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ABSTRACT

Dissertation on Economic Theory

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In this thesis, I study the effects of spillovers in all-pay auctions and the effects of regulating wages and hours on the labor market. In the first chapter, I study a model of asymmetric all-pay auctions with spillovers. In this model, players compete for a prize, and the sunk effort players exert during the conflict can affect the value of the winner's reward. The link between participants' efforts and rewards yields novel effects – in particular, players with higher costs and lower values than their opponent sometimes extract larger payoffs. In the second chapter, I study the problem of a labor market regulator who knows that workers prefer to work fewer hours at their current wage, but lacks specific knowledge of production and labor disutility. We show that moderate regulation (such as a small minimum wage) is counterproductive in that it results in hours that exceed the efficient quantity. We find that a combination of the minimum wage, overtime pay, and a cap on hours is optimal in a novel robust regulatory setting where the regulator has neither a prior nor exogenous bounds on model parameters.

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CHAPTER 1

Asymmetric All-Pay Auctions with Spillovers (with Maria Betto)

1.1. Introduction

All-pay auctions, or contests, model strategic interactions among players who must expend some non-refundable effort in order to win a prize. They have been applied in diverse settings such as labor (Rosen, 1986), R&D races (Che and Gale, 1998; Dasgupta, 1986), and litigation (Baye et al., 2005). For tractability, the recent literature mostly assumes that players' actions affect their opponent's probability of winning, but not the value of the prize. Yet, in many settings, such *spillover effects* on the prizes themselves arise naturally.

For example, consider the setting in Che and Gale (1998), where two lobbyists compete in an all-pay auction to win an incumbent politician's favor through campaign contributions. If the politician were instead a candidate running for office, then she would only be able to provide the reward if successfully elected. In this case, it is natural to assume that total campaign contributions increase the candidate's chances of prevailing. Therefore, each lobbyist's contributions increase her opponent's value for winning the politician's political favor. This raises new questions: is it better to curb one's own contributions to make their opponent lose interest? Or is it preferable to ramp up the competition? These questions have been largely left unanswered.

In other settings, spillovers may be designed. Consider an all-pay version of a standard labor tournament, in which division managers apply effort towards some production technology in order to win a promotion awarded to the most productive division. To maximize aggregate effort, a principal might choose to make the value of this promotion depend on everyone's performance in the contest. For example, if the promotion is for a partnership or involves stock options, the prize will be increasing in the efforts of all players. The effect that such compensation schemes have on the equilibrium has not yet been studied.

This paper fully identifies the equilibrium strategies and payoffs in general two-player auctions with spillovers and establishes their uniqueness.[1] We consider games with (i) complete information, (ii) deterministic prizes, (iii) at least partially sunk investment costs, and (iv) a general dependence of each participant's value for the prize on both players' actions. The key contribution of this paper lies in incorporating (iv). Indeed, all-pay contests without spillovers were extensively studied by Siegel (2009, 2010). These papers fully characterize equilibrium strategies and payoffs in games where contestants incur some (partially) unrecoverable cost, such as effort, in order to compete for prizes. We generalize the two-player, single-prize version of their model to allow for general spillovers to affect the winner's payoff. Our paper also has some overlap with the symmetric linear contests with spillovers studied in Baye et al. (2012). Unlike their work, however, we restrict attention to the all-pay case, but allow for asymmetric equilibria and nonlinear

[1]This paper also establishes the existence of equilibrium, though this result has already been proven; see Olszewski and Siegel (2022), for example. Our method, however, differs substantially from the previous literature.

payoffs. Even in the symmetric, linear all-pay auction with spillovers, we note that no previous paper that we are aware of has established equilibrium uniqueness.

The addition of spillovers can have a significant impact on equilibrium behavior. First, players with strictly higher costs can have higher payoffs than those with lower costs, even if their value functions for the prize are identical. In fact, in some settings, players could increase their payoffs if they were allowed to commit to a schedule of costly handicaps (See Section 1.4). Thus, trying to favor an "underdog" participant in a contest by means of reducing their costs may very well have the opposite effect, and in fact decrease their welfare in equilibrium. This is also important in settings in which players can commit to increasing their costs (e.g. by selecting an inefficient technology), as they may choose to do so.

