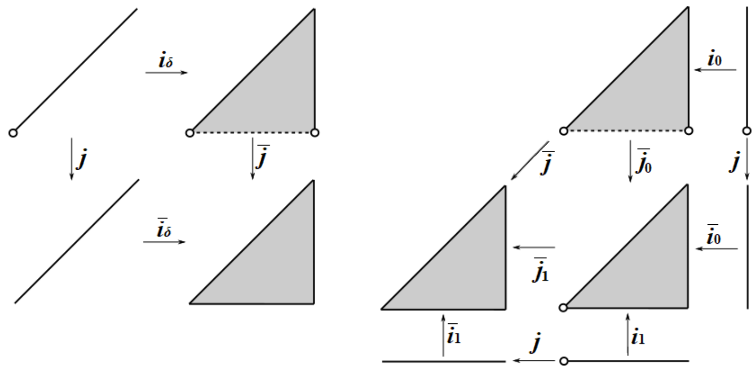

Figure 7.2. The diagram of maps in the proof of Theorem 7.0.7.

where $(\mathfrak{c}_{X_0})_{\overline{Z}'}$ is the Legendrian movie of $\mathfrak{c}_{X_0}$ under the contact flow $\phi_{\overline{Z}}^{\zeta,\eta}$ (in Definition 3.3.1). Applying the antimicrolocalization theorem [**124**, Theorem 6.28], we write $\Psi_{\overline{Z}}^{\zeta,\eta}(\mathscr{F})_{\mathrm{dbl}} \in Sh(N \times \Delta)$ for the image of $\Psi_{\overline{Z}}^{\zeta,\eta}(\mathscr{F})$ under the antimicrolocalization functor.

From Figure 7.2 one can notice that Φ_{02} and $\Phi_{12} \circ \Phi_{01}$ are (compositions of) nearby cycles along different boundary edges of $\overline{\Delta}$. Therefore it suffices to show that the nearby cycle functors commute and they agree with the 2-parametric nearby cycle functor. In order to apply Lemma 7.1.5 in our argument, note that firstly $SS^\infty(\Psi_{\bar{Z}'}^{\zeta,\eta}(\mathscr{F})) \cap \pi^* T^{*,\infty}\Delta = \varnothing$ since the singular support is the Legendrian movie under a contact flow, and secondly $\overline{SS_\pi^\infty(\Psi_{\bar{Z}'}^{\zeta,\eta}(\mathscr{F}))} \cap T^{*,\infty}([0,1] \times \{0\})$ is subanalytic Legendrian by the fact that

$$\lim_{\eta,\zeta\to 0} \phi_{\bar{Z}'}^{\zeta,\eta}(\mathfrak{c}_{X_0}) \subseteq \mathfrak{c}_{X_2},$$

$$\lim_{\eta\to 0} \phi_{\bar{Z}'}^{\zeta,\eta}(\mathfrak{c}_{X_0}) \subseteq \phi_Z^{\zeta}(\mathfrak{c}_{X_1}),$$

where the right hand sides are subanalytic Legendrian. Therefore, in all following cases Lemma 7.1.5 will apply.

(1) Firstly, we consider $\Phi_{02}(\mathscr{F})$ (Figure 7.2 left). Note that $\varphi^{z}_{Z_2}$ compresses $\mathfrak{c}_{X_0}$ to $\mathfrak{c}_{X_2}$. Write $i_\delta : N \times (0,1] \hookrightarrow N \times \Delta$, $(x,\zeta) \mapsto (x,\zeta,\zeta)$, $j : N \times (0,1] \hookrightarrow N \times [0,1]$ and $i : N \times \{0\} \hookrightarrow N \times [0,1]$. Then since $\phi^{\zeta,\zeta}_{\underline{Z}} = \phi^{\zeta}_{Z}$,

$$\Phi_{02}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} j_* \Psi^{\zeta}_{Z_2}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} j_* \big(i_\delta^{-1} \Psi^{\zeta,\eta}_{\underline{Z}}(\mathscr{F})\big)_{\mathrm{dbl}}.$$

Write $\bar{i}_\delta : N \times [0,1] \hookrightarrow N \times \overline{\Delta}$, $(x,\zeta) \mapsto (x,\zeta,\zeta)$, $\bar{j} : N \times \Delta \to N \times \overline{\Delta}$ and $\bar{i} : N \times \{(0,0)\} \hookrightarrow N \times \overline{\Delta}$. By Lemma 7.1.5 and Remark 3.3.3, we know that in fact

$$\Phi_{02}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} \bar{i}_\delta^{-1} \bar{j}_* \Psi^{\zeta,\eta}_{\underline{Z}}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} \bar{i}^{-1} \bar{j}_* \Psi^{\zeta,\eta}_{\underline{Z}}(\mathscr{F})_{\mathrm{dbl}}.$$

(2) Secondly, we consider $\Phi_{12}(\mathscr{F})$ (Figure 7.2 right). Note that $\varphi^{y}_{Z_1}$ compresses $\mathfrak{c}_{X_0}$ to $\mathfrak{c}_{X_1}$. Therefore,

$$\Phi_{01}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} j_* \Psi^{\eta}_{Z_1}(\mathscr{F})_{\mathrm{dbl}}.$$

Write $i_0 : N \times (0,1] \hookrightarrow N \times \Delta$, $(x,\eta) \mapsto (x,1,\eta)$. Since $\phi^{1,\eta}_{\underline{Z}} = \phi^{\eta}_{Z'}$, we know that

$$\Phi_{01}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} j_* \Psi^{\eta}_{Z_1}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} j_* \big(i_0^{-1} \Psi^{\zeta,\eta}_{\underline{Z}}(\mathscr{F})\big)_{\mathrm{dbl}}.$$

Write $\bar{j}_0 : N \times \Delta \hookrightarrow N \times \overline{\Delta}_0$ where $\overline{\Delta}_0 = \overline{\Delta} \backslash \{(0,0)\}$, and $\bar{i}_0 : N \times [0,1] \hookrightarrow N \times \overline{\Delta}$, $(x,\eta) \mapsto (x,1,\eta)$. By Lemma 7.1.5 and Remark 3.3.3, we know that in fact

$$\Phi_{01}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} \bar{i}_0^{-1} \bar{j}_{0,*} \Psi^{\zeta,\eta}_{\underline{Z}}(\mathscr{F})_{\mathrm{dbl}}.$$

Then we consider $\Phi_{12} \circ \Phi_{01}(\mathscr{F})$ (Figure 7.2 right). Write $i_1 : N \times (0,1] \hookrightarrow N \times \overline{\Delta}_0$, $(x,\zeta) \mapsto (x,\zeta,0)$ where $\overline{\Delta}_0 = \overline{\Delta} \backslash \{(0,0)\}$. Let $\varphi^z_{\overline{Z}}$ be the contact flow on $T^{*,\infty}(N \times [0,1])$ defined by the pull back vector field $\pi^* Z_2$ for $\pi : \overline{\Delta}_0 \cong (0,1] \times [0,1] \to (0,1]$, and $\phi^\zeta_{\overline{Z}} = \varphi^{\ln \zeta}_{\overline{Z}}$. Let $\Psi^\zeta_{\overline{Z}} : Sh(N \times \{1\} \times [0,1]) \to Sh(N \times \overline{\Delta}_0)$ be the Hamiltonian isotopy functor as in Theorem 3.3.1. Thus by Lemma 7.1.5

$$\left(\Psi^\zeta_{Z_2} \circ \Phi_{01}(\mathscr{F})\right)_{\mathrm{dbl}} \xrightarrow{\sim} \Psi^\zeta_{Z_2}\left(i^{-1}\overline{i}_0^{-1}\overline{j}_{0,*}\Psi^{\zeta,\eta}_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}\right)$$
$$\xrightarrow{\sim} i_1^{-1}\Psi^\zeta_{\overline{Z}_{\mathrm{st}}}\left(\overline{i}_0^{-1}\overline{j}_{0,*}\Psi^{\zeta,\eta}_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}\right) \xrightarrow{\sim} i_1^{-1}\overline{j}_{0,*}\Psi^{\zeta,\eta}_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}.$$

Therefore, by Lemma 7.1.5 again, we can show that

$$\Phi_{12} \circ \Phi_{01}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1}j_*\left(\Psi^\zeta_{Z_2} \circ \Phi_{01}(\mathscr{F})\right)_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1}j_* i_1^{-1}\overline{j}_{0,*}\Psi^{\zeta,\eta}_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}$$
$$\xrightarrow{\sim} i^{-1}\overline{i}_1^{-1}\overline{j}_{1,*}\overline{j}_{0,*}\Psi^{\zeta,\eta}_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} \overline{i}^{-1}\overline{j}_*\Psi^{\zeta,\eta}_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}.$$

Therefore, we can conclude that $\Phi_{02}(\mathscr{F}) \simeq \Phi_{12} \circ \Phi_{01}(\mathscr{F})$.

(3). On the level of morphisms, the base change formulas provide natural transformations between the morphism spaces, and the gapped full faithfulness theorem for nearby cycles Theorem 7.1.3 [**124**, Theorem 4.1] shows that the natural transformations are quasi-isomorphisms, and hence completes the proof. $\square$

As a corollary, we can immediately get the invariance of the microlocal sheaf category under any Liouville homotopies of Weinstein sectors.

Corollary 7.1.8. *Let X, X' be Weinstein domains with Lagrangian skeleta $\mathfrak{c}_X, \mathfrak{c}_{X'}$. Suppose the Liouville forms λ, λ' are homotopic through Liouville forms. Then*

$$\mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X) \simeq \mu Sh_{\mathfrak{c}_{X'}}(\mathfrak{c}_{X'}).$$

Proof. We view X, X' as Weinstein domains with contact boundary. By choosing a sufficiently small Weinstein neighbourhood (with contact boundary) of $\mathfrak{c}_{X'}$, we get a Liouville embedding $X' \hookrightarrow X$, and thus a functor

$$\Phi_{X',X} : \mu Sh_{\mathfrak{c}_{X'}}(\mathfrak{c}_{X'}) \simeq \mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X).$$

Then by choosing a sufficiently small Weinstein neighbourhood (with contact boundary) of $\mathfrak{c}_X$, we also get a Liouville embedding $X \hookrightarrow X'$, and thus a functor

$$\Phi_{X,X'} : \mu Sh_{\mathfrak{c}_X}(\mathfrak{c}_X) \hookrightarrow \mu Sh_{\mathfrak{c}_{X'}}(\mathfrak{c}_{X'}).$$

Then the theorem implies that $\Phi_{X,X'} \circ \Phi_{X',X} = \mathrm{id}$ and $\Phi_{X',X} \circ \Phi_{X,X'} = \mathrm{id}$. Hence they define inverse equivalences of categories. □

Remark 7.1.8. *Oleg Lazarev has pointed out to the author that* [**109**, Proposition 2.42] *has shown that for any Liouville homotopy between two different Weinstein structures on X, there is a Weinstein structure on the Liouville movie $X \times T^*[0,1]$ which agrees with the two Weinstein structures on the two ends. With this proposition, one can show that the argument in* [**124**, Theorem 9.14] *implies the above corollary as well. However, to the author's knowledge, when there is only a Liouville* *embedding of Weinstein manifolds $X_0 \hookrightarrow X_1$, it is not true that $X_1 \backslash X_0$ carries a Weinstein structure, and hence for Liouville embeddings, it still seems necessary to use our main result.*

7.2. Lagrangian Cobordism Functor by Specialization

The goal in this section is to apply the functorial specialization constructions in the previous section to the setting of Lagrangian cobordisms between Legendrian submanifolds. Using the full faithfulness of the cobordism functor, we prove a number of exact sequences and applications, which are analogous to results by Chantraine-Dimitroglou Rizell-Ghiggini-Golovko [**31**].

Moreover, we also compare our construction with Guillermou-Kashiwara-Schapira [**88**] in the case when the Lagrangian cobordism is induced by a Legendrian isotopy, and compare with Jin-Treumann [**94**] in the case when the Lagrangian cobordism is a Lagrangian filling of a standard brane.

7.2.1. Construction of cobordism functor

In this section we construct the Lagrangian cobordism and prove full faithfulness, which is the first part of Theorem 7.0.8. The proof here will be relatively concise, yet it still includes an outline of the constructions in Section 7.1.1. The reader may find more detailed explanation in those sections.

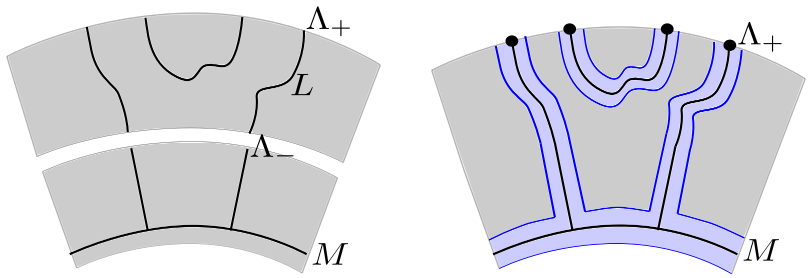

Figure 7.3. A schematic picture on how attaching Lagrangian cobordism L to the relative Lagrangian skeleton $M \cup \Lambda_- \times \mathbb{R}_{>0}$ of the sector (T^*M, Λ_-) gives a Weinstein subsector in (T^*M, Λ_+).

Proof of Theorem 7.0.8 Part 1. By Section 7.1.1, Gromov's h-principle [**65**, Theorem 12.3.1] enables us to embed the contactization of the Weinstein manifold $X \times \mathbb{R}$ into the contact boundary of a higher dimensional cotangent bundle $T^{*,\infty}N$.

Consider the symplectic normal bundle $\nu_{X\times\mathbb{R}}(T^{*,\infty}N)$ of $X \times \mathbb{R} \hookrightarrow T^{*,\infty}N$, and as in Remark 7.1.3 we assume that there is a Lagrangian subbundle $(X \times \mathbb{R})_\sigma \subset \nu_{X\times\mathbb{R}}(T^{*,\infty}N)$ by choosing a section in the Lagrangian Grassmannian of the normal bundle $\nu_{X\times\mathbb{R}}(T^{*,\infty}N)$. Consider the subanalytic Lagrangian skeleta $\mathfrak{c}_X \cup \Lambda_\pm \times \mathbb{R}$ and the Legendrian lifts $(\mathfrak{c}_X \cup \Lambda_\pm \times \mathbb{R}) \times \{0\}$ in $X \times \mathbb{R}$. Let the microlocal sheaf category supported on $\mathfrak{c}_X \cup \Lambda_\pm \times \mathbb{R}$ (which is independent of the embedding we choose) be

$$\mu Sh_{\mathfrak{c}_X \cup \Lambda_\pm \times \mathbb{R}} = \mu Sh_{(\mathfrak{c}_X \cup \Lambda_\pm \times \mathbb{R})_\sigma}|_{\mathfrak{c}_X \cup \Lambda_\pm \times \mathbb{R}}.$$

Since $(X, d\lambda)$ is a Weinstein manifold, we know that outside a compact subset K, we have $X\backslash K \cong \partial_\infty X \times \mathbb{R}$ where the Liouville flow $Z_\lambda = e^r\partial/\partial s$. Suppose $L \cap \partial_\infty X \times (-\infty, -r_0] = \Lambda_- \times (-\infty, 0]$. Glue $L \cap \partial_\infty X \times [-r_0, +\infty)$ with $\Lambda_- \times (-\infty, -r_0]$ along $\Lambda_- \times \{-r_0\}$, and denote by $\Lambda_- \times \mathbb{R} \cup L$ their concatenation in X. Note that this is the same as L, but we use the notation to emphasize that we will view it as the union of two separate parts to apply the cosheaf property later.

We can glue the Legendrian lift $\widetilde{L}$ of the Lagrangian L to the skeleton $\mathfrak{c}_X \cup \Lambda_\pm \times \mathbb{R}$ in the contactization $X \times \mathbb{R}$. As the primitive of L defined by $df_L = \lambda|_L$ is a constant when the $\mathbb{R}$ coordinate in $\partial_\infty X \times \mathbb{R}$ satisfies $r < -r_0$, we may assume that $f_L = 0$ when $r < -r_0$. The Legendrian lift of L is defined by

$$\widetilde{L} = \{(x, f_L(x)) | x \in L\} \subset X \times \mathbb{R}.$$

Then we consider the sheaf of categories $\mu Sh_{\widetilde{L}}$. Since $\widetilde{L}$ coincides with $\Lambda_- \times \mathbb{R} \subset X \times \{0\}$ when $r < -r_0$, we can glue $\widetilde{L} \cap \partial_\infty X \times [-r_0, +\infty) \times \mathbb{R}$ with $\Lambda_- \times (-\infty, -r_0] \times \{0\}$, and get their concatenation in $X \times \mathbb{R}$. Denote it by $\Lambda_- \times \mathbb{R} \cup \widetilde{L}$. We can thus consider the sheaf of categories $\mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}}$.

Since μSh_- is a sheaf and cosheaf of dg categories, we have a quasi-equivalence of categories

$$\mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}}\big(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}\big) \xrightarrow{\sim} \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_{\widetilde{L}}\big(\widetilde{L}\big).$$

We construct the specialization functor by the inclusion (also explained in Section 7.1.1 after Theorem 7.1.1)

$$\Phi_L : \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}}\big(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}\big) \longrightarrow \mu Sh_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}).$$

Consider the Liouville flow $\varphi_Z^z\,(z \in \mathbb{R})$ on X for negative time, which will compress $\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}$ onto $\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}$ as $z \to -\infty$. The Liouville flow on X extends to a contact Hamiltonian φ_Z^z in $T^{*,\infty}N$ with

$$d\varphi_Z^z/dz = t\partial/\partial t + Z_\lambda.$$

Write $\phi_Z^\zeta = \varphi_Z^{\ln\zeta}$, and consider the Legendrian movie of $\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}$ under the flow $\varphi_Z^z\,(z \in (-\infty, 0])$ or $\phi_Z^\zeta\,(\zeta \in (0,1])$. Since $M \cup \Lambda_- \times \mathbb{R}$ is the Legendrian lift of a Lagrangian skeleton while $\widetilde{L}$ is the lift of an embedded Lagrangian, there are no self Reeb chords and the gapped condition automatically holds. By Theorem 7.1.1 [**124**, Theorem 8.3], the nearby cycle functor gives us a fully faithful embedding of microlocal sheaves on the Legendrian movie of $\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}$ to sheaves on

$$\lim_{z\to-\infty} \varphi_Z^z\big(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}\big) = \lim_{\zeta\to 0} \phi_Z^\zeta\big(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}\big) \subset \mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}.$$

The full faithfulness of Φ_L follows from Theorem 7.1.1. The special case when $X = T^*M$ follows from Lemma 3.2.1. □

Remark 7.2.1. *The functor Φ_L can also be obtained in the setting of partially wrapped Fukaya categories. Indeed one can consider Weinstein manifolds with stops $(X, \Lambda_\pm)$ and view T^*L as a Weinstein sector. First apply the cosheaf property of partially wrapped Fukaya categories* [**75**, Theorem 1.27] *to get*

$$\mathcal{W}(X, \Lambda_-) \otimes_{\mathcal{W}(T^*(\Lambda\times[-1,1]))} \mathcal{W}(T^*L) \xrightarrow{\sim} \mathcal{W}(X \cup_{T^*(\Lambda\times[-1,1])} T^*L)$$

or in other words

$$\mathcal{W}(X,\Lambda_-)\otimes_{Loc^c(\Lambda)} Loc^c(L) \xrightarrow{\sim} \mathcal{W}(X\cup_{T^*(\Lambda\times[-1,1])} T^*L).$$

Then one can view $X\cup_{T^*(\Lambda\times[-1,1])}T^*L$ *as a Liouville subsector of* (T^*X,Λ_+) *(the compliment is a Liouville cobordism). Since* $X\cup_{T^*(\Lambda\times[-1,1])}T^*L$ *is Weinstein, following* [**75**, Section 8.3] *or* [**152**] *one can define a Viterbo restriction functor*

$$\mathcal{W}(X,\Lambda_+)\longrightarrow \mathcal{W}(X\cup_{T^*(\Lambda\times[-1,1])} T^*L).$$

Remark 7.2.2. *In fact the main theorem works in more general settings, as long as the gapped condition in Definition 7.1.1 is satisfied. For example, when* $i: L\hookrightarrow \partial_\infty X\times\mathbb{R}$ *is an exact Lagrangian cobordism with vanishing action self intersection points, i.e. for the primitive* $i^*\lambda = df_L$, $f_L(x)=f_L(x')$ *whenever* $i(x)=i(x')$, *then* L *can be lifted to an immersed Legendrian with no Reeb chords and the theorem still holds. Similarly, when* $\Lambda_\pm$ *are subanalytic Legendrians and* L *is the Lagrangian projection of a subanalytic Legendrian cobordism, the theorem still applies as long as the gapped condition holds.*

Then we show that concatenations of Lagrangian cobordisms give rise to compositions of our Lagrangian cobordism functors. Therefore our cobordism functor defines a functor from the category of Lagrangian cobordisms to the category of (small) dg categories.