Another contribution of this paper is the procedure to construct equilibrium strategy profiles. The equilibrium strategy distributions of asymmetric all-pay contests have two distinct parts: the densities and a mass point at zero. In the literature on all-pay contests without spillovers, starting with Baye et al. (1996), expected payoffs are obtained independently of the equilibrium distribution. This independence is exploited to derive the probability mass at zero for the weaker player from the payoffs, which is then used to compute the densities. In the presence of spillovers, however, a player's payoffs cannot be derived without the equilibrium strategy of their opponent. Because of this, the same process cannot be followed. To overcome this difficulty, we introduce an algorithm that works in exactly the opposite order: first, it solves for the density independently of the mass point, and then uses this density to find the probability mass at zero.

Our method capitalizes on the theory of Volterra Integral Equations (VIEs), which are integral equations with a unique fixed-point that can be obtained via iteration. To the best of our knowledge, these techniques have not previously been applied to the determination of equilibrium mixed-strategy profiles.[2]

The game we study is general enough to encompass many different applications in which spillovers matter. In particular, investment wars, contests with winner's regret, and military conflicts all fit our framework, since spillovers are key in each of these settings. Our model also subsumes a natural extension to the war of attrition which, unlike the classical model, yields a unique equilibrium on a bounded support. We are also able to use the same framework to describe wars of attrition where rational agents face uncompromising (never-yielding) types with positive probability, as in Abreu and Gul (2000) and Kambe (2019). Our approach identifies why these games admit unique equilibria when the regular war of attrition does not: the addition of an uncompromising type introduces an unavoidable cost that depends on a player's own score, and we show this single characteristic is sufficient in ensuring a unique equilibrium.

Finally, we extend the analysis to more than two players. The uniqueness result does not hold when the number of bidders exceeds two. We are nonetheless able to characterize a class of asymmetric equilibria when (appropriately normalized) costs are ranked. In this case, we show only two players participate in equilibrium. In addition, we are able to fully characterize the unique symmetric equilibrium for all-pay auctions with (i) more than two identical players, (ii) multiple homogeneous prizes, and (iii) spillovers generated by the

[2]Few other works in Economics use VIE methods in general. We note McAfee, McMillan, et al. (1989) and McAfee and Reny (1992) as some early examples. More recently, Gomes and Sweeney (2014) also used VIEs, to compute the unique efficient equilibrium bidding functions in generalized second-price auctions.

first runner-up. This setting accommodates a broad class of games including wars of attrition and auctions with winner's regret with any number of players and prizes. This extends the usefulness of our novel methodology.

The paper is organized as follows. We introduce the model, the equilibrium concept and the assumptions in Section 1.2. We construct the equilibrium and prove its uniqueness in Section 1.3. Section 1.4 presents sufficient conditions under which a player has a positive expected payoff. This includes an example where a player with higher costs and lower values receives a positive expected payoff, while her opponent receives zero. Section 1.5 contains useful results on closed forms that allows for simplified equilibrium computations in certain special cases. We illustrate their usage in the following Section 1.6, which is dedicated to applications. Sections 1.6.2, 1.6.4 and 1.6.3 in particular showcase closed-form solutions. Section 1.6.1 introduces a general perturbation of the classic war of attrition that ensures the equilibrium is unique. This perturbation admits the war of attrition with the possibility of an uncompromising type as a special case. In Section 1.7, we extend the analysis to contests with more than two players. Uniqueness no longer holds generally, though we are still able to find the unique symmetric equilibrium of a n-player, m-prize all-pay auction with spillovers. Finally, in Section 1.8, we review the related literature and discuss the results.

1.2. Model

We focus, for now, on auctions with two participants. Extensions with more players are considered in Section 1.7, where we show that the symmetric equilibrium of an auction with any number of identical players and prizes is just a transformation of the equilibrium of the two-player case.

An asymmetric auction with spillovers is a family $\{I, \{\tilde{S}_i\}_{i\in I}, \{u_i\}_{i\in I}\}$, where

(1) $I := \{1, 2\}$ is the index set of players.

(2) For each $i \in I$, $\tilde{S}_i := [0, \infty)$ is Player i's action space, i.e. her set of available scores (or bids). We use a tilde because a later assumption will allow us to replace the action set with a bounded interval. We let s_{-i},$\tilde{S}_{-i}$ denote the action and action space, respectively, of Player $j \neq i$.

(3) For each $i \in I$, $u_i : \tilde{S} \to \mathbb{R}$ is Player i's payoff, where $\tilde{S} := \prod_{i\in I} \tilde{S}_i$.