We recall how concatenations of Lagrangian cobordisms are defined. Let $L_0 \subset \partial_\infty X \times \mathbb{R}$ be a Lagrangian cobordism from Λ_0 to Λ_1, and L_1 be a Lagrangian cobordism from Λ_1 to Λ_2. Suppose $L_{0,1} \cap \partial_\infty X \times (-\infty, -r_0) \cup (r_0, +\infty)$ are standard cylinders. Then the concatenation $L_0 \cup L_1$ is an exact Lagrangian such that

(1) $(L_0 \cup L_1) \cap \partial_\infty X \times (-\infty, 0) \cong \varphi_Z^{-r_0}(L_0) \cap \partial_\infty X \times (-\infty, 0)$;
(2) $(L_0 \cup L_1) \cap \partial_\infty X \times (0, +\infty) \cong \varphi_Z^{r_0}(L_1) \cap \partial_\infty X \times (0, +\infty)$.

Here φ_Z^z is the Liouville flow on $\partial_\infty X \times \mathbb{R} \subset X$.

Our strategy is as follows. $\Phi_{L_0 \cup L_1}$ is defined by using the Liouville flow to compress $L_0 \cup L_1$ to the skeleton all at once, and $\Phi_{L_1} \circ (\Phi_{L_0} \times \mathrm{id}_{Loc(L_1)})$ is defined by first compressing L_0 to the skeleton while fixing L_1, and next compressing L_1 to the skeleton. We will try to define a 2-parametric family of contact flow that interpolates between them. Then following the construction, $\Phi_{L_0 \cup L_1}$ and $\Phi_{L_1} \circ (\Phi_{L_0} \times \mathrm{id}_{Loc(L_1)})$ are two different compositions of nearby cycles, and the theorem is reduced to commutativity of the nearby cycle functors.

Proof of Theorem 7.0.8 Part 2. Consider the lifting of the Liouville flow φ_Z^z in $T^{*,\infty}N$ that satisfies

$$d\varphi_Z^z/dz = t\partial/\partial t + Z_\lambda$$

on $X \times \mathbb{R}$. Suppose that the concatenation $(L_0 \cup L_1) \cap \partial_\infty X \times (-\epsilon, \epsilon) = \Lambda_1 \times (-\epsilon, \epsilon)$. Let $\eta : \mathbb{R} \to [0, 1]$ be a cut-off function such that $\eta|_{(-\infty, -\epsilon]} \equiv 0$ and $\eta|_{[\epsilon, +\infty)} \equiv 0$. Then we consider a flow $\varphi_{Z'}^z$ on $\partial_\infty X \times \mathbb{R}$ defined by $Z' = \eta(r) Z_\lambda = \eta(r) e^r \partial/\partial r$, such that

$$\varphi_{Z'}^z|_{\partial_\infty X\times(-\infty,-\epsilon)} = \varphi_Z^z,\ \ \varphi_{Z'}^z|_{\partial_\infty X\times(\epsilon,+\infty)} = \mathrm{id}.$$

Note that $\varphi_{Z'}^z$ defines an exact Lagrangian isotopy of $L_0 \cup L_1$, which can be lifted to a Legendrian isotopy of $\widetilde{L}_0 \cup \widetilde{L}_1$. Therefore, lift $\varphi_{Z'}^z$ to a contact flow on $X \times \mathbb{R}$ and still denote it by $\varphi_{Z'}^z$. As a contact flow,

$$d\varphi_{Z'}^z/dz|_{\partial_\infty X\times(-\infty,-\epsilon)\times\mathbb{R}} = t\partial/\partial t + Z_\lambda.$$

Write $\phi_Z^\zeta = \varphi_Z^{\ln\zeta}$ and $\phi_{Z'}^\zeta = \varphi_{Z'}^{\ln\zeta}$. Consider the 2-parameter family of contact Hamiltonian $\phi_{\overline{Z}'}^{\zeta,\eta} = \phi_Z^\zeta \circ \phi_{Z'}^{\eta-\zeta}$. Then $\phi_{\bar{Z}'}^{\zeta,\zeta} = \varphi_Z^\zeta,\ \ \phi_{\bar{Z}'}^{1,\eta} = \varphi_{Z'}^\eta$. In particular, the limits satisfy

$$\lim_{\zeta\to 0}\phi_{\bar{Z}'}^{\zeta,\zeta}(-) = \lim_{\zeta\to 0}\phi_Z^\zeta(-) = \lim_{z\to-\infty}\varphi_Z^z(-),$$
$$\lim_{\eta\to 0}\phi_{\bar{Z}'}^{\zeta,\eta}(-) = \phi_Z^\zeta\Big(\lim_{\eta\to 0}\phi_{Z'}^\eta(-)\Big) = \phi_Z^\zeta\Big(\lim_{y\to-\infty}\varphi_{Z'}^y(-)\Big).$$

Write $\Delta = \{(\zeta,\eta)|0 < \eta \leq \zeta \leq 1\}$, $\overline{\Delta} = \{(\zeta,\eta)|0 \leq \eta \leq \zeta \leq 1\}$ and $\overline{\Delta}_0 = \overline{\Delta}\backslash\{(0,0)\}$.

From Figure 7.2 one can notice that $\Phi_{L_0\cup L_1}$ and $\Phi_{L_0}\circ(\Phi_{L_0}\times \mathrm{id}_{Loc(L_1)})$ are (compositions of) nearby cycles along different boundary edges of $\overline{\Delta}$. Under the perspective of Theorem 7.0.7, we can write

$$\widetilde{\mathfrak{c}}_0 = \widetilde{\mathfrak{c}}_X \cup \Lambda_0 \times \mathbb{R}_{>0},\ \widetilde{\mathfrak{c}}_1 = \widetilde{\mathfrak{c}}_X \cup \Lambda_0 \times \mathbb{R}_{>0} \cup \widetilde{L}_0,\ \widetilde{\mathfrak{c}}_2 = \widetilde{\mathfrak{c}}_X \cup \Lambda_0 \times \mathbb{R}_{>0} \cup \widetilde{L}_0 \cup \widetilde{L}_1.$$

Since the limits under the Liouville flow is the Legendrian lift of the Lagrangian skeleton,

$$\lim_{\eta,\zeta\to 0} \phi_{\bar{Z}'}^{\zeta,\eta}(\mathfrak{c}_X \cup \Lambda_0 \times \mathbb{R} \cup \widetilde{L}_0 \cup \widetilde{L}_1) \subseteq \mathfrak{c}_X \cup \Lambda_2 \times \mathbb{R},$$

$$\lim_{\eta\to 0} \phi_{\bar{Z}'}^{\zeta,\eta}(\mathfrak{c}_X \cup \Lambda_0 \times \mathbb{R} \cup \widetilde{L}_0 \cup \widetilde{L}_1) \subseteq \phi_Z^{\zeta}(\mathfrak{c}_X \cup \Lambda_1 \times \mathbb{R} \cup \widetilde{L}_1),$$

we can apply Lemma 7.1.5 and Theorem 7.0.7. More precisely, following the notation there, when computing $\Phi_{L_0\cup L_1}$ we have

$$\Phi_{L_0\cup L_1}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} i^{-1} j_*\big(i_\delta^{-1}\Psi_{\overline{Z}}^{\zeta,\eta}(\mathscr{F})\big)_{\mathrm{dbl}} \xrightarrow{\sim} \bar{i}^{-1}\bar{j}_*\Psi_{\overline{Z}}^{\zeta,\eta}(\mathscr{F})_{\mathrm{dbl}}.$$

When computing the composition of Φ_{L_0} and $(\Phi_{L_0} \times \mathrm{id}_{Loc(L_1)})$ we have

$$\begin{aligned}\Phi_{L_0} \circ (\Phi_{L_0} \times \mathrm{id}_{Loc(L_1)})(\mathscr{F})_{\mathrm{dbl}} &\xrightarrow{\sim} i^{-1} j_*\big(\Psi_Z^{\zeta}(\Phi_{L_0} \times \mathrm{id}_{Loc(L_1)})(\mathscr{F})_{\mathrm{dbl}}\big) \\ &\xrightarrow{\sim} i^{-1} j_* i_1^{-1} \bar{j}_{0,*}\Psi_{\overline{Z}}^{\zeta,\eta}(\mathscr{F})_{\mathrm{dbl}} \xrightarrow{\sim} \bar{i}^{-1}\bar{j}_*\Psi_{\overline{Z}}^{\zeta,\eta}(\mathscr{F})_{\mathrm{dbl}}.\end{aligned}$$

Therefore we can conclude that concatenations of Lagrangian cobordisms induce compositions of the functors. □

Finally, we remark that when L is a Lagrangian concordance from Λ_- to Λ_+, i.e. L is diffeomorphic to $\Lambda_- \times \mathbb{R}$, we have in particular the following simple fully faithful embedding.

Corollary 7.2.1. *Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, $\Lambda_-, \Lambda_+ \subset \partial_\infty X$ be Legendrian submanifolds, and $L \subset \partial_\infty X \times \mathbb{R}$ a Lagrangian concordance from Λ_- to Λ_+. Then there is a fully faithful functor between the categories*

$$\Phi_L : \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \hookrightarrow \mu Sh_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}).$$

In particular, when $X = T^*M$, there is a fully faithful functor between proper sheaves

$$\Phi_L : Sh_{\Lambda_-}(M) \hookrightarrow Sh_{\Lambda_+}(M).$$

7.2.2. Full Faithfulness of Lagrangian Cobordism Functor

For Lagrangian cobordisms L_0, L_1 from Λ_- to Λ_+, Chantraine-Dimitroglou Rizell-Ghiggini-Golovko [**31**] constructed an acyclic Cthulhu complex $\mathrm{Cth}(\Lambda_\pm, L_0, L_1)$ consisting of linearized contact homologies of $\Lambda_\pm$ and the Floer chain complex of L_0, L_1, and hence produced a number of exact sequences. Similar to Chantraine-Dimitroglou Rizell-Ghiggini-Golovko [**31**], we are able to get a series of exact triangles from a Lagrangian cobordism, most of which are simple corollaries of the full faithfulness of our functor Φ_L.

We will always assume in this section that the smooth Legendrians and Lagrangians have vanishing Maslov class and relative second Stiefel-Whitney class. Then by Theorem 3.2.4 [**84**, Theorem 11.5], $\mu Sh_{\Lambda_-}(\Lambda_-) \simeq Loc(\Lambda_-), \mu Sh_{\widetilde{L}}(\widetilde{L}) \simeq Loc(L)$. Hence we have a quasi-equivalence

$$\mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup L}\big(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}\big) \xrightarrow{\sim} \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}(\Lambda_-)} \mu Sh_L(L).$$

First, we state the results we can get and their applications.

Corollary 7.2.2 (Mayer-Vietoris exact triangle)**.** *Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, and $\Lambda_-, \Lambda_+ \subset \partial_\infty X$ be Legendrian submanifolds. Suppose there is an exact Lagrangian cobordism $L \subset \partial_\infty X \times \mathbb{R}$ from Λ_- to Λ_+. Suppose there are sheaves $\mathscr{F}^-, \mathscr{G}^- \in \mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})$ with trivial monodromy along Λ_-, and their microstalks at Λ_- are F, G. Denoting by*

$$\mathscr{F}^+ = \Phi_L(\mathscr{F}^-),\ \ \mathscr{G}^+ = \Phi_L(\mathscr{G}^-),$$

the images of $\mathscr{F}^-, \mathscr{G}^-$ glued with trivial local systems on L with stalks F, G, then there is an exact triangle

$$\Gamma(\mu hom(\mathscr{F}^+, \mathscr{G}^+)) \to \Gamma(\mu hom(\mathscr{F}^-, \mathscr{G}^-)) \oplus C^*(L; Hom(F, G))$$
$$\to C^*(\Lambda_-; Hom(F, G)) \xrightarrow{+1} .$$

A flexible Weinstein manifold [**40**, Chapter 11] is a Weinstein manifold whose attaching spheres of index-n critical points are all loose Legendrian submanifolds [**116**]. Similar to the result in [**31**], we are able to prove a stronger result that any Legendrian submanifold in the boundary of a flexible Weinstein manifold whose microlocal sheaf category of proper objects over $\Bbbk = \mathbb{Z}/2\mathbb{Z}$ is nontrivial does not admit a Lagrangian cap. Assuming the equivalence between partially wrapped Fukaya categories and Legendrian contact homologies, this means that any Legendrian submanifold whose contact homology over $\Bbbk = \mathbb{Z}/2\mathbb{Z}$ has a proper module does not admit a Lagrangian cap.

Corollary 7.2.3. *Let X be a flexible Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, and $\Lambda_- \subset \partial_\infty X$ be a connected Legendrian submanifold. Suppose that $\mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})$ contains a nontrivial object with trivial monodromy along Λ_-. Then there is no Lagrangian cobordism from Λ_- to $\varnothing$ with Maslov data.*

Remark 7.2.3. *Since there are examples whose partially wrapped Fukaya category only has higher dimensional representations* [**106**,**107**]*, by the equivalence between Fukaya categories and sheaf categories* [**74**] *and the fact that* [**121**, Theorem 3.21] *(or Section 3.4)*

$$\mu Sh^b_{\mathfrak{c}_X}(\mathfrak{c}_X) \simeq \mathrm{Fun}^{ex}(\mu Sh^c_{\mathfrak{c}_X}(\mathfrak{c}_X)^{op}, \mathrm{Perf}(\Bbbk)),$$

this corollary is expected to be stronger than the result in [**31**]*. Note that there are also examples whose Legendrian contact homology is nontrivial but has only higher dimensional representations* [**149**].

Remark 7.2.4. *The assumption that the sheaf has trivial monodromy along Λ_- is necessary. For example, the Clifford Legendrian torus Λ_{Cliff} discussed in Theorem 7.0.10 does admit a microlocal rank 1 sheaf. However, there is a Lagrangian cobordism from a loose Legendrian sphere to Λ_{Cliff}* [**26**, Example 4.26]*, and hence there is a Lagrangian cap by* [**66**].

Proof of Corollary 7.2.3. Let $\mathscr{F}^- \in \mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})$ be a nonzero object with stalk at Λ_- being F. Suppose there is an exact Lagrangian cobordism from Λ_- to $\varnothing$. Then since $\mathscr{F}^-$ has trivial monodromy and the stalk F at Λ_- is nonzero, it can be extended to a local system on L with nonzero stalk. Glue $\mathscr{F}^-$ with the local system and write $\mathscr{F}^+ = \Phi_L(\mathscr{F}^-)$. Since X is flexible, $\Gamma(\mu hom(\mathscr{F}^+, \mathscr{F}^+)) \simeq 0$. From the Mayer-Vietoris exact triangle we know that (by setting $\mathscr{G}^- = \mathscr{F}^-$)

$$\Gamma(\mu hom(\mathscr{F}^-, \mathscr{F}^-)) \oplus C^*(L; Hom(F,F)) \simeq C^*(\Lambda_-; Hom(F,F)).$$

However, the fact that $H^0(L; Hom(F,F)) \simeq H^0(\Lambda_-; Hom(F,F))$ will force

$$H^0(\mu hom(\mathscr{F}^-, \mathscr{F}^-)) = 0,$$

i.e. $\mathrm{id}_{\mathscr{F}^-} = 0$, which gives a contradiction. □

Remark 7.2.5. *The fact that flexible Weinstein domains have trivial microlocal sheaf categories follows from* [**74**]*, the vanishing result for their symplectic cohomologies* [**118**, Theorem 3.2] *(using the embedding trick* [**66**, Corollary 6.3]*) and Abouzaid's generation criterion* [**1**]*. In fact using the embedding trick and the restriction functor* [**124**] *(or Section 7.1.1) we can also get a sheaf theoretic proof of this fact.*

The next exact sequence is the following, analogous to results in [**31**, Theorem 1.1] and Pan [**128**, Theorem 1.2].

Corollary 7.2.4. *Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, and $\Lambda_-, \Lambda_+ \subset \partial_\infty X$ be Legendrian submanifolds. Suppose there is an exact Lagrangian cobordism $L \subset \partial_\infty X \times \mathbb{R}$ from Λ_- to Λ_+. Suppose there are sheaves $\mathscr{F}^-, \mathscr{G}^- \in \mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})$ with trivial monodromy along Λ_-, and their* *stalks at* Λ_- *are* F, G*. Denoting by*

$$\mathscr{F}^+ = \Phi_L(\mathscr{F}^-),\ \ \mathscr{G}^+ = \Phi_L(\mathscr{G}^-),$$

the images of $\mathscr{F}^-, \mathscr{G}^-$ *glued with trivial local systems on* L *with stalks* F, G*, then there is an exact triangle*

$$\Gamma(\mu hom(\mathscr{F}^+, \mathscr{G}^+)) \to \Gamma(\mu hom(\mathscr{F}^-, \mathscr{G}^-)) \to C^*(L, \Lambda_-; Hom(F, G))[1] \xrightarrow{+1}.$$

Remark 7.2.6. *Following* [**128**, Theorem 1.6]*, restricting to the subcategory* $\mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})_{tri} \subset \mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})$ *with trivial monodromy along* Λ_-*, the functor defined by gluing with the trivial local system on* L

$$\mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})_{tri} \to \mu Sh^b_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R})_{tri}$$

is injective on objects as long as $H^0(L, \Lambda_-) = 0$*. The proof is the same as* [**128**]*, where one uses the fact that*

$$H^0(\mu hom(\mathscr{F}^+, \mathscr{G}^+)) \xrightarrow{\sim} H^0(\mu hom(\mathscr{F}^-, \mathscr{G}^-))$$

preserves the identity.

In particular, when $\Lambda_- = \varnothing$, i.e. when L is an exact Lagrangian filling of Λ_+, by choosing the trivial rank 1 local system on L, we are able to get a sheaf quantization $\mathscr{F}^+$ of L and this recovers the Seidel isomorphism [**51**]. The first proof in sheaf theory when $X = T^*M$ is obtained by Jin-Treumann [**94**].

Note that in contrary to [**51**], the proof in sheaf theory does not require $\mathcal{W}(X)$ or $\mu Sh^c_{\mathfrak{c}_X}(\mathfrak{c}_X)$ to vanish (because the sheaf categories are always identified with Fukaya categories, but they are expected to be the Chekanov-Eliashberg dg algebra or its representations only when the ambient manifold is flexible).

Corollary 7.2.5 (Nadler-Shende)**.** *Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, and $\Lambda_+ \subset \partial_\infty X$ be a Legendrian submanifold. Let $\Bbbk$ be a ring. Suppose there is an exact Lagrangian filling $L \subset X$ of Λ_+ with vanishing Maslov class (and relatively spin when $\mathrm{char}\Bbbk \neq 2$). Then there is $\mathscr{F}^+ \in \mu Sh^b_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R})$ such that*

$$\Gamma(\mu hom(\mathscr{F}^+, \mathscr{F}^+)) \simeq C^*(L; \Bbbk).$$

Proof. Pick the rank 1 trivial local system on $\mu Sh^b_L(L) \simeq Loc^b(L)$. Then Corollary 7.2.4 gives the result. □

After stating the long exact sequences and their applications, we explain their proofs, which all follow immediately from the following lemma as a corollary of full faithfulness.

Lemma 7.2.6. *Let X be a Weinstein manifold with subanalytic skeleton $\mathfrak{c}_X$, and $\Lambda_-, \Lambda_+ \subset \partial_\infty X$ be Legendrian submanifolds. Suppose there is an exact Lagrangian cobordism $L \subset \partial_\infty X \times \mathbb{R}$ from Λ_- to Λ_+. Suppose there are sheaves $\mathscr{F}^-, \mathscr{G}^- \in \mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R})$ with trivial monodromy along Λ_-, and their stalks at Λ_-*

are F, G. Denoting by

$$\mathscr{F}^+ = \Phi_L(\mathscr{F}^-),\ \ \mathscr{G}^+ = \Phi_L(\mathscr{G}^-),$$

the images of $\mathscr{F}^-, \mathscr{G}^-$ glued with the trivial local systems on L with stalks F, G, then there is a homotopy pullback diagram

$$\begin{array}{ccc} \Gamma(\mu hom(\mathscr{F}^+, \mathscr{G}^+)) & \longrightarrow & \Gamma(\mu hom(\mathscr{F}^-, \mathscr{G}^-)) \\ \downarrow & & \downarrow \\ C^*(L; Hom(F,G)) & \longrightarrow & C^*(\Lambda_-; Hom(F,G)). \end{array}$$

Proof. Denote by $\widetilde{\mathscr{F}}^+, \widetilde{\mathscr{G}}^+$ the sheaves in $\mu Sh^b_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}}\big(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}\big)$ obtained by gluing $\mathscr{F}^-, \mathscr{G}^-$ by the constant sheaf on L with stalk F, G. Then by the sheaf property of $\mu Sh^b(-)$, we have a pullback diagram

$$\begin{array}{ccc} \Gamma\big(\mu hom(\widetilde{\mathscr{F}}^+, \widetilde{\mathscr{G}}^+)\big) & \longrightarrow & \Gamma(\mu hom(\mathscr{F}^-, \mathscr{G}^-)) \\ \downarrow & & \downarrow \\ C^*(L; Hom(F,G)) & \longrightarrow & C^*(\Lambda_-; Hom(F,G)). \end{array}$$

By full faithfulness of Φ_L, we know that

$$\Gamma\big(\mu hom(\widetilde{\mathscr{F}}^+, \widetilde{\mathscr{G}}^+)\big) \xrightarrow{\sim} \Gamma(\mu hom(\mathscr{F}^+, \mathscr{G}^+)).$$

This proves the assertion. $\square$

Proof of Corollary 7.2.2. The result immediately follows from the lemma.

□

Proof of Corollary 7.2.4. Note that the restriction map on cohomology $C^*(L; Hom(F,G)) \to C^*(\Lambda_-; Hom(F,G))$ fits into an exact triangle

$$C^*(L; Hom(F,G)) \to C^*(\Lambda_-; Hom(F,G)) \to C^*(L, \Lambda_-; Hom(F,G))[1] \xrightarrow{+1} .$$

Since a pullback diagram preserves (co)fibers, this gives the exact sequence

$$\Gamma(\mu hom(\widetilde{\mathscr{F}}^+, \widetilde{\mathscr{G}}^+)) \to \Gamma(\mu hom(\mathscr{F}^-, \mathscr{G}^-)) \to C^*(L, \Lambda_-; Hom(F,G))[1] \xrightarrow{+1},$$

and hence completes the proof. □

7.2.3. Hamiltonian invariance of Cobordism Functor

We show the basic property that the Lagrangian cobordism functor is invariant under Hamiltonian isotopies in the symplectization that fix the positive (convex) and negative (concave) ends.

Theorem 7.2.7 (Hamiltonian invariance)**.** *Let X be a Weinstein manifold, $\Lambda_\pm \subset \partial_\infty X$ be Legendrian submanifolds, and $L \subset \partial_\infty X \times \mathbb{R}$ be a Lagrangian cobordism from Λ_- to Λ_+. Suppose there is a compactly supported Hamiltonian isotopy φ_H^s $(s \in I)$ on $\partial_\infty X \times \mathbb{R}$. Then*

$$\Phi_L^* \simeq \Phi_{\varphi_H^1(L)}^*, \;\; \Phi_L \simeq \Phi_{\varphi_H^1(L)}.$$

Again, we can only consider the dg categories $\mu Sh_{\mathfrak{c}_X\cup\Lambda_\pm\times\mathbb{R}}(\mathfrak{c}_X\cup\Lambda_\pm\times\mathbb{R})$ and $Loc(L)$, and show that

$$\Phi_L \simeq \Phi_{\varphi_H^1(L)} : \mu Sh_{\,\mathfrak{c}_X\cup\Lambda_-\times\mathbb{R}}(\mathfrak{c}_X\cup\Lambda_-\times\mathbb{R}) \times_{Loc(\Lambda_-)} Loc(L)$$
$$\to \mu Sh_{\mathfrak{c}_X\cup\Lambda_+\times\mathbb{R}}(\mathfrak{c}_X\cup\Lambda_+\times\mathbb{R}).$$

Then the results will immediately follow from the properties of adjoint functors.

Our strategy is to compare Φ_L and $\Phi_{\varphi_H^s(L)}$ by constructing a 1-parametric family of Lagrangian cobordism functors, and then Theorem 3.3.1 [**88**] will allow us to show that $\Phi_L \simeq \Phi_{\varphi_H^1(L)}$ from the initial condition $\Phi_L \simeq \Phi_{\varphi_H^0(L)}$.

Identify $\mathfrak{c}_X\cup\Lambda_\pm\times\mathbb{R}$ and L with their Legendrian image in some higher dimensional contact manifold $T^{*,\infty}N$, and lift φ_H^s to a contact Hamiltonian flow on $T^{*,\infty}N$. Consider $(\mathfrak{c}_X\cup\Lambda_-\times\mathbb{R})\times I$. Then we have a Lagrangian cobordism functor

$$\Phi_{L\times I} : \mu Sh_{((\mathfrak{c}_X\cup\Lambda_-\times\mathbb{R})\times I)\cup(L\times I)}\big(((\mathfrak{c}_X\cup\Lambda_-\times\mathbb{R})\times I)\cup(L\times I)\big)$$
$$\to \mu Sh_{(\mathfrak{c}_X\cup\Lambda_+\times\mathbb{R})\times I}\big((\mathfrak{c}_X\cup\Lambda_+\times\mathbb{R})\times I\big).$$

On the other hand, let $\widetilde{L}_H$ be the Legendrian movie of $\widetilde{L}$ (in Definition 3.3.1) under the Hamiltonian flow φ_H^s. Then we have a Lagrangian cobordism functor

$$\Phi_{L_H} : \mu Sh_{((\mathfrak{c}_X\cup\Lambda_-\times\mathbb{R})\times I)\cup\widetilde{L}_H}\big(((\mathfrak{c}_X\cup\Lambda_-\times\mathbb{R})\times I)\cup\widetilde{L}_H\big)$$
$$\to \mu Sh_{(\mathfrak{c}_X\cup\Lambda_+\times\mathbb{R})\times I}\big((\mathfrak{c}_X\cup\Lambda_+\times\mathbb{R})\times I\big).$$

For $\mathscr{F} \in \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}}\big(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}\big)$, write $\pi : T^{*,\infty}(N \times I) \to T^{*,\infty}N$. We consider

$$\pi^{-1}(\mathscr{F}) \in \mu Sh_{((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup (\widetilde{L} \times I)}\big(((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup (\widetilde{L} \times I)\big).$$

On the other hand, by Theorem 3.3.2 the Hamiltonian isotopy φ_H^s defines a canonical sheaf

$$\Psi_H^0(\mathscr{F}) \in \mu Sh_{((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup \widetilde{L}_H}\big(((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup \widetilde{L}_H\big).$$

Lemma 7.2.8. *Let $\pi : N \times I \to N$ be the projection, $i_s : N \times \{s\} \hookrightarrow N \times I$ be the inclusion, and $\widetilde{L}_H$ be the Legendrian movie of $\widetilde{L}$ under the Hamiltonian flow φ_H^s $(s \in I)$. Then for any $\mathscr{F} \in \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L})$,*

$$i_s^{-1}\Phi_{L \times I}(\pi^{-1}(\mathscr{F})) = \Phi_L(\mathscr{F}),\ \ i_s^{-1}\Phi_{L_H}(\Psi_H^0(\mathscr{F})) = \Phi_{\varphi_H^s(L)}(\mathscr{F}).$$

Proof. First of all, let $\varphi_{\overline{Z}}^z$ be the Liouville flow on $T^{*,\infty}(N \times I)$ defined by $(\varphi_Z^z, \mathrm{id})$. Let $\Psi_{\overline{Z}}^\zeta$ be the equivalence functor defined by the Liouville flow $\varphi_{\overline{Z}}^z\,(z \in (-\infty, 0])$ or $\phi_{\overline{Z}}^\zeta\,(\zeta \in (0,1])$ on $T^{*,\infty}(N \times I)$, and

$$N \times \{0\} \xrightarrow{i_Z} N \times [0,1] \xleftarrow{j_Z} N \times (0,1],$$

$$N \times I \times \{0\} \xrightarrow{\bar{i}_{\overline{Z}}} N \times I \times [0,1] \xleftarrow{\bar{j}_{\overline{Z}}} N \times I \times (0,1].$$

Write $(\Psi^{\zeta}_{\overline{Z}}(\Psi^0_H(\mathscr{F})))_{\mathrm{dbl}} \in Sh(N \times I \times (0,1])$ for the image of $\Psi^{\zeta}_{\overline{Z}}(\Psi^0_H(\mathscr{F}))$ under the antimicrolocalization functor in Theorem 7.1.2 [**124**]. Then by Remark 3.3.3,

$$i_s^{-1}\big(\Psi^{\zeta}_{\overline{Z}}(\Psi^0_H(\mathscr{F}))\big)_{\mathrm{dbl}} = \Psi^{\zeta}_{Z}(i_s^{-1}\Psi^0_H(\mathscr{F}))_{\mathrm{dbl}}.$$

Similarly, write $i_s^{-1}\Phi_{L\times I}(\pi^{-1}(\mathscr{F}))_{\mathrm{dbl}} \in Sh(N\times\{s\})$ for the image of $i_s^{-1}\Phi_{L\times I}(\pi^{-1}(\mathscr{F}))$ under the antimicrolocalization functor. By Lemma 7.1.5 we have

$$\begin{aligned} i_s^{-1}\Phi_{L\times I}(\pi^{-1}(\mathscr{F}))_{\mathrm{dbl}} &\xrightarrow{\sim} i_s^{-1} i_{\overline{Z}}^{-1} \overline{j}_{\overline{Z},*}\big(\Psi^{\zeta}_{\overline{Z}}(\pi^{-1}(\mathscr{F}))\big)_{\mathrm{dbl}} \\ &\xrightarrow{\sim} i_Z^{-1} j_{Z,*}\big(\Psi^{\zeta}_{Z}(i_s^{-1}\pi^{-1}(\mathscr{F}))\big)_{\mathrm{dbl}} \xrightarrow{\sim} \Phi_L(\mathscr{F})_{\mathrm{dbl}}. \end{aligned}$$

On the other hand, write $i_s^{-1}\Phi_{L_H}(\pi^{-1}(\mathscr{F}))_{\mathrm{dbl}} \in Sh(N \times \{s\})$ for the image of $i_s^{-1}\Phi_{L_H}(\pi^{-1}(\mathscr{F}))$ under the antimicrolocalization functor. By Lemma 7.1.5 again we also have

$$\begin{aligned} i_s^{-1}\Phi_{L_H}(\pi^{-1}(\mathscr{F}))_{\mathrm{dbl}} &\xrightarrow{\sim} i_s^{-1} i_{\overline{Z}}^{-1} \overline{j}_{\overline{Z},*}\big(\Psi^{\zeta}_{\overline{Z}}(\Psi^0_H(\mathscr{F}))\big)_{\mathrm{dbl}} \\ &\xrightarrow{\sim} i_Z^{-1} j_{Z,*}\big(\Psi^{\zeta}_{Z}(i_s^{-1}\Psi^0_H(\mathscr{F}))\big)_{\mathrm{dbl}} \xrightarrow{\sim} \Phi_{\varphi^s_H(L)}(\mathscr{F})_{\mathrm{dbl}}. \end{aligned}$$

Therefore the proof is completed. □

Proof of Theorem 7.2.7. For $\mathscr{F} \in \mu Sh_{\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L}}(\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R} \cup \widetilde{L})$, we consider

$$\pi^{-1}(\mathscr{F}) \in \mu Sh_{((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup (\widetilde{L} \times I)}\big(((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup (\widetilde{L} \times I)\big).$$

On the other hand, for the Hamiltonian isotopy φ_H^s we consider by Theorem 3.3.2

$$\Psi_H^0(\mathscr{F}) \in \mu Sh_{((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup \widetilde{L}_H}\big(((\mathfrak{c}_X \cup \Lambda_- \times \mathbb{R}) \times I) \cup \widetilde{L}_H\big).$$

There is a natural morphism $\pi^{-1}(\mathscr{F}) \to \Psi_H^0(\mathscr{F})$, and thus a natural morphism

$$\Phi_{L\times I}\big(\pi^{-1}(\mathscr{F})\big) \to \Phi_{L_H}\big(\Psi_H^0(\mathscr{F})\big).$$

We will show that this is a natural quasi-isomorphism. In fact,

$$\mathrm{Cone}\left(\Phi_{L\times I}\big(\pi^{-1}(\mathscr{F})\big) \to \Phi_{L_H}\big(\Psi_H^0(\mathscr{F})\big)\right) \in \mu Sh_{(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}) \times I}\big((\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}) \times I\big).$$

By Lemma 7.2.8, we also know that when $s = 0$,

$$\begin{aligned} i_0^{-1}\mathrm{Cone}&\left(\Phi_{L\times I}\big(\pi^{-1}(\mathscr{F})\big) \to \Phi_{L_H}\big(\Psi_H^0(\mathscr{F})\big)\right) \\ &\simeq \mathrm{Cone}\left(i_0^{-1}\Phi_{L\times I}\big(\pi^{-1}(\mathscr{F})\big) \to i_0^{-1}\Phi_{L_H}\big(\Psi_H^0(\mathscr{F})\big)\right) \\ &\simeq \mathrm{Cone}\left(\Phi_L\big(i_0^{-1}\pi^{-1}(\mathscr{F})\big) \to \Phi_L\big(i_0^{-1}\Psi_H^0(\mathscr{F})\big)\right) \simeq 0. \end{aligned}$$

As by Theorem 3.3.2, $i_0^{-1} : \mu Sh_{(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}) \times I}\big((\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}) \times I\big) \to \mu Sh_{\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R}}(\mathfrak{c}_X \cup \Lambda_+ \times \mathbb{R})$ defines an equivalence, we can conclude that the mapping cone is identically zero, and thus

$$\Phi_{L\times I}\big(\pi^{-1}(\mathscr{F})\big) \xrightarrow{\sim} \Phi_{L_H}\big(\Psi_H^0(\mathscr{F})\big).$$

Therefore by restricting to $s = 1$ and applying Lemma 7.2.8 again the proof is completed. $\square$

7.2.4. Comparison with the Isotopy Functor

When there is a Legendrian isotopy $\varphi_H^s\,(s \in I)$ from Λ_0 to Λ_1, it will define a Lagrangian cobordism L from Λ_0 to Λ_1 [**27**] or [**68**, Section 4.2.3]. Hence we have a fully faithful Lagrangian cobordism functor

$$\Phi_L : Sh_{\Lambda_0}(M) \hookrightarrow Sh_{\Lambda_1}(M).$$

On the other hand, Guillermou-Kashiwara-Schapira [**88**] constructed a sheaf quantization functor Ψ_H from a Hamiltonian isotopy given by taking convolution with an integral kernel. We will prove the following comparison theorem.

Theorem 7.2.9. *Let $\Lambda_s \subset T^{*,\infty}M\,(s \in I)$ be a Legendrian isotopy induced by $\varphi_H^s\,(s \in I)$, with vanishing Maslov class, and L the Lagrangian cobordism from Λ_0 to Λ_1 coming from the isotopy. Then for Φ_L the Lagrangian cobordism functor and Ψ_H the sheaf quantization functor,*

$$\Phi_L \simeq \Psi_H : Sh_{\Lambda_0}(M) \to Sh_{\Lambda_1}(M).$$

In this section we show that when the Lagrangian cobordism L from Λ_- to Λ_+ is induced by a Hamiltonian isotopy in Theorem 3.3.1 [**88**], i.e. $\Lambda_- = \Lambda$ and $\Lambda_+ = \varphi_H^1(\Lambda)$, then our Lagrangian cobordism functor agrees with the sheaf quantization functor given by the Hamiltonian isotopy[2].

[2]The author is very grateful to Vivek Shende, who essentially explains to the author the strategy of the proof that appears here.

Our strategy is similar to the proof of Theorem 7.2.7 (Hamiltonian invariance), that is, to realize the Lagrangian cobordism as a functor

$$Sh^b_{\Lambda\times I}(M\times I)\to Sh^b_{\Lambda_H}(M\times I)$$

where Λ_H is the Legendrian movie of Λ under the Hamiltonian flow φ_H^s $(s\in I)$. Then Theorem 3.3.1 [**88**] will enable us to conclude that $\Phi_L\simeq\Psi_H$ at $M\times\{1\}$ from the initial condition at $M\times\{0\}$.