Let $s := (s_i; s_{-i})$ denote an arbitrary element of $\tilde{S}$. Then, for each $(s_i; s_{-i})$, we further define

$$u_i(s_i; s_{-i}) := p_i(s_i; s_{-i})v_i(s_i; s_{-i}) - c_i(s_i)$$

where (i) $p_i(s_i; s_{-i})$ denotes the probability that i wins the prize given the score profile $(s_i; s_{-i})$, with $p_i(s_i; s_{-i}) = 1 - p_{-i}(s_{-i}; s_i)$ and

$$p_i(s_i; s_{-i}) = \begin{cases} 1 & \text{if } s_i > s_{-i}, \\ \alpha_i \in [0, 1] & \text{if } s_i = s_{-i}, \\ 0 & \text{if } s_i < s_{-i}; \end{cases}$$

(ii) $v_i : \tilde{S} \to \mathbb{R}_+$ maps each score profile $(s_i; s_{-i})$ to Player i's value $v_i(s_i; s_{-i})$ from winning the prize, and (iii) $c_i : \tilde{S}_i \to \mathbb{R}_+$ outputs Player i's private cost $c_i(s_i)$ given her submitted score s_i.

Definition 1.1 (Two-player all-pay auction with spillovers)**.** A two-player all-pay auction is said to have *spillovers* if, for some $i \in I$ and $s_i \in \tilde{S}_i$, there exists $s_{-i}, \hat{s}_{-i} \in \tilde{S}_{-i}$ such that

$$v_i(s_i, s_{-i}) \neq v_i(s_i, \hat{s}_{-i})$$

i.e., the prize's value for at least one player and an action of that player is not constant in their opponent's action.

Accommodating spillovers is the distinguishing feature of our analysis. As is standard, we are interested in characterizing the Nash equilibrium of these general contests.

Definition 1.2 (Best-responses)**.** Consider a two-player all-pay auction $\{I, \{\tilde{S}_i\}_{i\in I}, \{u_i\}_{i\in I}\}$. For each $i \in I$, let $\Delta\tilde{S}_i$ denote the set of probability distributions on $\tilde{S}_i$ and let $\Delta\tilde{S} := \prod_{i\in I} \Delta\tilde{S}_i$. Player i's best response set $b_i(G_{-i})$ to $G_{-i} \in \Delta\tilde{S}_{-i}$ is given by

$$b_i(G_{-i}) := \arg\max_{s\in\tilde{S}_i} \int_{\tilde{S}_{-i}} u_i(s; s_{-i}) dG_{-i}(s_{-i})$$

Definition 1.3 (Nash equilibrium)**.** Consider the two-player all-pay auction $\{I, \{\tilde{S}_i\}_{i\in I}, \{u_i\}_{i\in I}\}$. A Nash equilibrium of this game is a profile $\mathbf{G}^\star := (G_i^\star)_{i\in I} \in \Pi_{i\in I}(\Delta\tilde{S}_i)$ where, for each $i \in I$, $G_i^\star$'s induced probability measure assigns measure one to $b_i(G_{-i}^\star)$.

1.2.1. Assumptions

The following assumptions are imposed throughout whenever a two-player all-pay auction is invoked. A.3 shows that none of these assumptions are superfluous to our results.

Assumption 1.1 (A1.1, Smoothness)**.** *The function* $v_i(s_i; y)$ *is continuously differentiable in* s_i *and continuous in* y *for all* $i \in I$*,* $s_i \in \tilde{S}_i$*, and* $y \in \tilde{S}_{-i}$ *with* $s_i \geq y$*. The function* $c_i(s_i)$ *is continuously differentiable in* s_i *for all* $i \in I$*,* $s_i \in \tilde{S}_i$*.*

Assumption 1.2 (A1.2, Monotonicity)**.** *For all* $i \in I$ *and* $s_i > 0$*,* $c_i'(s_i) > 0$ *and*

$$v_i'(s_i; y) < c_i'(s_i)$$

for almost all y*, where* $v_i'(s; y) := \partial v_i(s_i; y)/\partial s_i$*.*

Assumption 1.3 (A1.3, Interiority)**.** *For all* $i \in I$*,*

$$v_i(0,0) > c_i(0) = 0 \quad \textit{and} \quad \lim_{s_i \to \infty} \sup_{y \in \tilde{S}_{-i}} v_i(s_i; y) < \lim_{s_i \to \infty} c_i(s_i).$$

Versions of assumptions A1.1, A1.2, and A1.3 are adopted by most papers in the all-pay auction literature. A1.2 formalizes the sense in which these contests are all-pay, since bids are costly for both the winner and the loser.[3] A1.3 ensures that bids are positive and bounded.

Note that, for each $i \in I$, there exist $T_i \in \tilde{S}_i$ such that Player i will never choose a score $s \geq T_i$. Thus, we can restrict the action space to $S_i := [0, T_i]$.