First, recall the construction of Lagrangian cobordisms induced by a Hamiltonian isotopy [**27**]. Suppose the contact Hamiltonian is $H: T^{*,\infty}M\to\mathbb{R}$. Then consider the homogeneous symplectic Hamiltonian to be $\widehat{H}(x,\xi)=|\xi|H(x,\xi/|\xi|): T^*M\to\mathbb{R}$. Let $\eta:[0,+\infty)\to[0,1]$ be a cut-off function such that $\eta(r)=0$ when r is small, and $\eta(r)=1$ when r is large. Then the Lagrangian cobordism induced by H is

$$L=\varphi^1_{\eta(|\xi|)\widehat{H}(x,\xi)}(\Lambda\times\mathbb{R}_{>0}).$$

One can see that L coincides with $\Lambda\times\mathbb{R}_{>0}$ when $|\xi|$ is small, and coincides with $\varphi_H^1(\Lambda)\times\mathbb{R}_{>0}$ when $|\xi|$ is large.

Now we try to construct a Lagrangian cobordism $\overline{L}$ from $\Lambda\times I$ to Λ_H, so that the restriction to $T^*M\times\{0\}$ is just $\Lambda\times\mathbb{R}_{>0}$, and the restriction to $T^*M\times\{1\}$ is L. Let

$$\overline{\varphi}^t_{\overline{H}}: T^{*,\infty}(M\times I)\to T^{*,\infty}(M\times I);\ (x,\xi,s,\sigma)\mapsto(\varphi_H^{st}(x,\xi),s,\sigma-sH\circ\varphi_H^{st}(x,\xi)).$$

Then the Lagrangian cobordism $\overline{L}$ induced by $\overline{\varphi}^t_{\overline{H}}\,(t \in I)$ will satisfy our conditions.

Recall that to define the Lagrangian cobordism functor, we need to consider a proper embedding $e : T^*M \hookrightarrow T^{*,\infty}(M \times \mathbb{R})$. For example, consider a Riemannian metric g, let φ_g^t be the geodesic flow, and define

$$e(x,\xi) = (\varphi_g^{-1}(x,\xi), |\xi|_g^2/2, 1).$$

Then we are going to work with the (singular) Legendrians $(M \cup \Lambda \times \mathbb{R}_{>0})_\epsilon^\prec$ and $(M \cup \varphi_H^1(\Lambda) \times \mathbb{R}_{>0})_\epsilon^\prec \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$.

Let $\mathscr{F} \in \mu Sh^b_{M\cup\Lambda\times\mathbb{R}_{>0}}(M \cup \Lambda \times \mathbb{R}_{>0})$. Let $\varphi^s_{\eta\widehat{H}}$ be the Hamiltonian flow on T^*M that extends to $T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$. Then by Theorem 3.3.2, there is a canonical sheaf $\Psi_{\eta\widehat{H}}(\mathscr{F}) \in \mu Sh^b_{M\cup\varphi^1_{\eta\widehat{H}}(\Lambda\times\mathbb{R}_{>0})}(M \cup \varphi^1_{\eta\widehat{H}}(\Lambda \times \mathbb{R}_{>0}))$ whose restriction to $M \cup \Lambda \times \mathbb{R}_{>0}$ is $\mathscr{F}$, this means $\Psi_{\eta\widehat{H}}(\mathscr{F})$ is the unique lifting of $\mathscr{F}$ under the (restriction) functor

$$\begin{aligned}\mu Sh_{M\cup\varphi^1_{\eta\widehat{H}}(\Lambda\times\mathbb{R}_{>0})} &\xrightarrow{\sim} \mu Sh_{M\cup\Lambda\times\mathbb{R}_{>0}}(M \cup \Lambda \times \mathbb{R}_{>0}) \times_{Loc(\Lambda)} Loc(L) \\ &\xrightarrow{\sim} \mu Sh_{M\cup\Lambda\times\mathbb{R}_{>0}}(M \cup \Lambda \times \mathbb{R}_{>0}).\end{aligned}$$

In other words, by abusing notations, we can write

$$\Phi_L(\mathscr{F}) = \Phi_L(\Psi_{\eta\widehat{H}}(\mathscr{F})).$$

Lemma 7.2.10. *Let $\overline{L}$ be the Lagrangian cobordism from $\Lambda \times I$ to Λ_H induced by $\overline{\varphi}^s_{\overline{H}}$, $i_s : T^{*,\infty}(M \times \mathbb{R}) \times \{s\} \hookrightarrow T^{*,\infty}(M \times \mathbb{R} \times I)$ be the inclusion and $\pi : T^{*,\infty}(M \times \mathbb{R} \times$*

$I) \to T^{*,\infty}(M\times\mathbb{R})$ be the projection. Then for any $\mathscr{F} \in \mu Sh_{M\cup\Lambda\times\mathbb{R}_{>0}}(M\cup\Lambda\times\mathbb{R}_{>0})$,

$$i_s^{-1}\Phi_{\overline{L}}(\pi^{-1}(\mathscr{F})) = \Phi_{L_s}(\mathscr{F}),$$

where $L_s = \varphi^1_{s\eta\widehat{H}}(\Lambda\times\mathbb{R}_{>0})$ is the Lagrangian cobordism induced by φ^t_{sH}.

Proof. First of all, $\varphi^z_{\overline{Z}}$ be the Liouville flow on $T^{*,\infty}(M\times\mathbb{R}\times I)$ defined by $(\varphi^z_Z, \mathrm{id})$. Let $\Psi^\zeta_{\overline{Z}}$ be the equivalence functor defined by the Liouville flow $\varphi^z_{\overline{Z}}(z \in (-\infty,0])$ or $\phi^\zeta_{\overline{Z}}(\zeta\in(0,1])$ on $T^{*,\infty}(M\times\mathbb{R}\times I)$, and

$$M\times\mathbb{R}\times\{0\} \xrightarrow{i_Z} M\times\mathbb{R}\times[0,1] \xleftarrow{j_Z} M\times\mathbb{R}\times(0,1],$$

$$M\times\mathbb{R}\times I\times\{0\} \xrightarrow{\bar{i}_{\overline{Z}}} M\times\mathbb{R}\times I\times[0,1] \xleftarrow{\bar{j}_{\overline{Z}}} M\times\mathbb{R}\times I\times(0,1].$$

Write $(\Psi^\zeta_{\overline{Z}}(\Psi^0_{\eta\widehat{H}}(\mathscr{F})))_{\mathrm{dbl}} \in Sh(M\times\mathbb{R}\times I)$ for the image of $\Psi^\zeta_{\overline{Z}}(\Psi^0_{\eta\widehat{H}}(\mathscr{F}))$ under the antimicrolocalization functor in Theorem 7.1.2 [**124**]. Then by Remark 3.3.3,

$$i_s^{-1}\big(\Psi^\zeta_{\overline{Z}}(\Psi^0_{\eta\widehat{H}}(\mathscr{F}))\big)_{\mathrm{dbl}} = \Psi^\zeta_{\overline{Z}}(i_s^{-1}\Psi^0_{\eta\widehat{H}}(\mathscr{F}))_{\mathrm{dbl}}.$$

We can write down the Lagrangian cobordism functor as a series of compositions

$$\Phi_{\overline{L}}(\pi^{-1}(\mathscr{F}))_{\mathrm{dbl}} = \bar{i}_{\overline{Z}}^{-1}\bar{j}_{\overline{Z},*}\big(\Psi^\zeta_{\overline{Z}}\Psi^0_{\eta H}(\pi^{-1}(\mathscr{F}))\big)_{\mathrm{dbl}}.$$

Note that $\Psi^{\zeta}_{\overline{Z}}$ is the equivalence functor defined by the Liouville flow on $T^{*,\infty}(M \times \mathbb{R} \times I)$. Then by Lemma 7.1.5 there is a natural morphism

$$\begin{aligned} i_s^{-1}\Phi_{\overline{L}}(\pi^{-1}(\mathscr{F}))_{\mathrm{dbl}} &\xrightarrow{\sim} i_s^{-1}\overline{i}_{\overline{Z}}^{-1}\overline{j}_{\overline{Z},*}\big(\Psi^{\zeta}_{\overline{Z}}\Psi^{0}_{\eta\widehat{H}}(\pi^{-1}(\mathscr{F}))\big)_{\mathrm{dbl}} \\ &\xrightarrow{\sim} i_Z^{-1}j_{Z,*}\Psi^{\zeta}_{Z}\big(i_s^{-1}\Psi^{0}_{\eta\widehat{H}}(\pi^{-1}(\mathscr{F}))\big)_{\mathrm{dbl}} \\ &\xrightarrow{\sim} i_Z^{-1}j_{Z,*}\big(\Psi^{\zeta}_{Z}\Psi^{0}_{s\eta\widehat{H}}(\mathscr{F})\big)_{\mathrm{dbl}} \xrightarrow{\sim} \Phi_{L_s}(\mathscr{F})_{\mathrm{dbl}}. \end{aligned}$$

and thus we complete the proof. □

Proof of Theorem 7.2.9. Consider the Lagrangian cobordism $\overline{L}$ induced by $\overline{\varphi}^t_{\overline{H}}$. By Lemma 7.2.10, we know that for $i_0 : T^{*,\infty}(M\times\mathbb{R})\times\{0\} \hookrightarrow T^{*,\infty}(M\times\mathbb{R}\times I)$ and $\pi : T^{*,\infty}(M \times \mathbb{R} \times I) \to T^{*,\infty}(M \times \mathbb{R})$,

$$i_0^{-1}\Phi_{\overline{L}}(\mathscr{F}) = \Phi_{\Lambda\times\mathbb{R}_{>0}}(\mathscr{F}) = \mathscr{F}.$$

By Theorem 3.3.2 and Remark 3.3.3, $i_0^{-1} : \mu Sh_{((M\cup\Lambda\times\mathbb{R}_{>0})\times I)\cup\overline{L}}(((M\cup\Lambda\times\mathbb{R}_{>0})\times I)\cup \overline{L}) \to \mu Sh_{M\cup\Lambda\times\mathbb{R}_{>0}}(M\cup\Lambda\times\mathbb{R}_{>0})$ is an equivalence and its inverse is the Hamiltonian isotopy functor Ψ^0_H in Theorem 3.3.1 [**88**]. Therefore

$$\Phi_{\overline{L}}(\pi^{-1}(\mathscr{F})) = \Psi^0_H(\mathscr{F}).$$

Finally, by restricting to $M \times \{1\}$ and apply Lemma 7.2.10 again, we can conclude that $\Phi_L(\mathscr{F}) = \Psi_H(\mathscr{F})$. □

7.2.5. Comparison with the Filling Functor

When $\Lambda_- = \emptyset$, a Lagrangian cobordism from Λ_- to Λ_+ is a Lagrangian filling. Jin-Treumann [**94**] constructed a sheaf quantization functor $Loc(L) \to Sh_{\Lambda_+}(M)$ from any Lagrangian filling L of Λ_+, that is, a fully faithful embedding

$$\Psi_L^{\mathrm{JT}} : Loc(L) \hookrightarrow Sh_{\Lambda_+}(M),$$

as we have explained in Section 4.4. We will show the following comparison result.

Proposition 7.2.11. *Let $U \subset M$ be an open subset with subanalytic boundary, $\Lambda_+ = \nu_{U,-}^{*,\infty} M$ be the inward unit conormal and L the standard Lagrangian brane associated to U with Legendrian boundary Λ_+. Then for Φ_L the Lagrangian cobordism functor and Ψ_L^{JT} the Jin-Treumann sheaf quantization functor,*

$$\Phi_L \simeq \Psi_L^{JT} : Loc(L) \hookrightarrow Sh_{\Lambda_+}(M).$$

In fact, using Nadler-Zaslow correspondence [**119, 126**] or Viterbo's sheaf quantization construction [**159**], if one can prove additionally the functoriality of Φ_L and Ψ_L^{JT} as functors from infinitesimal Fukaya categories, then $\Phi_L \simeq \Psi_L^{\mathrm{JT}}$ for any Lagrangian filling of any Legendrians Λ_+.

When $\Lambda_- = \emptyset$, L is a Lagrangian filling of Λ_+. In this section we basically show that for costandard Lagrangian branes, our fully faithful functor

$$\Phi_L : Loc(L) \hookrightarrow Sh_{\Lambda_+}(M)$$

coincides with the functor Jin-Treumann constructed [**94**]. Again, the reader may skip this section.

Fix an embedding $e : T^*M \hookrightarrow T^{*,\infty}(M\times\mathbb{R})$. For example, consider a Riemannian metric g, let φ_g^t be the geodesic flow, and define

$$e(x,\xi) = (\varphi_g^{-1}(x,\xi), |\xi|_g^2/2, 1).$$

Then $M \cup \Lambda \times \mathbb{R}_{>0} \subset T^{*,\infty}(M \times \mathbb{R})$ is a (singular) Legendrian.

Let $U \subset M$ be an open subset with subanalytic boundary ∂U. The outward conormal of U is denoted by $\nu^*_{U,+}M$. Then the Lagrangian skeleton $M \cup \nu^*_{U,+}M$ is shown in Figure 7.4.

Definition 7.2.1. *Let $m_U : \overline{U} \to [0, +\infty)$ be the defining function of ∂U such that $m_U^{-1}(0) = \partial U$. Let $f_U = -\ln(m_U)$. Then the graph of the exact 1-form $L = L_U = L_{df_U} \subset T^*M$ is the costandard Lagrangian brane associated to U.*

When L intersect the ideal contact boundary [**81**] of T^*M at $\nu^{*,\infty}_{U,+}M$ such that it is tangent to $\nu^*_{U,+}M$ to infinite order (for example, when 0 is a regular value of m_U), it can be viewed as a Lagrangian filling of $\nu^{*,\infty}_{U,+}M$, equipped with a different primitive f'_U that is bounded on $L = L_U$. Via the embedding e, its image $\widetilde{L}$ will be a Legendrian in $T^{*,\infty}(M \times \mathbb{R})$ that coincides with $\nu^*_{U,+}M$ at infinity.

Proof of Proposition 7.2.11. Consider a complex of local systems $\mathscr{F}_L$ on L with stalk F. Note that the projection $\pi_L : L \hookrightarrow T^*M \to M$ defines a diffeomorphism $L \cong U$. Write $\mathscr{F}_U = \pi_{L,*}\mathscr{F}_L$. We will show that both functors send $\mathscr{F}_L$ to $\mathscr{F}_U$.

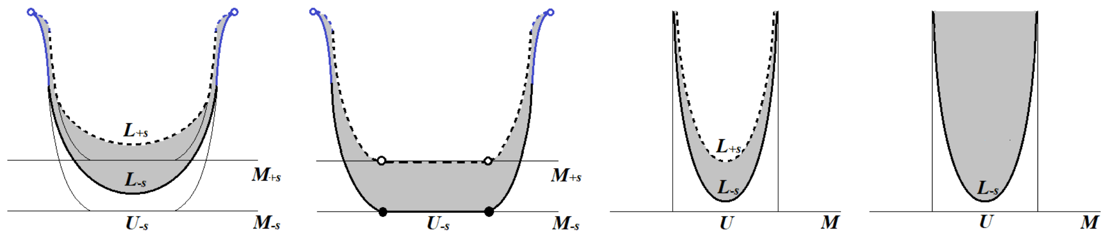

Figure 7.4. The Nadler-Shende construction (left) and the Jin-Treumann construction (right). The grey regions are the supports of the corresponding sheaves. The thin lines on the left are the skeleton $M \cup \nu^*_{U,+}M$ embedded in $J^1(M)$, and the thick lines there are the two copies of Lagrangian fillings. The blue lines are the family of cusps $\partial\Lambda \times \prec$.

(1) We first compute $\Phi_L : Loc(L) \to Sh(M)$. Let the vertical vector field $\partial/\partial t$ be the Reeb vector field. Consider the skeleton $M \cup \nu^*_{U,-}M$ and its positive/negative Reeb pushoff $(M \cup \nu^*_{U,-}M)_{\pm\epsilon}$. Lift L to a Legendrian $\widetilde{L}$ that coincides with $M \cup \nu^*_{U,-}M$ when the radius coordinate $r = \ln|\xi|$ in T^*M is sufficiently large. When r is large, we cut off the Legendrian $(\nu^{*,\infty}_{U,-}M)_{\pm\epsilon}$ and connect them by a family of cusps $\nu^{*,\infty}_{U,-}M \times \prec$, and also cut off $\tilde{L}_{\pm\epsilon}$ and connect them by a family of cusps $\nu^{*,\infty}_{U,-}M \times \prec$. We consider

$$Loc(L) \xrightarrow{\sim} \mu Sh_{\widetilde{L}}(\widetilde{L}) \hookrightarrow Sh_{(\widetilde{L},\partial\widetilde{L})^{\prec}_{\epsilon}}(M \times \mathbb{R})_0.$$

Here the subscript 0 means the subcategory of sheaves with 0 stalk outside a compact set. Hence there is a sheaf $\mathscr{F}_{\mathrm{dbl}}$ with singular support in $(\widetilde{L}, \partial\widetilde{L})^{\prec}_{\epsilon}$ whose microlocalization along $\widetilde{L}_{-\epsilon}$ is given by $\mathscr{F}_L$, given by the antimicrolocalization functor Theorem 7.1.2 [**124**].

Running the Liouville flow φ_Z^z for $z \in (-\infty, 0]$ or ϕ_Z^ζ for $\zeta \in (0,1]$, we can get a sheaf on $M \times \mathbb{R} \times (0,1]$. Note that the end $(\nu_{U,-}^{*,\infty} M)_{\pm\epsilon}$ (which coincides with $\partial \widetilde{L}_{\pm\epsilon}$) is preserved by Liouville flow up to a Reeb translation (due to change of the primitive f'_U of L_U), and the limit points

$$\lim_{z \to -\infty} \varphi_Z^z(\widetilde{L}, \partial\widetilde{L})_\epsilon^\prec = \lim_{\zeta \to 0} \phi_Z^\zeta(\widetilde{L}, \partial\widetilde{L})_\epsilon^\prec \subset T_{\tau>0}^{*,\infty}(M \times \mathbb{R}) \times \{0\}$$

are exactly $(U \cup \nu_{U,-}^* M)_\epsilon^\prec$. The resulting sheaf is therefore in $Sh_{(\overline{U} \cup \nu_{U,-}^* M)_\epsilon^\prec}(M \times \mathbb{R})$.

Now we apply the microlocalization functor

$$Sh_{(\overline{U} \cup \nu_{U,-}^* M)_\epsilon^\prec}(M \times \mathbb{R})_0 \to \mu Sh_{(\overline{U} \cup \nu_{U,-}^* M)_{-\epsilon}}((\overline{U} \cup \nu_{U,-}^* M)_{-\epsilon}) \xrightarrow{\sim} Sh_{\nu_{U,-}^* M}(M)_0.$$

The microstalks for points in $\overline{U}_{-\epsilon}$ are F, and those for points in $M_{-\epsilon} \backslash \overline{U}_{-\epsilon}$ are 0. The microlocal monodromy along U is determined by $\mathscr{F}_U = \pi_{L,*}\mathscr{F}_L$ because topologically taking the limit $\lim_{z\to-\infty} \varphi_Z^z(L)$ under the Liouville flow gives a homotopy equivalence $L \simeq \lim_{z\to-\infty} \varphi_Z^z(L) \simeq U \cup \nu_{U,-}^* M \simeq U$ that is homotopic to the projection $\pi_L : L \xrightarrow{\sim} U$.

(2) Then we consider $\Psi_L^{\mathrm{JT}} : Loc(L) \to Sh(M)$. In [**94**] they considered the Legendrian lift $\widetilde{L}$ of L whose front projection onto $M \times \mathbb{R}$ is the graph of the function f_U. Then consider the positive/negative Reeb pushoff $\widetilde{L}_{\pm\epsilon}$, which are the graphs of the functions $f_U \pm \epsilon$. We have [**94**]

$$Loc(L) \xrightarrow{\sim} \mu Sh_{\widetilde{L}}(\widetilde{L}) \hookrightarrow Sh_{\widetilde{L}_{\pm\epsilon}}(M \times \mathbb{R})_0.$$

Then there is a sheaf $\mathscr{F}'_{\mathrm{dbl}}$ with singular support in $\widetilde{L}_{-\epsilon} \cup \widetilde{L}_{\epsilon}$, given by the antimicrolocalization functor [**94**, Section 3.15], whose microlocalization along $\widetilde{L}_{-\epsilon}$ gives the local system $\mathscr{F}_L$. Write $D_{\pm\epsilon} = \{(x,t)|t = f_U(x) \pm \epsilon\}$. Indeed the sheaf is supported in the region

$$D_{[-\epsilon,\epsilon)} = \{(x,t)|f_U(x) - \epsilon \leq t < f_U(x) + \epsilon\}$$

with stalk F. This is because the sheaf has zero stalk below $D_{-\epsilon} = \{(x,t)|t = f_U(x) - \epsilon\}$ and hence for sufficiently small $\epsilon' > \epsilon$ (as in Example 3.2.4)

$$\mathscr{F}_L = m_{\widetilde{L}_{-\epsilon}}(\mathscr{F}'_{\mathrm{dbl}}) \simeq \mathrm{Tot}(\mathscr{F}'_{\mathrm{dbl}}|_{D_{-\epsilon}} \to \mathscr{F}'_{\mathrm{dbl}}|_{D_{-\epsilon'}}) \simeq \mathscr{F}'_{\mathrm{dbl}}|_{D_{-\epsilon}}.$$

In addition, the monodromy of the local system in the region $D_{[-\epsilon,\epsilon)}$ bounded by $\pi(\widetilde{L}_{-\epsilon})$ and $\pi(\widetilde{L}_{\epsilon})$ is also determined by $\mathscr{F}_L$, since for $\pi_M : M \times \mathbb{R} \to M$,

$$\mathscr{F}'_{\mathrm{dbl}}|_{D_{[-\epsilon,\epsilon)}} = \pi_M^{-1}(\mathscr{F}'_{\mathrm{dbl}}|_{D_{-\epsilon}})|_{D_{[-\epsilon,\epsilon)}}.$$

Now we consider a Legendrian isotopy to move $\widetilde{L}_{\epsilon}$ along the positive Reeb direction. Jin-Treumann showed that [**94**, Section 3.18] for $S > T > 0$ sufficiently large we have

$$Sh_{\widetilde{L}_{-\epsilon} \cup \widetilde{L}_{\epsilon+S}}(M \times \mathbb{R}) \xrightarrow{j^{-1}_{M\times(-\infty,T)}} Sh_{\widetilde{L}_{-\epsilon}}(M \times (-\infty, T)) \xleftarrow{\sim} Sh_{\widetilde{L}_{-\epsilon}}(M \times \mathbb{R}),$$

and hence one can get a sheaf $\mathscr{F}'$ in $Sh_{\widetilde{L}_{-\epsilon}}(M \times \mathbb{R})$ with stalk F supported in the region $D_{[-\epsilon,+\infty)} = \{(x,t)|t \geq f_U(x) - \epsilon\}$ above $D_{-\epsilon}$, and the monodromy in this region determined by $\mathscr{F}_L$ since

$$\mathscr{F}'|_{D_{[-\epsilon,+\infty)}} = \pi_M^{-1}(\mathscr{F}'|_{D_{-\epsilon}})|_{D_{[-\epsilon,+\infty)}}.$$

Finally we push forward the sheaf to $Sh_{\nu^*_{U,-}M}(M)_0$ via the projection $\pi_M : M \times \mathbb{R} \to M$. Note that in the fiber of the projection $\{x\} \times \mathbb{R}$ $(x \in U)$, the sheaf is $F_{r \geq f_U(x)}$, and $\pi_{x,*}(F_{r\geq f_U(x)}) = F$. Hence the projection will give a sheaf supported on U with stalk F. In addition we claim that the monodromy defines the local system $\mathscr{F}_U = \pi_{L,*}\mathscr{F}_L$ on $\overline{U}$ because the projection of L onto M via $\widetilde{L} \hookrightarrow T^{*,\infty}_{\tau>0}(M \times \mathbb{R}) \to M \times \mathbb{R} \to M$ coincides with the projection $\pi_L : L \hookrightarrow T^*M \to M$ which gives the diffeomorphism $L \cong U$.

Hence $\Phi_L \simeq \Psi_L^{\mathrm{JT}} : Loc(L) \to Sh(M)$ when $L = L_U$ is a standard Lagrangian brane corresponding to the open subset $U \subset M$. $\square$

7.3. Examples and Applications in Lagrangian Cobordisms

We now focus on some concrete examples of Legendrian submanifolds and Lagrangian cobordisms and explain what the Lagrangian cobordism functor on sheaves will tell us. In particular, we will prove Theorem 7.0.10.

7.3.1. Examples of cobordism functors

We consider the elementary Lagrangian cobordism given by attaching a Lagrangian k-handle $(0 \leq k \leq n)$. The local model of the front projection in $\mathbb{R}^{n+1}$ is shown in Figure 7.5 and 7.6.

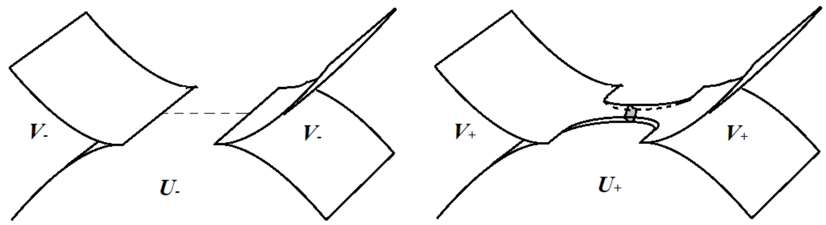

Figure 7.5. On the left is the front projection of Λ_-, and on the right is the front projection of Λ_+ after attaching a Lagrangian 1-handle connecting the two cusps along the dashed line, where in the middle of the tube (the grey slice) there is a unique Reeb chord.

The front projection of $\Lambda_\pm$ gives a stratification on $\mathbb{R}^{n+1}$, such that on each stratum the sheaf is locally constant. Hence we are able to get a combinatoric model given by stalks on each stratum and the transition maps given by the microlocal Morse lemma as in Example 3.1.5 and 3.2.4. We will explain how objects behave under the cobordism functor.

For the stratification given by $\Lambda_\pm$, denote by $V_\pm \subset \mathbb{R}^{n+1}$ the domain whose x_{n+1}-coordinate is bounded by the front projection of $\Lambda_\pm$ and $U_\pm \subset \mathbb{R}^{n+1}$ the domain whose x_{n+1}-coordinate is unbounded on each vertical slice $\{(x_1, \dots, x_n)\} \times \mathbb{R}$ (see Figure 7.5 and 7.6). For a sheaf in $Sh^b_{\Lambda_-}(\mathbb{R}^{n+1})$, suppose the stalk in the region V_- is B and the stalk in U_- is A (Figure 7.7). Suppose the microstalk of $\mathscr{F}$ is

$$F \simeq \mathrm{Tot}(A \to B).$$

Instead of doing concrete computations, we will describe objects under the Lagrangian cobordism functor in three steps by only using a few properties of our functor:

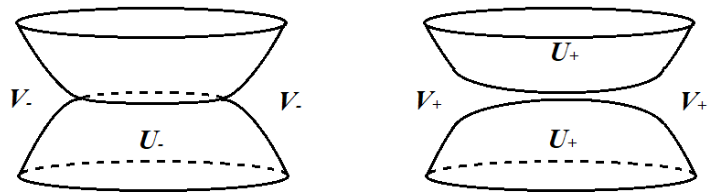

Figure 7.6. On the left is the front projection of Λ_-, and on the right is the front projection of Λ_+ after attaching a Lagrangian 2-handle connecting the S^1-family of cusps along the disk.

(1) determine the stalks in U_+ and V_+ using the fact that the cobordism functor is identity outside a compact set in $\mathbb{R}^{n+1}$ and hence the stalks are preserved;

(2) determine the microlocalization along Λ_+ (relative to boundary), which is a local system with stalk F, using the fact that the Liouville flow fixes the end Λ_+ and hence the cobordism functor preserves the microlocalization;

(3) determine the local system in V_+ using the fact that $B \simeq A \oplus F$, and hence the local system with stalk F on Λ_+ determines a local system with stalk F on V_+ and a local system with stalk B on V_+ (relative to boundary at infinity in $\mathbb{R}^{n+1}$).

The information above will uniquely determine the sheaf.

Before starting to determine the sheaf $\mathscr{F}^+ \in Sh^b_{\Lambda_+}(\mathbb{R}^{n+1})$, we first note that $\mathscr{F}^- \in Sh^b_{\Lambda_-}(\mathbb{R}^{n+1})$ has an image in $Sh^b_{\Lambda_+}(\mathbb{R}^{n+1})$ via the cobordism functor iff it can be lifted into

$$Sh^b_{\Lambda_-}(\mathbb{R}^{n+1}) \times_{Loc^b(\Lambda_-)} Loc^b(L).$$

Since $\Lambda_- \cong S^{k-1} \times D^{n-k+1}$ while $L \cong D^k \times D^{n-k+1}$, this is the same as saying that the microlocalization $m_{\Lambda_-}(\mathscr{F}^-)$ can be trivialized over S^{k-1}.

Remark 7.3.1. *Note that not all complexes of local systems in* $Loc^b(S^k)$ $(k \geq 2)$ *are trivial. For example for the Hopf fibration* $\pi : S^3 \to S^2$, $\pi_* \Bbbk_{S^3}$ *is a nontrivial complex of local system on* S^2*. The reason is that although* $H^1(S^k) = 0$, $H^k(S^k) \neq 0$ *and that will give extension classes in* $Ext^1(\Bbbk_{S^k}[1-k], \Bbbk_{S^k})$*.*

Here is how the sheaf $\mathscr{F}^+$ is determined. (1) Firstly, note that away from the cusps, the Lagrangian cobordism is just a standard embedded cylinder $\Lambda_0 \times \mathbb{R}$, and hence is fixed by the Liouville flow. The functor

$$\mu Sh^b_{\mathbb{R}^{n+1} \cup \Lambda_0 \times \mathbb{R} \cup L}(\mathbb{R}^{n+1} \cup \Lambda_0 \times \mathbb{R} \cup L) \to \mu Sh^b_{\mathbb{R}^{n+1} \cup \Lambda_0 \times \mathbb{R}}(\mathbb{R}^{n+1} \cup \Lambda_0 \times \mathbb{R})$$

is the identity. This shows that the sheaf should remain the same away from compact subsets in $\mathbb{R}^{n+1}$. Then one can see explicitly that the stalks of $\mathscr{F}^+$ are determined by $\mathscr{F}^-$, where the stalk in the region V_+ must be B and the stalk in U_+ must be A.

(2) Secondly, note that the complex of local systems $m_{\Lambda_-}(\mathscr{F}^-)$ on Λ_- has stalk F. After gluing with a local system $\mathscr{L}_L$ on L, by restriction we can determine a complex of local systems on $\Lambda_+ \times \{+\infty\}$. Note that the restriction of the local system along $\partial L = \partial \Lambda_\pm \times \mathbb{R}$ is determined by the microlocalization on $\partial \Lambda_-$.

Since $\Lambda_+ \times \{+\infty\}$ is preserved by the negative time Liouville flow φ_Z^z up to a Reeb translation (due to the change of the primitive f_L of L), the functor

$$\begin{aligned} &\mu Sh^b_{M \cup \Lambda_- \times \mathbb{R}_{>0} \cup L}(M \cup \Lambda_- \times \mathbb{R}_{>0} \cup L) \\ &\quad \to \mu Sh^b_{\lim_{z \to -\infty} \varphi_Z^z(M \cup \Lambda_- \times \mathbb{R}_{>0} \cup L)}(\lim{}_{z \to -\infty} \varphi_Z^z(M \cup \Lambda_- \times \mathbb{R}_{>0} \cup L)) \\ &\quad \to \mu Sh^b_{M \cup \Lambda_+ \times \mathbb{R}_{>0}}(M \cup \Lambda_+ \times \mathbb{R}_{>0}) \end{aligned}$$

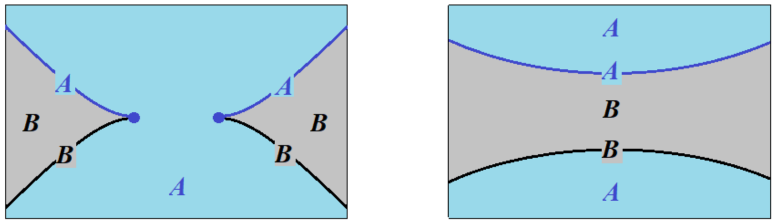

Figure 7.7. The microlocal sheaf on $Sh^b_{\Lambda_-}(\mathbb{R}^{n+1})$ (left) and $Sh^b_{\Lambda_+}(\mathbb{R}^{n+1})$ (right) before and after the Lagrangian 1-handle attachment. Here we assume $\Lambda_\pm \subset T^{*,\infty}_{\tau>0}(\mathbb{R}^n \times \mathbb{R}_\tau)$.

is an equivalence on $\Lambda_+ \times \{+\infty\}$ induced by the Reeb translation (Theorem 3.3.2). Hence the complex of local systems on $\Lambda_+ \times \{+\infty\}$ is preserved by the nearby cycle functor. Therefore, after applying Φ_L, the microstalk on Λ_+ is still F, where the microlocal monodromy is still the same as the restriction of the local system $\mathscr{L}_L$ onto Λ_+.

Note that the restriction of the local system to boundary $\mathscr{L}_L|_{\partial\Lambda_\pm\times\mathbb{R}}$ is the pull back of the given local system $m_{\partial\Lambda_-}(\mathscr{F})$. Therefore, after applying the cobordism functor we get the microlocalization in the fiber of $Loc^b(\Lambda_+) \to Loc^b(\partial\Lambda_+)$ at the point $m_{\partial\Lambda_-}(\mathscr{F})$.

(3) Finally, we determine the local system in the region V_+. Note that V_+ is not contractible relative to boundary at infinity $\partial V_+ = S^{k-1} \times D^{n-k+1}$. In particular, globally there could be nontrivial monodromy coming from our choice of the local monodromy relative to boundary, parametrized by the fiber of $Loc^b(V_+) \to Loc^b(\partial V_+)$. Because there are transition maps

$$A \to B \to A$$

whose composition is a quasi-isomorphism. Without loss of generality, we assume that it is the identity [**148**, Corollary 3.18]. Then there is a splitting of chain complexes

$$B \simeq A \oplus \mathrm{Tot}(A \to B) \simeq A \oplus F.$$

Therefore since the microlocal monodromy along Λ_+ has been determined by the local system on L we chose, so is the monodromy of the sheaf in V_+ if we identify $\mathscr{F}^+|_{V_+}$ with $A_{V_+} \oplus \mathscr{L}|_{V_+}$, where A_{V_+} is just the constant local system and $\mathscr{L}_{V_+}$ is a local system on V_+ with stalk F that extends $\mathscr{L}_L|_{\Lambda_+}$.

In fact topologically $(V_+, \partial V_+) \simeq (L, \Lambda_-) \simeq (D^k \times D^{n-k+1}, S^{k-1} \times D^{n-k+1})$ by considering the projection map $L \hookrightarrow \mathbb{R}^{2n+1} \times \mathbb{R} \to \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$. We claim that $\mathscr{L}|_{V_+} \cong \mathscr{L}_L$ relative to the boundary $\partial V_+ \cong S^{k-1} \times D^{n-k+1} \cong \Lambda_-$, meaning that they live in the same fiber of the restriction functor. Indeed, the restriction of $\mathscr{L}_{V_+}$ and $\mathscr{L}_L$ to Λ_+ should both be $\mathscr{L}_L|_{\Lambda_+}$, but $\mathscr{L}_L|_{\Lambda_+}$ extends uniquely to L since the inclusion $\Lambda_+ \hookrightarrow L$ is just $D^k \times S^{n-k} \hookrightarrow D^k \times D^{n-k+1}$. Therefore $\mathscr{L}|_{V_+} \simeq \mathscr{L}_L$ (respectively, the restriction of $\mathscr{L}_{\partial V_+}$ and $\mathscr{L}_{\Lambda_-}$ to $\partial \Lambda_+ \cong \partial \Lambda_-$ agree, but $\mathscr{L}_L|_{\partial \Lambda_+}$ extends uniquely to ∂V_+, so the local systems live in the same fiber).

Now we look at several different k-handle attachments to see what these data are in specific cases when $0 \leq k \leq 2$.

7.3.1.1. Lagrangian 1-handle attachment. When $k = 1$ there are 2 disconnected strata inside the cusps of Λ_- (Figure 7.5 and 7.7). The sheaf $\mathscr{F}_- \in Sh^b_{\Lambda_-}(\mathbb{R}^{n+1})$ can be extended only when the microlocal monodromy along $S^0 \times \mathbb{R}^n \subset \Lambda_-$ can be extended to a local system along $D^1 \times \mathbb{R}^n \subset L$. This is equivalent to saying that the microstalks on two components $F \simeq F'$.

Let the stalk in the region V_- bounded by the 2 cusps be B, B' and let the stalk outside be A. Then using the splitting of chain complexes

$$B \simeq A \oplus \mathrm{Tot}(A \to B) \simeq A \oplus F \simeq A \oplus \mathrm{Tot}(A \to B') \simeq B',$$

where $F = \mathrm{Tot}(A \to B) \simeq \mathrm{Tot}(A \to B')$ is the microstalk, we know that the condition implies that $B \simeq B'$. After applying the cobordism functor, the stalk in V_+ bounded by the front of Λ_+ is B and the stalk outside is A.

There is a choice we need to make for the quasi-isomorphism between all the B's, and that is coming from our choice for the local system on L. Different identifications may give different monodromies along Λ_+ relative to the boundary at infinity $\partial L = S^0 \times D^n$.

Namely, when gluing with a local system on L, we assign an extra quasi-isomorphism f_F bewteen the stalks F on both components of Λ_-. After applying Φ_L, the microstalk on Λ_+ is still F, where the quasi-isomorphism from F on the left to F on the right is given by f_F. Then by the quasi-isomorphism

$$B \simeq A \oplus F,$$

the transition map of B from left to right will be given by $f_B = (\mathrm{id}_A, f_F)$.

In particular, if the microstalk $F \simeq \Bbbk^r$ ($\mathscr{F}_-$ is pure), then the choices are classified by $GL_r(\Bbbk)$. When $F \simeq \Bbbk$ ($\mathscr{F}_-$ is simple), then the choices are classified by $\Bbbk^\times$.