Assumption 1.4 (A1.4, Discontinuity at ties)**.** *For all* $i \in I$ *and* $s \in S_i \cap S_{-i}$*,*

$$v_i(s; s) > 0.$$

[3]We note that A1.2 does exclude situations where a higher score is not necessarily more costly. Siegel (2014) discusses contests with nonmonotonic costs, allowing for competitors with head starts and the provision performance-based subsidies. These contingencies are excluded from our analysis.

Assumption A1.4 is a novel, yet natural assumption. It states that agents would prefer to win a tie than lose one. It is satisfied if the prize is always valuable (i.e. winning is better than losing), or if there are no spillovers. To see that it is never violated in the absence of spillovers, note that T_i is less than or equal to any x satisfying $v_i(x; y) \leq c_i(x)$ for all $y \leq x$. If there are no spillovers and $v_i(s) \leq 0$ for some $s \leq T_i$, then $c_i(s) \leq 0$. Therefore $s = 0$, which violates Assumption A1.3. Note that this assumption is equivalent to assuming a discontinuity in payoffs at ties because A1.1 and A1.3 guarantee that $v_i(s; s) \neq 0$ implies A1.4.

1.3. Characterization of equilibrium

By standard arguments contained in the Appendix, any pair of equilibrium strategies will be mixed with support on some interval $[0, \bar{s}]$, and at most one player will have a mass point at zero. Players must therefore be indifferent between all points of their interval support:

$$\bar{u}_i(G_{-i}) := \int_0^s v_i(s; y)\, dG_{-i}(y) - c_i(s) \qquad \text{for all } s \in [0, \bar{s}]. \tag{1.1}$$

Any pair of distributions (G_1, G_2) that satisfy (1.1) is an equilibrium. This paper's main contribution to the literature is in characterizing the solution to this system of equations, and in showing that it is unique.

Theorem 1.1. *Every two-player all-pay auction has a* ***unique Nash equilibrium*** $(G_i^\star)_{i \in I} \in \prod_{i \in I}(\Delta S_i)$ *in mixed strategies. Furthermore,*

$$G_i^\star(s) = \int_0^s \tilde{g}_i(y)\, dy + \int_{\bar{s}}^{\bar{s}_i} \tilde{g}_i(y)\, dy, \tag{1.2}$$

where $\tilde{g}_i(s)$ *solves*

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s;s)} - \int_0^s \frac{v'_{-i}(s;y)}{v_{-i}(s;s)}\tilde{g}_i(y)\,dy, \tag{1.3}$$

$\bar{s}_i$ *solves* $\int_0^{\bar{s}_i} \tilde{g}_i(y)dy = 1$ *and* $\bar{s} = \min_{i\in I} \bar{s}_i$. *The solution admits the following representation*

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s;s)} + \int_0^s r_{-i}(s;y)\frac{c'_{-i}(y)}{v_{-i}(y;y)}\,dy,$$

where

$$r_{-i}(s;y) := -k^0_{-i}(s;y) + k^1_{-i}(s,y) - k^2_{-i}(s;y) + \dots$$

for $k^0_{-i}(s;y) := \frac{v'_{-i}(s;y)}{v_{-i}(s;s)}$ *and* $k^n_{-i}(s;y)$, $n = 1, 2, \dots,$ *defined recursively by*

$$k^n_{-i}(s;y) := \int_y^s \frac{v'_{-i}(s;z)}{v_{-i}(s;s)}k^{n-1}_{-i}(z;y)\,dz.$$

We outline the proof here with an emphasis on the general methodology. We show in the appendix that in any equilibrium, players choose strictly increasing, continuous mixed strategies with common support on some interval $[0, \bar{s}]$, as in (1.1), and that at most one participant can have a mass point at zero. Moreover, differentiating (1.1) yields (1.3), which must be satisfied on $[0, \bar{s}]$ in equilibrium for some $\bar{s}$ (Lemma A.1 in the Appendix).

The key step is recognizing that we can apply results about Volterra Integral Equations (VIE) to show that (1.3) has a unique solution. The relevant result is summarized in the following Lemma. For a proof, see e.g. Brunner (2017).