Remark 7.3.2. *One can compare our computation with the computation in* [**23**, Section 5] *for Legendrian links and* [**26**, Section 5.5] *for Legendrian surfaces, by decomposing those cobordisms into a composition of Reidemeister moves and a handle attachment.*

What we described is only the local picture, globally there are different possibilities. Let's fix $F \simeq \Bbbk$ (this means $\mathscr{F}_-$ is simple). (1) When the 1-handle L connects two different components of Λ_-, then

$$H^1(\Lambda_-; \Bbbk^\times) \cong H^1(L; \Bbbk^\times).$$

Consider the moduli space of rank 1 local systems on Λ (coming from the derived moduli stacks of local systems) given by $Loc_1(\Lambda) = [H^1(\Lambda; \Bbbk^\times)/H^0(\Lambda; \Bbbk^\times)]$, and consider the framed moduli space of rank 1 local systems on a manifold Λ given by $Loc_1^{fr}(\Lambda) = H^1(\Lambda; \Bbbk^\times)$ with framing data, i.e. fixed trivializations of stalks, at each component. Then

$$Loc_1(\Lambda_-) \times [\Bbbk^\times/\Bbbk^\times] \cong Loc_1(L), \quad Loc_1^{fr}(\Lambda_-) \cong Loc_1^{fr}(L).$$

Consider the derived moduli stack of microlocal rank 1 sheaves $\mathbb{R}\mathcal{M}_1(\Lambda_\pm)$. Denote by $\mathcal{M}_1(\Lambda_\pm)$ the classical moduli stacks defined by the 1-rigid locus (the 1-rigid locus of $\mathbb{R}\mathcal{M}_1(\Lambda_\pm)$ consisting objects with no negative self-extensions is always an Artin stack, but they may not coincide with the derived stack)[3] [**147**, Section 2.4]. Assuming that

[3]The flag moduli space considered in [**26**, **157**] is, strictly speaking, slightly different as they do not remember the trivial $\Bbbk^\times$-action by only taking quotients of flags by $PGL_n(\Bbbk)$ instead of $GL_n(\Bbbk)$.

these classical moduli stacks coincide with the derived stacks, we have an embedding

$$\mathcal{M}_1(\Lambda_-) \times [\mathbb{k}^\times/\mathbb{k}^\times] = \mathcal{M}_1(\Lambda_-) \times_{Loc_1(\Lambda_-)} Loc_1(L) \hookrightarrow \mathcal{M}_1(\Lambda_+).$$

Consider $\mathcal{M}_1^{fr}(\Lambda_\pm)$ the classical moduli stacks defined by the 1-rigid locus with framing data at each component of $\Lambda_\pm$. Then we have an embedding

$$\mathcal{M}_1^{fr}(\Lambda_-) = \mathcal{M}_1^{fr}(\Lambda_-) \times_{Loc_1^{fr}(\Lambda_-;\mathbb{k}^\times)} Loc_1^{fr}(L;\mathbb{k}^\times) \hookrightarrow \mathcal{M}_1^{fr}(\Lambda_+).$$

(2) When the 1-handle L is attached on one component of Λ_-, then the moduli spaces of rank 1 local systems satisfy

$$H^1(\Lambda_-;\mathbb{k}^\times) \times \mathbb{k}^\times \cong H^1(L;\mathbb{k}^\times).$$

Therefore, for the moduli spaces of rank 1 local systems we know that

$$Loc_1(\Lambda_-) \times [\mathbb{k}^\times/\mathbb{k}^\times] \cong Loc_1(L), \quad Loc_1^{fr}(\Lambda_-) \times \mathbb{k}^\times \cong Loc_1^{fr}(L).$$

Hence assuming that the classical moduli stacks of microlocal rank 1 sheaves $\mathcal{M}_1(\Lambda_\pm)$ coincide with the derived stacks, we have an embedding

$$\mathcal{M}_1(\Lambda_-) \times [\mathbb{k}^\times/\mathbb{k}^\times] = \mathcal{M}_1(\Lambda_-) \times_{Loc_1(\Lambda_-)} Loc_1(L) \hookrightarrow \mathcal{M}_1(\Lambda_+).$$

The moduli spaces they consider are equal to $\mathcal{M}_1(\Lambda)$ considered here after further taking quotients by the trivial $\mathbb{k}^\times$-action.

For the moduli stacks of microlocal rank 1 sheaves with framing data at each component $\mathcal{M}_1^{fr}(\Lambda_\pm)$, we have an embedding

$$\mathcal{M}_1^{fr}(\Lambda_-) \times \Bbbk^\times = \mathcal{M}_1^{fr}(\Lambda_-) \times_{Loc_1^{fr}(\Lambda_-;\Bbbk^\times)} Loc_1^{fr}(L;\Bbbk^\times) \hookrightarrow \mathcal{M}_1^{fr}(\Lambda_+).$$

Remark 7.3.3. *In* [**78**] *the authors considered augmentation varieties for Legendrian links of positive n-braid closures, and for any such 2 Legendrian links connected by a 1-handle cobordism they showed that*

$$Aug(\Lambda_-) \times \Bbbk^\times \hookrightarrow Aug(\Lambda_+).$$

That is because when considering $Aug(\Lambda)$ they always fix n marked points and do not change the number of marked points when the number of components increases/decreases. This should be thought of as equivalent to the moduli space of microlocal rank 1 sheaves together with framing data at n base points [**147**, Section 2.4] *or equivalently trivialization data of microstalks at n base points.*

7.3.1.2. Lagrangian 2-handle attachment. When $k = 2$, the sheaf $\mathscr{F}_- \in Sh^b_{\Lambda_-}(\mathbb{R}^{n+1})$ can be extended only when the microlocal monodromy along $S^1 \times \mathbb{R}^{n-1} \subset \Lambda_-$ can be extended to a local system along $D^2 \times \mathbb{R}^{n-1} \subset L$. As $C^*(D^2;\Bbbk) \cong \Bbbk$, this is equivalent to saying that the microlocal monodromy is trivial along $S^1 \times \mathbb{R}^{n-1} \subset \Lambda_-$.

As in the case $k = 1$, there is a choice we need to take into consideration which is the contracting homotopy from the local system on S^1 to the trivial one, and the choice of the contracting homotopy will give different (higher) monodromies relative to the boundary at infinity $\partial L = S^1 \times D^{n-1}$. Consider a triangulation of $D^2 = \Delta^2$. Then this gives a stratification D^2. The 1-dimensional strata gives us quasi-isomorphisms

$$f_{01} : F \to F,\ f_{12} : F \to F,\ f_{02} : F \to F.$$

For the 2-dimensional stratum, we need to assign an extra chain homotopy H_{012} from f_{02} to $f_{12} \circ f_{01}$, i.e. $H_{012} : F \to F[-1]$ such that

$$H_{012}\delta_F - \delta_F H_{012} = f_{02} - f_{12} \circ f_{01}.$$

The (higher) monodromy along Λ_+ is preserved by the functor Φ_L and hence determines the microlocal monodromy of $\mathscr{F}_+$ along Λ_+. Using the quasi-isomorphism

$$B \simeq A \oplus F,$$

the monodromy data of F determines the monodromy data of the stalk B in $\mathscr{F}_+$.

When $F \simeq \Bbbk^r$ (the sheaf is pure), then the contracting homotopy data is trivial, and hence any such sheaf with trivial monodromy in $Sh^p_{\Lambda_-}(M)$ extends uniquely to a sheaf in $Sh^p_{\Lambda_+}(M)$.

Suppose the classical moduli stacks of microlocal rank r sheaves $\mathcal{M}_r(\Lambda_\pm)$ coincide with the derived stacks (with fixed framing data at a point). For $[\beta] \in \pi_1(\Lambda_-)$, let $\mathcal{M}_r^{[\beta]}(\Lambda_-)$ be the substack consisting of sheaves with trivial microlocal monodromy along β. Then for L a Lagrangian 2-handle cobordism attached along β, we have an embedding of algebraic stacks

$$\mathcal{M}_r^{[\beta]}(\Lambda_-) \hookrightarrow \mathcal{M}_r(\Lambda_+).$$

For the moduli stacks of microlocal rank r sheaves with framing data at each component, we get a similar embedding.

7.3.1.3. Lagrangian k-handle attachment ($k \geq 3$). When $k \geq 3$, we need to choose higher homotopy data to ensure that the monodromy of the complex of local systems along the attaching sphere $S^{k-1} \times D^{n-k+1} \subset \Lambda_-$ can be extended to $D^k \times D^{n-k+1} \subset L$. The monodromy along Λ_+ is preserved by the functor Φ_L and hence determines the monodromy of $\mathscr{F}_+$ along Λ_+. Using the quasi-isomorphism

$$B \simeq A \oplus F,$$

the (higher) monodromy data of F determines the (higher) monodromy data of B in $\mathscr{F}_+$.

However, in the special case when $F \simeq \Bbbk^r$, there will be no nontrivial higher homotopy data, and since the attaching sphere is changed from S^1 to S^{k-1} ($k \geq 3$), we know that any local system is trivial, so any such pure sheaf in $Sh^p_{\Lambda_-}(M)$ extends uniquely to a pure sheaf in $Sh^p_{\Lambda_+}(M)$.

Suppose the classical moduli stacks of microlocal rank r sheaves $\mathcal{M}_r(\Lambda_\pm)$ coincide with the derived stacks (with fixed framing data at a point). Then for L a Lagrangian

k-handle cobordism ($k \geq 3$), we have an embedding of algebraic stacks

$$\mathcal{M}_r(\Lambda_-) \hookrightarrow \mathcal{M}_r(\Lambda_+).$$

For the moduli stacks of microlocal rank r sheaves with framing data at each component, we get a similar embedding.

7.3.2. Applications to Legendrian surfaces

In this section we use the computation of the number of microlocal rank 1 sheaves to prove the results Theorem 7.0.10. We will frequently refer to [**157**] and [**26**] for the theory of Legendrian weaves (which are certain type of Legendrian surfaces) and their moduli space of microlocal rank 1 sheaves.

First, we recall that the correspondence between the front projection of Legendrian weaves and their planar graphs are illustrated in Figure **??**. The combinatoric constructions of Lagrangian handle attachments for Legendrian weaves are illustrated in Figure 7.8, and proved in [**26**, Theorem 4.10].

Proof of Theorem 7.0.10. (1) We start from $\Lambda_{g,k}$. Consider the local picture near a degree 3 vertex of the graph. Consider a Lagrangian 1-handle cobordism in the shadowed region (Figure 7.10 left). This will give a cobordism from $\Lambda_{g,k}$ to $\Lambda_{g+1,k}$. Then consider a Lagrangian 2-handle cobordism in the shadowed region (Figure 7.10 middle). This gives a cobordism from $\Lambda_{g+1,k}$ to $\Lambda_{g,k}$. For general $\Lambda_{g,k}, \Lambda_{g',k'}$, the cobordism can be constructed by concatenation.

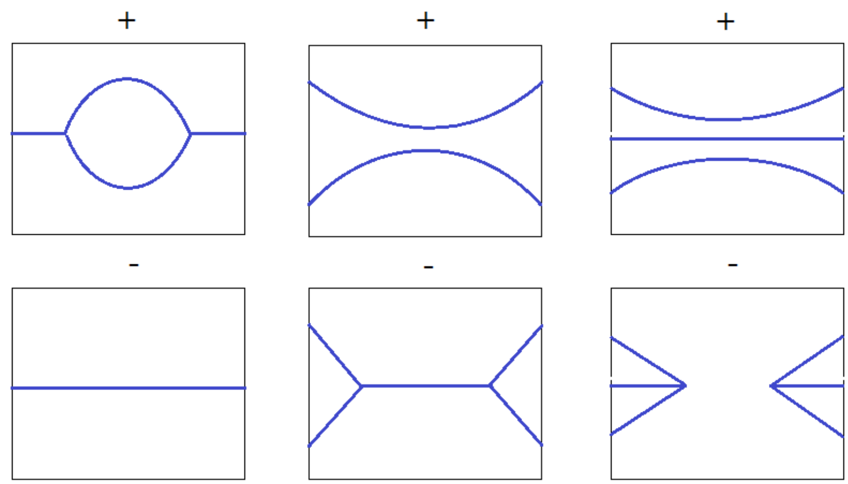

Figure 7.8. The graph on the left is a Lagrangian 1-handle attachment in Legendrian weaves; in the middle is a Lagrangian 2-handle attachment in Legendrian weaves; on the right is a Legendrian connected sum cobordism. Λ_+ are on the top while Λ_- are on the bottom.

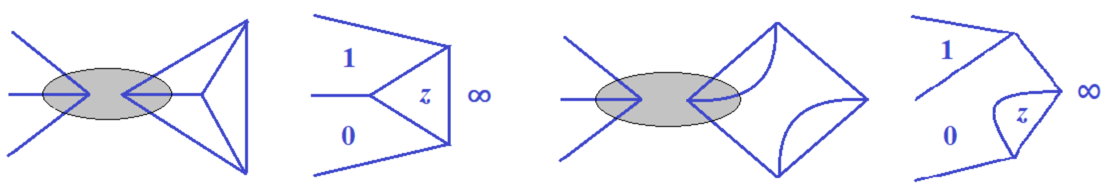

Figure 7.9. Taking connected sum with Λ_{Cliff} (left) and with Λ_{Unknot} (right). The cobordisms are from left to right in each picture. The labelling $0, 1, \infty, z$ is a $\Bbbk P^1$ coloring (so that regions sharing a common edge have different colors), which determines a microlocal rank 1 sheaf.

(2) This is essentially proved by Dimitroglou Rizell [**43**]. First of all, notice that $\Lambda_{g,0}$ admits an exact Lagrangian filling by taking a sequence of Lagrangian 1-handle cobordisms (Figure 7.11). Next, we claim that for any $k \geq 1$, $\Lambda_{g',k}$ does not admit exact Lagrangian fillings. Assuming the claim, then clearly there cannot be exact Lagrangian cobordisms from $\Lambda_{g,0}$ to $\Lambda_{g',k}$.

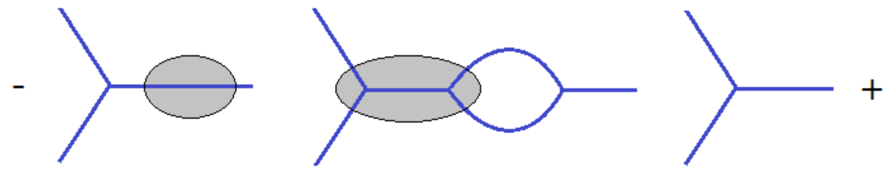

Figure 7.10. The cobordism from $\Lambda_{g,k}$ to $\Lambda_{g+1,k}$ to $\Lambda_{g,k}$ (from left to right). The grey regions are the places where we attach Lagrangian handles.

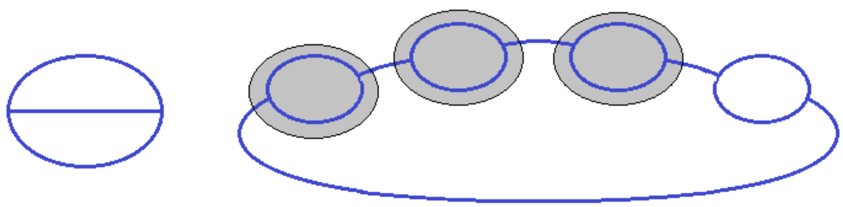

Figure 7.11. The Lagrangian filling of the Legendrian surface $\Lambda_{g,0}$ by Lagrangian 1-handle cobordisms in all the shadowed regions and finally fill the unknot on the left by a Lagrangian disk.

We now prove the claim using sheaves. One way is to apply [**157**, Theorem 1.3]. An alternative approach is the following [**26**, Theorem 5.12]. In the cases we are considering here, we know that

$$\mathcal{M}_1(\Lambda) = [\mathcal{M}_1^{fr}(\Lambda)/\Bbbk^\times].$$

and hence the flag moduli spaces in [**26**, **157**] are identified with the framed moduli space of sheaves with framing data at a single point. When $k \geq 1$, one can consider locally a triangle in the graph. A microlocal rank 1 sheaf is characterized by a $\Bbbk P^1$ colorings of regions (such that any regions sharing a common edge have different

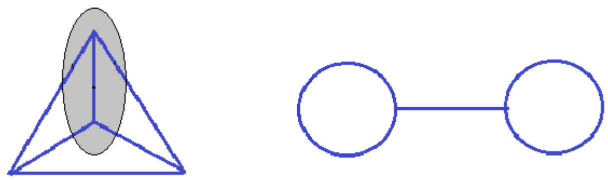

Figure 7.12. A Lagrangian 2-handle cobordism from $\Lambda_{S^2,\mathrm{loose}}$ (right) to Λ_{Cliff} (left).

colors). Without loss of generality, one can assume that outside the triangle, the 3 regions are colored by $0, 1$ and ∞ (Figure 7.9). Then the possible choices for colors in the triangle region are $\Bbbk^\times \backslash \{1\}$, i.e.

$$\mathcal{M}_1^{fr}(\Lambda_{g',k}) = \mathcal{M}_1^{fr}(\Lambda_{g'-1,k-1}) \times (\Bbbk^\times \backslash \{1\}).$$

When $\Bbbk = \mathbb{Z}/2\mathbb{Z}$, then there are no available choices and hence there are no microlocal rank 1 sheaves with $\mathbb{Z}/2\mathbb{Z}$-coefficients on $\Lambda_{g',k}$. Hence there cannot be any Lagrangian fillings. The claim is proved.

(3) First we should note that as explained in [**26**, Example 4.26] there is a Lagrangian cobordism L_0 from Λ_{Cliff} to a loose Legendrian 2-sphere $\Lambda_{S^2,\mathrm{loose}}$ by a Lagrangian 2-handle attachment (Figure 7.12), where the fact that $\Lambda_{S^2,\mathrm{loose}}$ is loose follows from [**26**, Proposition 4.24]. Hence there is a Lagrangian cobordism from $\Lambda_{g,k} = \Lambda_{g-1,k-1} \# \Lambda_{\mathrm{Cliff}}$ to a genus $g-1$ surface $\Lambda_{g-1,k-1} \# \Lambda_{S^2,\mathrm{loose}}$, and $\Lambda_{g-1,k-1} \# \Lambda_{S^2,\mathrm{loose}}$ is also loose.

We now apply [**66**, Theorem 2.2][4]. First we need to construct a formal Lagrangian cobordism, that is, a smooth cobordism $\psi : L_1 \hookrightarrow \mathbb{R}^6$ from $\Lambda_{g-1,k-1}\#\Lambda_{S^2,\text{loose}}$ to $\Lambda_{g,k'}$, and a family of bundle maps $\Psi_s : TL_1 \to T\mathbb{R}^6|_{L_1}$ such that $\Psi_0 = d\psi$, $\Psi_s \equiv d\psi$ near positive and negative ends, and Ψ_1 is a Lagrangian bundle map.

Note that $\Lambda_{g,k'}$ and $\Lambda_{g,k-1}$ are formally Legendrian isotopic for any $k \geq 1, k' \geq 0$. This means that there is a smooth isotopy $\psi'_t : \Lambda_t \hookrightarrow \mathbb{R}^5$, $t \in I$, together with a family of bundle maps $\Psi'_{s,t}$, $s, t \in I$, such that $\Psi'_{s,0} = d\psi'_0$, $\Psi'_{s,1} = d\psi'_1$, $\Psi'_{0,t} = d\psi'_t$, and $\Psi'_{1,t}$ are Lagrangian bundle maps into the contact distribution. Given a formal Legendrian isotopy, we can consider a smooth cobordism $L = \Lambda \times I$ from Λ_0 to Λ_1 being

$$\psi : L \hookrightarrow \mathbb{R}^5 \times I,\ \psi(x,t) = (\psi'_t(x), t),$$

and a family of bundle maps $\Psi_s : TL \to T(\mathbb{R}^5 \times I)|_L$ such that $\Psi_0 = d\psi$ and Ψ_1 is a Lagrangian bundle map by considering the homotopy such that

$$\Psi_s|_{T\Lambda_t} = \Psi_{s,t},\ \Psi_s|_{\langle \partial/\partial t\rangle} = (1-s)d\psi(\partial/\partial t) + s\,\partial/\partial r.$$

Therefore, we can get a formal Lagrangian concordance from $\Lambda_{g,k-1}$ to $\Lambda_{g,k'}$. By part (1) we know that there is a genuine Lagrangian cobordism from $\Lambda_{g-1,0}$ to $\Lambda_{g,k-1}$. Thus by concatenation, we will get a formal Lagrangian cobordism (L_1, ψ, Ψ_s) from $\Lambda_{g-1,k-1}\#\Lambda_{S^2,\text{loose}}$ to $\Lambda_{g,k'}$, and in fact

$$H^1(L_1) \xrightarrow{\sim} H^1(\Lambda_{g-1,k-1}\#\Lambda_{S^2,\text{loose}}).$$

[4]The author thanks Emmy Murphy for pointing out that the Lagrangian cap construction helps build cobordisms in this setting.

Then by [**66**, Theorem 2.2] there is a Lagrangian cobordism L_1 from $\Lambda_{g-1,k-1}\#\Lambda_{S^2,\mathrm{loose}}$ to $\Lambda_{g,k'}$ such that

$$H^1(L_1) \xrightarrow{\sim} H^1(\Lambda_{g-1,k-1}\#\Lambda_{S^2,\mathrm{loose}}) = \Bbbk^{2g-2}.$$

Taking the concatenation of L_0 and L_1, we will get a Lagrangian cobordism such that

$$\dim\operatorname{coker}(H^1(L_0 \cup L_1) \to H^1(\Lambda_{g,k})) = 2.$$

(4) We show that there cannot be Lagrangian cobordisms L with vanishing Maslov class from $\Lambda_{g,k}$ to $\Lambda_{g,k'}$ for $k < k'$ such that $H^1(L) \twoheadrightarrow H^1(\Lambda_{g,k})$. Indeed consider a degree 3 vertex in the graph of $\Lambda_{g-1,k-1}$. Taking connected sum with Λ_{Cliff} and $\Lambda_{\mathrm{Unknot}}$ will give $\Lambda_{g,k}$ and $\Lambda_{g,k-1}$. As explained in Part (2), a microlocal rank 1 sheaf is characterized by the number $\Bbbk P^1$ colorings of the graph (Figure 7.9). The possible choices for colors in the triangle region are $\Bbbk^\times\backslash\{1\}$,

$$\mathcal{M}_1^{fr}(\Lambda_{g,k}) = \mathcal{M}_1^{fr}(\Lambda_{g-1,k-1}) \times (\Bbbk^\times\backslash\{1\}).$$

On the other hand, for $\Lambda_{g,k-1}$, assume the upper half region and lower half region are colored $0, \infty$ (Figure 7.9). Then the possible choices for colors in the bi-gon region are $\Bbbk^\times$, i.e.

$$\mathcal{M}_1^{fr}(\Lambda_{g,k-1}) = \mathcal{M}_1^{fr}(\Lambda_{g-1,k-1}) \times \Bbbk^\times.$$

In particular, $|\mathcal{M}_1^{fr}(\Lambda_{g,k})(\mathbb{F}_q)| < |\mathcal{M}_1^{fr}(\Lambda_{g,k-1})(\mathbb{F}_q)|$, and by induction $|\mathcal{M}_1^{fr}(\Lambda_{g,k'})(\mathbb{F}_q)| < |\mathcal{M}_1^{fr}(\Lambda_{g,k})(\mathbb{F}_q)|$.

When $H^1(L) \twoheadrightarrow H^1(\Lambda_{g,k})$ is surjective, in the fiber product

$$\begin{array}{ccc} \mathcal{M}_1^{fr}(\Lambda_{g,k-1}) \times_{H^1(\Lambda_{g,k};\Bbbk^\times)} H^1(L;\Bbbk^\times) & \xrightarrow{\widehat{r}} & \mathcal{M}_1^{fr}(\Lambda_{g,k}) \\ \downarrow & & \downarrow P \\ H^1(L;\Bbbk^\times) & \xrightarrow{r} & H^1(\Lambda_{g,k};\Bbbk^\times), \end{array}$$

the horizontal map r at the bottom is a projection map, and hence so is $\widehat{r}$ (in fact the vertical map on the right P is called the period map in [**157**, Section 4.7]). Therefore

$$|(\mathcal{M}_1^{fr}(\Lambda_{g,k}) \times_{H^1(\Lambda_{g,k};\Bbbk^\times)} H^1(L;\Bbbk^\times))(\mathbb{F}_q)| \geq |\mathcal{M}_1^{fr}(\Lambda_{g,k})(\mathbb{F}_q)| > |\mathcal{M}_1^{fr}(\Lambda_{g,k'})(\mathbb{F}_q)|.$$

However, a fully faithful Lagrangian cobordism functor Φ_L should induce an embedding

$$\mathcal{M}_1^{fr}(\Lambda_{g,k}) \times_{H^1(\Lambda_{g,k};\Bbbk^\times)} H^1(L;\Bbbk^\times) \hookrightarrow \mathcal{M}_1^{fr}(\Lambda_{g,k'}).$$

That is a contradiction. □

Remark 7.3.4. *For the Lagrangian cobordism from $\Lambda_{g,k}$ to $\Lambda_{g+1,k}$ and to $\Lambda_{g,k}$, one can see (in Figure 7.10) that the ascending manifold of Lagrangian 1-handle and the descending manifold of the Lagrangian 2-handle have geometric intersection number 1. Since these Lagrangians are regular* [**62**]*, one can cancel the pair of critical points so that the Lagrangian is Hamiltonian isotopic to a cylinder.*

7.4. Lagrangian Cobordism Functor as Correspondence

When $\Lambda_\pm \subset T^{*,\infty}_{\tau>0}(M \times \mathbb{R})$, we have obtained two approaches of constructing Lagrangian cobordism functors. One is by composing the (conditional) sheaf quantization functor on $\widetilde{L}$ with the restriction functor to the positive end explained in Section 4.4

$$Sh_{\Lambda_-}(M \times \mathbb{R})_0 \times_{\mu Sh_{\Lambda_-}} \mu Sh_L \xrightarrow{\Psi_L} Sh_{\widetilde{L}}(M \times \mathbb{R} \times \mathbb{R}_{>0})_0 \xrightarrow{i_+^{-1}} Sh_{\Lambda_+}(M \times \mathbb{R})_0.$$

The other is given by the functorial specialization functor of Nadler-Shende [**124**]

$$Sh_{\Lambda_-}(M \times \mathbb{R}) \times_{\mu Sh_{\Lambda_-}} \mu Sh_L \xrightarrow{\Phi_L} Sh_{\Lambda_+}(M \times \mathbb{R}).$$

In this section, we prove Theorem 7.0.9, which will imply that they are identical. Indeed, since by Theorem 4.0.9, Ψ_L is the right inverse of $(i_-^{-1}, m_{\widetilde{L}})$ the fiber product of restriction to the negative end and microlocalization along the cobordism, once we prove that $\Phi_L \circ (i_-^{-1}, m_{\widetilde{L}}) = i_+^{-1}$, this will indicate that

$$i_+^{-1} \circ \Psi_L = \Phi_L \circ (i_-^{-1}, m_{\widetilde{L}}) \circ \Psi_L = \Phi_L.$$

7.4.1. Construction of a suspension Lagrangian cobordism

We note that in the statement of Theorem 7.0.9, the Lagrangian cobordism functor of Nadler-Shende is defined by attaching to the negative end the Lagrangian L along the radius (vertical) direction, while the conical Lagrangian cobordism is defined by connecting by the Legendrian $\widetilde{L}$ along the base (horizontal) direction. In order to investigate the relation between the two pictures we have to consider both directions (and connect them in a geometric way).

Our goal in this section is thus to define a suspension exact Lagrangian cobordism $\Sigma L \subset T^*(M \times (1, +\infty) \times \mathbb{R}_{>0})$ that is diffeomorphic to $L \times \mathbb{R}$, such that

(1) the symplectic reduction of the suspension to $\Sigma L_{s_-} \subset T^*(M \times \mathbb{R}_{>0}) \times \{s_-\}$ is the exact Lagrangian cobordism L;
(2) the symplectic reduction of the suspension to $\Sigma L_{s_+} \subset T^*(M \times \mathbb{R}_{>0}) \times \{s_+\}$ is the trivial Lagrangian cobordism $\Lambda_+ \times \mathbb{R}_{>0}$;
(3) the suspension $\Sigma L \subset T^*(M \times (1, +\infty) \times \mathbb{R}_{>0})$ is an exact Lagrangian cobordism between the Legendrian lifts from $\widetilde{L}$ to $\Lambda_+ \times (1, +\infty)$.

Here we view the Lagrangian cobordisms L and $\Lambda_+ \times (1, +\infty)$ as in a subdomain in the the symplectization $J^1(M) \times (1, +\infty) \cong T^*(M \times (1, +\infty))$. This will simplify some of the formulas in the discussion.

First, consider the trivial exact Lagrangian cobordism $\widetilde{L} \times \mathbb{R}_{>0} \subset T^*(M \times (1, +\infty) \times \mathbb{R}_{>0})$ of the Legendrian lift $\widetilde{L} \subset J^1(M \times (1, +\infty))$, i.e.

$$\widetilde{L} \times \mathbb{R}_{>0} = \big\{(x, s, r; sr\xi, rt, st + w) | (x, s; s\xi, t; st + w) \in \widetilde{L}\big\}.$$

Then, a natural approach is to define a diffeomorphism

$$\phi : (1, +\infty) \times \mathbb{R}_{>0} \xrightarrow{\sim} (1, +\infty) \times \mathbb{R}_{>0}$$

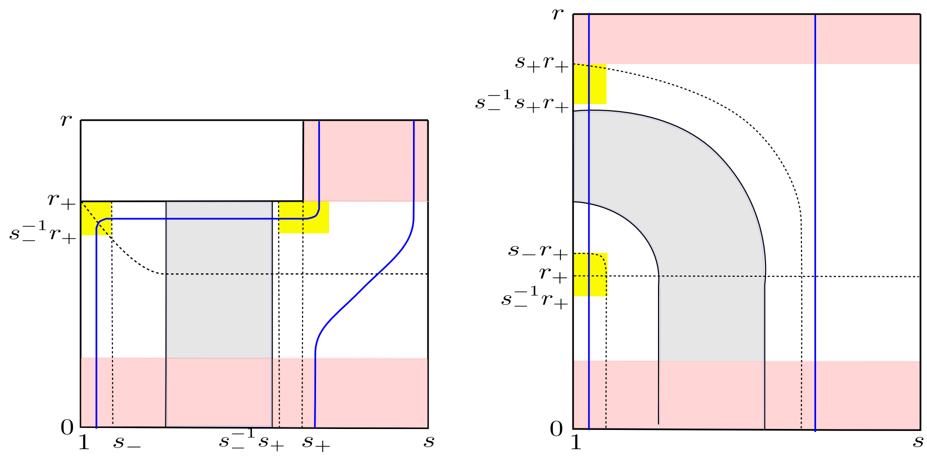

Figure 7.13. The diffeomorphism ϕ on a subdomain of $(1,+\infty)\times(0,+\infty)$. The grey region represents where the Lagrangian cobordism $L\subset T^*(M\times(1,+\infty))$ is not cylindrical. The pink regions are where Condition (1) & (2) are satisfied. The two blue lines are the preimage of $\phi^{-1}(\{s\}\times(0,+\infty))$ for $s<s_-$ and $s>s_+$. The two yellow regions are the regions in $\phi^{-1}((1,s_-)\times(0,+\infty))$ that are not controlled by Condition (3).

that sends the negative end $(1,+\infty)\times\{0\}$ to $(1,+\infty)\times\{0\}$, sends the positive end $(1,+\infty)\times\{+\infty\}$ to $\{1\}\times\mathbb{R}_{>0}$, sends $\{1\}\times\mathbb{R}_{>0}$ to $\{(1,0)\}$ and sends $\{+\infty\}\times\mathbb{R}_{>0}$ to $\{+\infty\}\times\mathbb{R}_{>0}\cup\mathbb{R}_{>0}\times\{+\infty\}$, which will extend to an exact symplectomorphism

$$\varphi: T^*(M\times(1,+\infty)\times\mathbb{R}_{>0})\xrightarrow{\sim}T^*(M\times(1,+\infty)\times\mathbb{R}_{>0}).$$

However, there is some technical difficulties to define the suspension cobordism using the diffeomorphism defined globally on $(1,+\infty)\times\mathbb{R}_{>0}$, and instead we will consider only a certain subdomain.

Fix $s_->1$ and $r_->0$ sufficiently small, and respectively $s_+>1$ and $r_+>0$ sufficiently large. Suppose the Lagrangian cobordism L is conical outside $T^*(M\times$

(s_-, s_+)). Consider the diffeomorphism from a subdomain on the plane to the plane $\phi : (1, +\infty) \times \mathbb{R}_{>0} \setminus (1, s_+] \times [r_+, +\infty) \to (1, +\infty) \times \mathbb{R}_{>0}$ that satisfies the following conditions:

(1) $\phi(s, r) = (s, r)$ for $s > 1$ and $r < r_-$ sufficiently small;

(2) $\phi(s, r) = (s_+^{-1} s, s_+ r)$ for $s > s_+$ and $r > r_+$ sufficiently large;

(3) $\phi(s, r) = (s, r)$ for $1 < s < s_-$ sufficiently small and $r < s_-^{-1} r_+$, $\phi(s, r) = (r_+ r^{-1}, s_-^{-1} r_+ s)$ for $s_- < s < s_+$ and $s_-^{-1} r_+ < r < r_+$.

See Figure 7.13. This induces a symplectomorphism (partially defined on the subdomain)

$$\varphi : T^*\big(M \times ((1, +\infty) \times \mathbb{R}_{>0} \setminus (1, s_+] \times [r_+, +\infty))\big) \xrightarrow{\sim} T^*\big(M \times (1, +\infty) \times \mathbb{R}_{>0}\big).$$

Using the (partially defined) symplectomorphism, we define the suspension Lagrangian cobordism as follows.

Definition 7.4.1. *Let $L \subset T^*(M \times (1, +\infty))$ be an exact Lagrangian cobordism between Legendrians. Then the suspension Lagrangian ΣL is the exact Lagrangian submanifold $\varphi(\widetilde{L} \times \mathbb{R}_{>0}) \subset T^*(M \times (1, +\infty) \times \mathbb{R}_{>0})$.*

Remark 7.4.1. *We explain the reason why the construction is more complicated than one may imagine. In fact, the Lagrangian $\widetilde{L} \times \mathbb{R}_{>0}$ is conical with respect to the r-direction. In particular, the Liouville flow along the r-direction determines the positive/convex end $(1, +\infty) \times (r_+, +\infty)$ and negative/concave end $(1, +\infty) \times (0, r_-)$.*

Then we need to deform $(1,+\infty)\times\mathbb{R}_{>0}$ *in a way so that the Liouville flow changes and the new positive/convex end becomes* $(s_+,+\infty)\times(r_+,+\infty)$ *while the negative/concave end stays the same. Moreover, after the deformation the Lagrangian* $\widetilde{L}\times\mathbb{R}_{>0}$ *should stay conical with respect to the* r*-direction, in order for it to be a Lagrangian cobordism.*

Therefore, it is in fact natural to come up with this complicated diffeomorphism, where Condition (1) & (2) ensure that the Lagrangian stay conical and the positive and negative end are exactly what we need. The reason to cut off the top left corner $(1,s_+]\times[r_+,+\infty)$ *is to make sure that the conical condition is not violated on the top left corner after deformation (so that we only need to move* $(s_+,+\infty)\times(r_+,+\infty)$ *horizontally along the* s*-direction without extra uncontrolled deformation).*

This discussion also explains why we do not try to define ΣL *as both a cobordism from bottom to top and a cobordism from left to right, but rather only require that the behaviour of its symplectic reduction on the left and right slices. It seems that requiring the Lagrangian* ΣL *to be tangent to both the horizontal and the vertical Liouville flow would make it too difficult to construct.*

Consider the suspension Lagrangian ΣL and its positive/negative ends with respect to the $\mathbb{R}_{>0}$ direction. Suppose $\widetilde{L}\times\mathbb{R}_{>0}\cap T^*(M\times(1,\infty)\times(0,r_-)) = \{(x,s,r;sr\xi,rt,st+w)|(x,s;s\xi,t;st+w)\in\widetilde{L}\}$. With the assumption that $\phi(s,r)=(s,r)$ for $r<r_-$ (which is the bottom pink region in Figure 7.13), we know that the

induced symplectomorphism on this region is the identity,

$$\Sigma L \cap T^*(M \times (1,\infty) \times (0, r_-)) = \big\{(x,s,r; sr\xi, rt, st+w) \big| (x,s;s\xi,t;st+w) \in \widetilde{L}\big\}.$$

With the assumption that $\phi(s,r) = (s_+^{-1}s, s_+r)$ for $s > s_+$ and $r > r_+$ (which is the top pink region in Figure 7.13), we know that the induced symplectomorphism sends $(x,s,r;\xi,\sigma,\rho)$ to $(x, s_+^{-1}s, s_+r; \xi, s_+\sigma, s_+^{-1}\rho)$, which, after coordinate changes, implies that

$$\Sigma L \cap T^*(M \times (1,\infty) \times (s_+r_+, \infty)) = \big\{(x,s,r; sr\xi, rt, st) \big| (x,\xi,t) \in \Lambda_+\big\}.$$

Hence we can conclude the following lemma:

Lemma 7.4.1. *Let $L \subset T^*(M \times \mathbb{R}_{>0})$ be a conical Legendrian cobordism from Λ_- to Λ_+. Then $\Sigma L \subset T^*(M \times (1,\infty) \times \mathbb{R}_{>0})$ is an exact Lagrangian cobordism from the Legendrian lift $\widetilde{L}$ to the trivial Legendrian cone $\Lambda_+ \times (1,\infty)$.*

Then consider the symplectic reduction of ΣL along the hypersurface $T^*M \times T^*_s(1,+\infty) \times T^*\mathbb{R}_{>0}$, i.e.

$$\Sigma L_s = \pi_s(\Sigma L \cap T^*M \times T^*_s(1,+\infty) \times T^*\mathbb{R}_{>0}) \subset T^*M \times T^*\mathbb{R}_{>0}$$

(which are the blue lines in Figure 7.13).