Lemma 1.1. *Let $K(s;y)$ and $f(s)$ be continuous functions. Then, the following integral equation*

$$g(s) = f(s) + \int_0^s K(s;y)g(y)dy \qquad \text{for all } s \in [0,\bar{s}] \tag{1.4}$$

has a solution, g, unique almost everywhere. Moreover, (1.4) *defines a contraction mapping, implying the solution can be found by iteration. This iteration reduces to:*

$$g(s) = f(s) + \int_0^s R(s;y)f(y)dy,$$

where $R(s;y)$ is the unique resolvent kernel defined by

$$R(s;y) = \sum_{m=0}^{\infty} K_m(s;y)$$

where $K_0 \equiv K$ and K_m is defined recursively for $m = 1, 2, \ldots$ as

$$K_m(s;y) = \int_y^s K_{m-1}(s;z)K(z;y)\,dz.$$

Note that (1.4) is the same as (1.3) for $f(s) := c'_{-i}(s)/v_{-i}(s;s)$ and $K(s;y) := -v'_{-i}(s;y)/v_{-i}(s;s)$. So, Lemma 1.1 implies that only one pair of functions $(\tilde{g}_1, \tilde{g}_2)$ solves (1.3). Next we show that the unique solutions are densities, i.e. for each i there is an interval $[0, \bar{s}_i]$ where $\tilde{g}_i$ is non-negative and integrates to one.

Lemma 1.2. *Assume a two-player all-pay auction where $(\tilde{g}_i)_{i\in I}$ satisfies the indifference condition in* (1.3). *Then, for each $i \in I$, there exists $\bar{s}_i \in S_i$ such that*

$$\int_0^{\bar{s}_i} \tilde{g}_i(y)dy = \tilde{G}_i(\bar{s}_i) = 1, \tag{1.5}$$

and $\tilde{g}_i(s)$ is positive for $s \leq \bar{s}_i$.

Lemma 1.2 is proven in the Appendix. We must now ensure the two densities have the same support. The next key insight is that there is exactly one way to do this. Recall that at most one player can have an mass point and that this mass point must be at zero (Lemma A.1). If $\bar{s}_1 = \bar{s}_2$, then there is a unique equilibrium without any mass point. Otherwise, order the players such that $\bar{s}_1 < \bar{s}_2$. Then, give Player 2 a mass point of size $1 - \tilde{G}_2(\bar{s}_1)$. By construction, both players' densities integrate to one on the common support $[0, \bar{s}_1]$.

The above can be performed via the following steps:

(1) Find each $\tilde{g}_i(s)$.[4]

(2) Integrate each $\tilde{g}_i(s)$ to find $\bar{s}_i$ given by equation 1.5.

(3) Take $\bar{s} = \min_i \bar{s}_i$ and give each player an mass point at zero of size

$$1 - \tilde{G}_i(\bar{s}),$$

which is positive for at most one player.

The three steps are illustrated by Figure 1.1.

Since the cumulative distribution functions are useful, we sometimes use the alternate expression presented in Corollary 1.1.1.

Corollary 1.1.1. *Consider a two-player all-pay auction where $v_i(s; y)$ is continuously differentiable in both arguments for all $i \in I$ (A1.1 guarantees differentiability in the first*

[4] Analytically, it can be expressed as a series or in closed form when possible – see Section 1.5 – or numerically – see Appendix A.4.

argument). Then, we can alternatively express the unique equilibrium as

$$G_i(s) = \left[\tilde{G}_i(\overline{s}_i) - \tilde{G}_i(\overline{s})\right] + \tilde{G}_i(s),$$

where

$$\tilde{G}_i(s) = \frac{c_{-i}(s)}{v_{-i}(s;s)} + \int_0^s \frac{\partial v_{-i}(s;y)}{\partial y} \frac{\tilde{G}_i(y)}{v_{-i}(s;s)} dy. \tag{1.6}$$

The solution admits the following series representation

$$\tilde{G}_i(s) = \frac{c_{-i}(s)}{v_{-i}(s;s)} + \int_0^s \frac{c_{-i}(y)}{v_{-i}(y;y)} \frac{R_{-i}(s;y)}{v_{-i}(s;s)} \, dy$$