First, consider the slice $T^*M \times \{s\} \times T^*\mathbb{R}_{>0}$ for $1 < s < s_-$ sufficiently small. On $T^*M \times \{s\} \times T^*(0, s_-^{-1}r_+)$, we know that the symplectomorphism is induced by

$\phi(s,r) = (s,r)$, so

$$\Sigma L_s \cap T^*M \times T^*(0, s_-^{-1}r_+) = \Lambda_- \times (0, s_-^{-1}r_+);$$

on $T^*M \times \{s\} \times T^*(s_+r_+, +\infty)$, we know that the symplectomorphism is induced by $\phi(s,r) = (s_+^{-1}s, s_+r)$, so since the first coordinate $s_+^{-1}s$ is fixed

$$\Sigma L_s \cap T^*M \times T^*(s_+r_+, +\infty) = \Lambda_+ \times (s_+r_+, +\infty);$$

on $T^*M \times \{s\} \times T^*(s_-r_+, s_-^{-1}s_+r_+)$, we know that the symplectomorphism is induced by $\phi(s,r) = (r_+r^{-1}, r_+s)$, so since the first coordinate r_+r^{-1} is fixed

$$\Sigma L_s \cap T^*M \times T^*(s_-r_+, s_-^{-1}s_+r_+) = L \cap T^*(M \times (s_-r_+, s_-^{-1}s_+r_+)).$$

Finally, consider $\Sigma L_s \cap T^*M \times T^*(s_-^{-1}r_+, s_-r_+)$ (which is the left/bottom yellow region in Figure 7.13). Suppose that $\gamma_1(\theta) = \phi^{-1}(s,\theta)$. Consider the isotopy from the curve γ_1 to the line segment γ_0 connecting $(s, s_-^{-1}r_+)$ and $(s_-, s^{-1}r_+^{-1})$. We use the following lemma (see for example [**114**, Section 3.3]):

Lemma 7.4.2. *Let X be an exact symplectic manifold. Let C_u, $0 \leq u \leq 1$ be a smooth family of coisotropic submanifolds in X and $\mathscr{F}_u = \ker(\omega_X|_{TC_u})$ be the characteristic foliation on TC_u. Then there is a family of exact symplectomorphisms*

$$\varphi_u : C_0/\mathscr{F}_0 \xrightarrow{\sim} C_u/\mathscr{F}_u.$$

Moreover, let $L \subset X$ *be an exact Lagrangian. Suppose that* $L \pitchfork C_u$ *for any* $0 \leq u \leq 1$. *Let* $L_u = \pi_u(L \cap C_u) \subset C_u/\mathscr{F}_u$ *be the symplectic reduction. Then*

$$\varphi_u(L_0) = L_u.$$

Proof. Using the isotopy extension theorem, there is a family of exact symplectomorphisms $\varphi_u : X \to X$ such that $\varphi_u(C_0) = C_u$. Properties of symplectomorphisms then ensure that it preserves characteristic foliations $\varphi_u(\mathscr{F}_0) = \mathscr{F}_u$. Then we obtain the first part of the claim.

For the second part of the claim, note that $L \pitchfork \varphi_u(C_0)$ if and only if $\varphi_u^{-1}(L) \pitchfork C_0$. Since

$$L \cap \varphi_u(C_0) = \varphi_u(\varphi_u^{-1}(L) \cap C_0),$$

we can conclude that $\pi_0(\varphi_u^{-1}(L) \cap C_0)$ are exact Lagrangian isotopic and thus are Hamiltonian isotopic. □

Since $\widetilde{L} \times \mathbb{R}_{>0} \cap T^*(M \times (1, s_-) \times (s_-^{-1}r_+, r_+))$ is the trivial suspension of the Legendrian cone $\Lambda_- \times (1, s_-) \times (s_-^{-1}r_+, r_+)$, where

$$\Lambda_- \times (1, +\infty) \times \mathbb{R}_{>0} = \{(x, s, r; sr\xi, rt, st) | (x, \xi, t) \in \Lambda_-\}.$$

Thus $\widetilde{L} \times \mathbb{R}_{>0} \pitchfork T^*M \times T^*_{\gamma_u}((1, s_-) \times (s_-^{-1}r_+, r_+))$ for any $0 \leq u \leq 1$. Using Lemma 7.4.2, we know that the symplectic reduction of $\widetilde{L} \times \mathbb{R}_{>0}$ along $T^*M \times T^*_{\gamma_1}((1, s_-) \times (s_-^{-1}r_+, s_-r_+))$, is Hamiltonian isotopic to the reduction along $T^*M \times T^*_{\gamma_0}((1, s_-) \times (s_-^{-1}r_+, s_-r_+))$. We can easily tell that the symplectic reduction along

$T^*M \times T^*_{\gamma_0}((1, s_-) \times (s_-^{-1}r_+, s_-r_+))$ is the trivial Legendrian cone, i.e.

$$\Sigma L_s \cap T^*M \times T^*(s_-^{-1}r_+, s_-r_+) \cong \Lambda_- \times (s_-^{-1}r_+, s_-r_+).$$

Similarly, we consider $\Sigma L_s \cap T^*M \times T^*(s_-^{-1}s_+r_+, s_+r_+)$ (which is the right/top yellow region in Figure 7.13). Let $\gamma_1(\theta) = \phi^{-1}(s, \theta)$. We can connect γ_1 to the line segment γ_0. Since $\widetilde{L} \times \mathbb{R}_{>0} \cap T^*(M \times (s_-^{-1}s_+, s_-s_+) \times (s_-^{-1}r_+, r_+))$ is the trivial suspension of the Legendrian cone $\Lambda_+ \times (s_-^{-1}s_+, s_-s_+) \times (s_-^{-1}r_+, r_+)$, we can apply Lemma 7.4.2 again. Therefore, we can conclude the following lemma:

Lemma 7.4.3. *Let $L \subset T^*(M \times (1, +\infty))$ be an exact Lagrangian cobordism from Λ_- to Λ_+. Then for $1 < s < s_-$ sufficiently small, the symplectic reduction $\Sigma L_s \subset T^*(M \times \{s\} \times \mathbb{R}_{>0})$ is Hamiltonian isotopic to the Lagrangian cobordism $L \subset T^*(M \times \mathbb{R}_{>0})$ under a compactly supported Hamiltonian isotopy.*

Then, consider the slice $T^*M \times \{s\} \times T^*\mathbb{R}_{>0}$ for $s > s_+$ sufficiently large. Let $\gamma_1(\theta) = \phi^{-1}(s, \theta)$. Since $\widetilde{L} \times \mathbb{R}_{>0} \cap T^*(M \times (s_+, +\infty) \times \mathbb{R}_{>0})$ is the trivial suspension of the Legendrian cone $\Lambda_+ \times (s_+, +\infty) \times \mathbb{R}_{>0}$, we can apply Lemma 7.4.2 for the deformation induced by the isotopy from γ_1 to the line segment γ_0. We can easily tell that the symplectic reduction along $T^*M \times T^*_{\gamma_0}((s_+, +\infty) \times (r_-, r_+))$ is the trivial Legendrian cone, i.e.

$$\Sigma L_s \cap T^*(M \times (r_-, r_+)) \cong \Lambda_+ \times (r_-, r_+).$$

Therefore, we can conclude the following lemma:

Lemma 7.4.4. *Let $L \subset T^*(M \times (1, +\infty))$ be an exact Lagrangian cobordism from Λ_- to Λ_+. Then for $s > s_+$ sufficiently large, the symplectic reduction $\Sigma L_s \subset T^*(M \times \{s\} \times \mathbb{R}_{>0})$ is Hamiltonian isotopic to the trivial Lagrangian cobordism $\Lambda_+ \times \mathbb{R}_{>0} \subset T^*(M \times \mathbb{R}_{>0})$ under a compactly supported Hamiltonian isotopy.*

7.4.2. Comparison between the two constructions

Given the suspension exact Lagrangian cobordism $\Sigma L \subset T^*(M \times (1, +\infty) \times \mathbb{R}_{>0})$ from $\widetilde{L}$ to $\Lambda_+ \times \mathbb{R}_{>0}$, we will prove the following commutative diagram of sheaves of categories

$$\begin{array}{ccccc} \mu Sh_{M \cup (\Lambda_- \times \mathbb{R}_{>0})} \times_{\mu Sh_{\Lambda_-}} \mu Sh_L & \xleftarrow{\tilde{j}_-^{-1}} & \mu Sh_{(M \times \mathbb{R}_{>0}) \cup (\widetilde{L} \times \mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L} & \xrightarrow{\tilde{j}_+^{-1}} & \mu Sh_{M \cup (\Lambda_+ \times \mathbb{R}_{>0})} \\ \downarrow \Phi_L & & \downarrow \Phi_{\Sigma L} & & \downarrow \Phi_{\Lambda_+ \times \mathbb{R}_{>0}} \\ \mu Sh_{M \cup (\Lambda_+ \times \mathbb{R}_{>0})} & \xleftarrow{j_-^{-1}} & \mu Sh_{(M \times \mathbb{R}_{>0}) \cup (\Lambda_+ \times \mathbb{R}^2_{>0})} & \xrightarrow{j_+^{-1}} & \mu Sh_{M \cup (\Lambda_+ \times \mathbb{R}_{>0})}, \end{array}$$

where $\Phi_{\Lambda_+ \times \mathbb{R}_{>0}} \simeq \mathrm{id}$, and j_-^{-1} and j_+^{-1} are (obvious) equivalences with composition being the identity. Hence the diagram simplifies to

$$\begin{array}{ccc} \mu Sh_{(M \times \mathbb{R}_{>0}) \cup (\widetilde{L} \times \mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L} & \xrightarrow{\tilde{j}_-^{-1}} & \mu Sh_{M \cup (\Lambda_- \times \mathbb{R}_{>0})} \times_{\mu Sh_{\Lambda_-}} \mu Sh_L \\ & \searrow^{\tilde{j}_+^{-1}} & \downarrow \Phi_L \\ & & \mu Sh_{M \cup (\Lambda_+ \times \mathbb{R}_{>0})}. \end{array}$$

Using the fact that the cobordism $\widetilde{L}$ to $\Lambda_+ \times \mathbb{R}_{>0}$ diffeomorphic to $L \times \mathbb{R}$ and the negative end is equal to $\widetilde{L}$, we have a canonical identification of categories

$$\mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L} \simeq \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})}.$$

Therefore, we will show that $\tilde{j}_-^{-1} \simeq (i_-^{-1}, m_{\widetilde{L}})$ and $\tilde{j}_+^{-1} \simeq i_+^{-1}$, and hence complete the proof of Theorem 7.0.9.

Lemma 7.4.5. *Let L be an exact Lagrangian cobordism from Λ_- to Λ_+ and ΣL the suspension exact Lagrangian cobordism. There is a commutative diagram*

$$\begin{array}{ccccc}
\mu Sh_{M\cup(\Lambda_-\times\mathbb{R}_{>0})} \times_{\mu Sh_{\Lambda_-}} \mu Sh_L & \xleftarrow{\tilde{j}_-^{-1}} & \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L} & \xrightarrow{\tilde{j}_+^{-1}} & \mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})} \\
\downarrow \Phi_L & & \downarrow \Phi_{\Sigma L} & & \downarrow \Phi_{\Lambda_+\times\mathbb{R}_{>0}} \\
\mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})} & \xleftarrow{j_-^{-1}} & \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\Lambda_+\times\mathbb{R}^2_{>0})} & \xrightarrow{j_+^{-1}} & \mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})},
\end{array}$$

Proof. Recall from Section 7.1.1 the construction of the Lagrangian cobordism functor. Fix a contact embedding $T^*(M \times \mathbb{R}_{>0}) \hookrightarrow T^{*,\infty}N$. Write

$$N \times \{0\} \xrightarrow{i} N \times [0,1] \xleftarrow{j} N(0,1]$$

and let ϕ_Z^ζ, $0 < \zeta \leq 1$ is the contact Hamiltonian flow that lifts the Liouville flow on $T^*(M \times \mathbb{R})$. For $\mathscr{F} \in \mu Sh_{M\cup(\Lambda\times\mathbb{R}_{>0})} \times_{\mu Sh_\Lambda} \mu Sh_L$, the image under the Lagrangian cobordism functor Φ_L is defined by

$$\Phi_L(\mathscr{F})_{\mathrm{dbl}} = i^{-1} j_*(\Psi_Z(\mathscr{F})_{\mathrm{dbl}}),$$

Then consider $\mathscr{F} \in \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L}$. Write

$$N \times \mathbb{R}_{>0} \times \{0\} \xrightarrow{\bar{i}} N \times \mathbb{R}_{>0} \times [0,1] \xleftarrow{\bar{j}} N \times \mathbb{R}_{>0} \times (0,1]$$

and let $\phi_{\bar{Z}}^{\zeta}$, $0 < \zeta \leq 1$ is the contact Hamiltonian flow that lifts the pull back Liouville flow on $T^*(M \times \mathbb{R} \times \mathbb{R}_{>0})$. We apply Proposition 7.1.5 and Remark 3.3.3 and get

$$\begin{aligned}\Phi_L(\tilde{j}_-^{-1}\mathscr{F})_{\mathrm{dbl}} &= i^{-1}j_*\big(\Psi_Z(\tilde{j}_-^{-1}\mathscr{F})_{\mathrm{dbl}}\big) = i^{-1}j_*\big(\tilde{j}_-^{-1}\Psi_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}\big)\\ &= i^{-1}\tilde{j}_-^{-1}\bar{j}_*\big(\Psi_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}\big) = j_-^{-1}\bar{i}^{-1}\bar{j}_*\big(\Psi_{\overline{Z}}(\mathscr{F})_{\mathrm{dbl}}\big) = j_-^{-1}\Phi_{\Sigma L}(\mathscr{F})_{\mathrm{dbl}}.\end{aligned}$$

This prove the commutativity for the square on the left. Using the same argument, we have commutativity for the square on the right. $\square$

In the next lemma, we explain why there is an identification $\tilde{j}_+^{-1} \simeq i_+^{-1}$.

Lemma 7.4.6. *Let L be an exact Lagrangian cobordism from Λ_- to Λ_+ and ΣL the suspension exact Lagrangian cobordism. There is a commutative diagram*

$$\begin{array}{ccc} \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L} & & \\ \downarrow\wr & \searrow^{\tilde{j}_+^{-1}} & \\ & & \mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})}. \\ & \nearrow_{i_+^{-1}} & \\ \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} & & \end{array}$$

Proof. Using the property that the symplectic reduction $\Sigma L_s \cong \Lambda_+ \times \mathbb{R}_{>0}$, there is a commutative diagram

$$\begin{array}{ccccc} \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} & \longrightarrow & \mu Sh_L & \longleftarrow & \mu Sh_{\Sigma L} \\ \downarrow i_+^{-1} & & \downarrow i_+^{-1} & & \downarrow i_+^{-1} \\ \mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})} & \longrightarrow & \mu Sh_{\Lambda_+} & \longleftarrow & \mu Sh_{\Lambda_+\times\mathbb{R}_{>0}} \end{array}$$

The restriction $\tilde{j}_+^{-1}$ is the restriction functor on the homotopy pull back which commutes with the above diagram

$$\tilde{j}_+^{-1} : \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L} \to \mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})} \times_{\mu Sh_{\Lambda_+}} \mu Sh_{\Lambda_+\times\mathbb{R}_{>0}}.$$

However, since ΣL is diffeomorphic to $L \times \mathbb{R}$, the restriction is an equivalence $\mu Sh_{\Sigma L} \xrightarrow{\sim} \mu Sh_L$. Thus there are equivalences of the homotopy pull back given by natural restriction functors

$$\begin{array}{ccc} \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \times_{\mu Sh_L} \mu Sh_{\Sigma L} & \xrightarrow{\sim} & \mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \\ \tilde{j}_+^{-1} \downarrow & & \downarrow i_+^{-1} \\ \mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})} \times_{\mu Sh_{\Lambda_+}} \mu Sh_{\Lambda_+\times\mathbb{R}_{>0}} & \xrightarrow{\sim} & \mu Sh_{M\cup(\Lambda_+\times\mathbb{R}_{>0})}. \end{array}$$

Therefore, the diagram in the statement commutes by the commutativity of the restriction functors. $\square$

In the final lemma, we explain why there is an identification between $\tilde{j}_-^{-1}$ and $(i_-^{-1}, m_{\widetilde{L}})$.

Lemma 7.4.7. *Let L be an exact Lagrangian cobordism from Λ_- to Λ_+ and ΣL the suspension exact Lagrangian cobordism. There is a commutative diagram*

$$\begin{array}{ccc}
\mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})}\times_{\mu Sh_L}\mu Sh_{\Sigma L} & & \\
\wr\downarrow & \searrow^{\tilde{j}_-^{-1}} & \\
 & & \mu Sh_{M\cup(\Lambda_-\times\mathbb{R}_{>0})}\times_{\mu Sh_{\Lambda_-}}\mu Sh_L. \\
 & \nearrow_{(i_-^{-1},m_{\widetilde{L}})} & \\
\mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} & &
\end{array}$$

Proof. Using the property that the symplectic reduction $\Sigma L_s \cong L$, there is a commutative diagram

$$\begin{array}{ccccc}
\mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} & \xrightarrow{m_{\widetilde{L}}} & \mu Sh_L & \longleftarrow & \mu Sh_{\Sigma L} \\
\downarrow{\scriptstyle i_-^{-1}} & & \downarrow{\scriptstyle i_-^{-1}} & & \downarrow{\scriptstyle i_-^{-1}} \\
\mu Sh_{M\cup(\Lambda_-\times\mathbb{R}_{>0})} & \longrightarrow & \mu Sh_{\Lambda_-} & \longleftarrow & \mu Sh_L
\end{array}$$

The restriction $\tilde{j}_-^{-1}$ is the restriction functor on the homotopy pull back which commutes with the above diagram

$$\tilde{j}_-^{-1}:\mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})}\times_{\mu Sh_L}\mu Sh_{\Sigma L}\to\mu Sh_{M\cup(\Lambda_-\times\mathbb{R}_{>0})}\times_{\mu Sh_{\Lambda_-}}\mu Sh_L.$$

However, since ΣL is diffeomorphic to $L\times\mathbb{R}$, the restriction is an equivalence $\mu Sh_{\Sigma L}\xrightarrow{\sim}\mu Sh_L$, and the composition determined by the trivial cobordism is the identity $\mu Sh_L\xleftarrow{\sim}\mu Sh_{\Sigma L}\to\mu Sh_L$. Therefore, the composition is the microlocalization $m_{\widetilde{L}}$

$$\mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})}\xrightarrow{m_{\widetilde{L}}}\mu Sh_L\xleftarrow{\sim}\mu Sh_{\Sigma L}\to\mu Sh_L.$$

Therefore, the diagram in the statement commutes by the equivalence of the homotopy pull back

$$\mu Sh_{(M\times\mathbb{R}_{>0})\cup(\widetilde{L}\times\mathbb{R}_{>0})} \xrightarrow{\sim} \times_{\mu Sh_L}\mu Sh_{\Sigma L}$$

given by natural restriction functor and commutativity of all the restriction functors.

□

Combining Lemma 7.4.5, 7.4.6 and 7.4.7, we can conclude Theorem 7.0.9, which, by the discussion at the beginning of this section, implies that the Lagrangian cobordism functor obtained by the specialization functor over Lagrangian skeleta in Section 7.2, and the functor obtained by the conditional sheaf quantization functor on the Legendrian cobordism in Section 4.4 agree with each other.

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