where

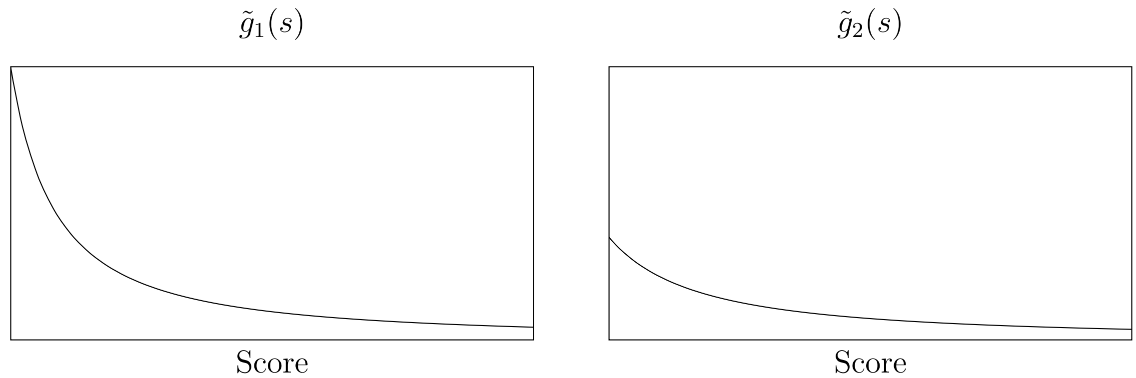
$$R_{-i}(s;y) := K^0_{-i}(s;y) + K^1_{-i}(s,y) + K^2_{-i}(s;y) + \dots$$

for $K^0_{-i}(s;y) := \partial v_{-i}(s;y)/\partial y$ *and* $K^n_{-i}(s;y)$, $n = 1, 2, \dots,$ *defined recursively by*

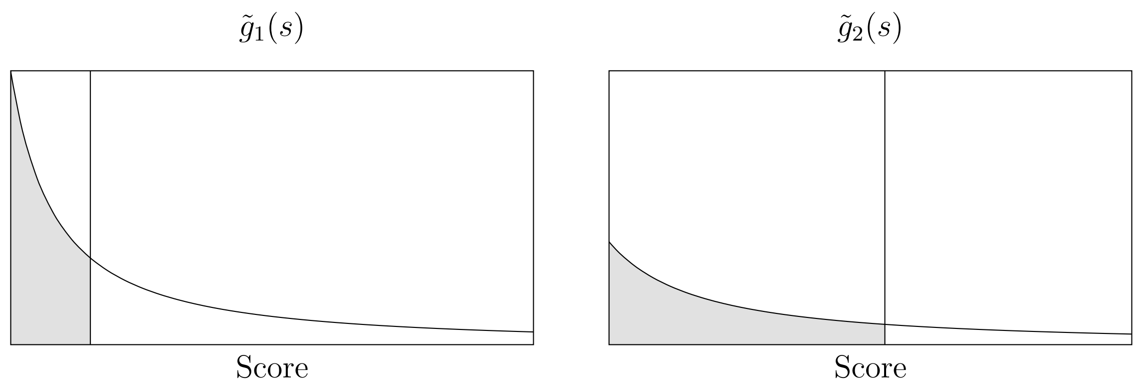
$$K^n_{-i}(s;y) := \int_y^s \frac{\partial v'_{-i}(s;z)}{\partial z} \frac{K^{n-1}_{-i}(z;y)}{v_{-i}(z;z)} \, dz.$$

We end this section with a note on parallels between our methodology and the one used in the incomplete information, all-pay auction literature. The similarities are formal in nature, and arise because both problems involve solving a pair of differential (in the incomplete information case) or integral (in our case) equations.

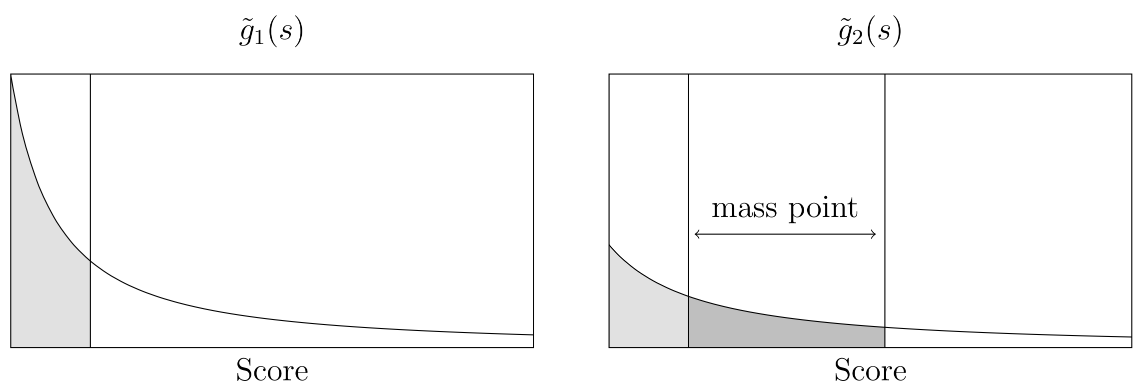
In the incomplete information setting of e.g. Amann and Leininger (1996), the unique equilibrium – in *pure strategies* – is obtained as the solution to a pair of differential equations, which arise from taking first-order conditions of each players' expected payoffs. To back out the mass of players types' that bid zero, the authors then make use of the



(a) Step 1: begin with $\tilde{g}_1$ and $\tilde{g}_2$



(b) Step 2: find the cutoffs where each $\tilde{g}_i$ integrates to 1



(c) Step 3: enforce identical supports by transferring any excess density to zero

Figure 1.1. Steps to find equilibrium strategies for a contest with $v_1(s_1; s_2) = 2 + s_1 + 2s_2$, $v_2(s_2; s_1) = 1 + s_2 + 2s_1$, $c_1(s_i) = 3s_1$, and $c_2(s_2) = 4s_2$.

boundary condition where each players' top type must, in equilibrium, choose identical top bids.

In our setting with complete information, the unique equilibrium is instead in *mixed strategies*. It is obtained as the solution to a pair of integral equations, which arise from *indifference*, rather than first-order conditions: players' payoffs must be invariant to any choice of bids within their mixed-strategy supports (Equation 1.1). We are then able to pin down the probability mass with which one of the two players bids zero through a different sort of "boundary condition"– specifically, the fact that both players' bidding distributions' supports must be identical, and integrate to 1 in that support.

1.4. Payoffs

Since payoffs are constant on the interval $[0, \bar{s}]$, each player i receives an expected payoff of $v_i(0;0)G_{-i}(0) \geq 0$. Only one player can have a mass point (at zero), so there can be at most one player – their opponent – with a positive payoff. In a symmetric contest, both players receive an expected payoff of zero. Theorem 1.1 immediately implies a necessary and sufficient condition for a player to have a positive payoff.

Corollary 1.1.2. *Consider a two-player all-pay auction. Player i has a positive payoff if, and only if, there exists an $\bar{s}_i$ such that*

$$\int_0^{\bar{s}_i} \left(\frac{c_i'(x)}{v_i(x;x)} + \int_0^x c_i'(y) \frac{r_i(x;y)}{v_i(y;y)}\, dy \right) dx < \int_0^{\bar{s}_i} \left(\frac{c_{-i}'(x)}{v_{-i}(x;x)} + \int_0^x c_{-i}'(y) \frac{r_{-i}(x;y)}{v_{-i}(y;y)}\, dy \right) dx = 1,$$

where $r_i(x;y)$, $r_{-i}(x;y)$ are defined as in Theorem 1.1.

Moreover, Player i's positive expected payoffs are given by:

$$\left[1-\int_0^{\bar{s}_i}\left(\frac{c_i'(x)}{v_i(x;x)}+\int_0^x c_i'(y)\frac{r_i(x;y)}{v_i(y;y)}\,dy\right)\,dx\right]v_i(0;0).$$

Corollary 1.1.2 fully characterizes the payoffs of any two-payer all-pay auction with spillovers in terms of the model's primitives. While it is very general, it is not easily verifiable, justifying the use of simpler sufficient conditions.

In contests without spillovers, as pointed out in Siegel (2009, 2010), it is easy to identify the player with a positive payoff when normalized costs (i.e. the cost-value ratio) are ranked. That is, if, for all $s > 0$,

$$\frac{c_i(s)}{v_i(s)} < \frac{c_{-i}(s)}{v_{-i}(s)} \tag{1.7}$$

holds in an auction with no spillovers, then player i has a positive payoff.

This is consistent with Corollary 1.1.2. We can combine Corollaries 1.1.2 and 1.1.1 to obtain the equivalent condition:

$$\frac{c_i(\bar{s})}{v_i(\bar{s};\bar{s})}+\int_0^{\bar{s}}\frac{\partial v_i(\bar{s};y)}{\partial y}\frac{\tilde{G}_{-i}(y)}{v_i(\bar{s};\bar{s})}dy < \frac{c_{-i}(\bar{s})}{v_{-i}(\bar{s};\bar{s})}+\int_0^{\bar{s}}\frac{\partial v_{-i}(\bar{s};y)}{\partial y}\frac{\tilde{G}_i(y)}{v_{-i}(\bar{s};\bar{s})}dy$$

In the absence of spillovers, the integral terms are equal to zero and the condition is implied by (1.7). In the presence of spillovers, one must impose a condition on the integrals.

Theorem 1.2. *Consider a two-player all-pay auction where $v_i(s;y)$ is continuously differentiable in s and y for all $i \in I$. Suppose that the following two conditions hold:*

$$\frac{c_i(s)}{v_i(s;s)} < \frac{c_{-i}(s)}{v_{-i}(s;s)} \tag{1.8}$$

$$\frac{1}{v_i(s;s)}\left|\frac{\partial v_i(s;y)}{\partial y}\right| \leq \frac{1}{v_{-i}(s;s)}\frac{\partial v_{-i}(s;y)}{\partial y} \tag{1.9}$$

for all $s \in (0, \bar{s}]$ and $y \in [0, s]$. Then, Player i has a positive payoff.

Theorem 1.2 gives an analogue of (1.7) for some contests with spillovers. The proof is in the Appendix. Condition (1.8) is the same as (1.7), while Condition (1.9) additionally imposes two extra requirements: (1) spillovers increase the value of the prize for Player $-i$ and (2) player i is less dependant on these spillovers than her opponent.

Unlike contests without spillovers, condition 1.7 is not sufficient for the conclusion of Theorem 1.2 to hold. Example 1.1, for instance, contains a situation where a player with strictly higher costs receives positive expected payoffs even when both players have exactly the same value function v for the prize.

Corollary 1.2.1 shows that the negative effect of spillovers is indeed a necessary condition for a reversal in contests with symmetric prize values.

Corollary 1.2.1. *Consider a two-player all-pay auction with spillovers. Suppose the players have the same value $v(s;y) \equiv v_1(s;y) = v_2(s;y)$, which is continuously differentiable in both arguments and $c_2(s) > c_1(s)$ for all s. Then Player 2 has a positive payoff only if*

$$\frac{\partial v(s;y)}{\partial y} < 0$$

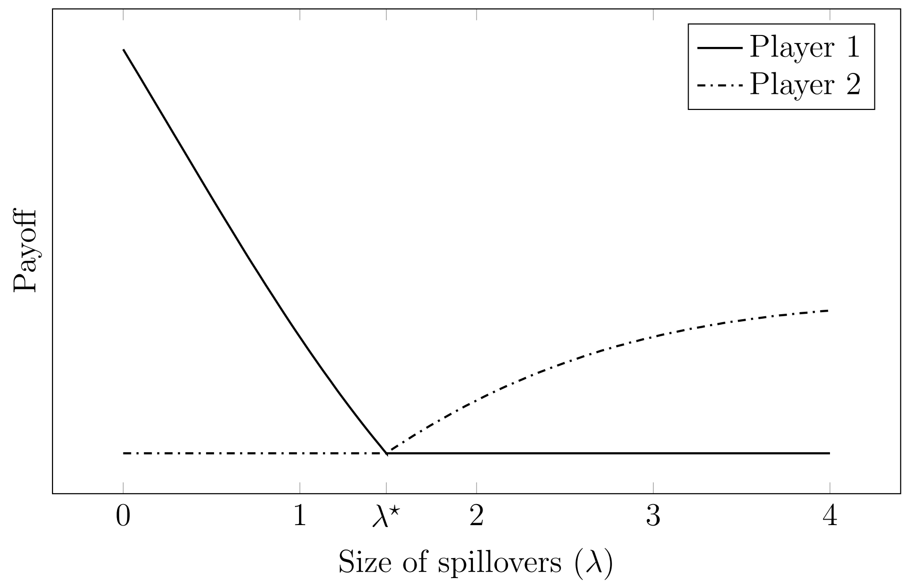


Figure 1.2. Player 2 has strictly higher costs. However Player 2 receives a positive payoff when spillovers are sufficiently large ($\lambda > \lambda^\star \approx 1.489$).

for some s, y.

Intuitively, in these cases, a marginal increase in effort reduces the prize's attractiveness to the opponent by eroding its value. Thus, a player with higher absolute costs may nevertheless have lower marginal costs at value ranges where the value erosion inflicted on the opponent is substantial enough to suppress her incentives to win.

Example 1.1 (Higher cost player has positive payoffs)**.** Consider a two-player contest with spillovers. Let $c_1(s) = s^2$, $c_2(s) = s$, and $v(s; y) := v_1(s; y) = v_2(s; y)$ be given by:

$$v(s; y) = \frac{2}{5} + \frac{1}{1 + e^{\lambda(2y-1)}}$$

where $\lambda \geq 0$ is an exogenous parameter that determines the size of the spillovers. The support of the players strategies is contained in $[0, 0.9]$. On this interval, Player 1 has a strict cost advantage.

When $\lambda = 0$, the prize is constant such that $v(s; y) = 0.9$. In this case, Player 1 receives a positive payoff. When we increase λ, this payoff decreases until we reach some $\lambda^\star \approx 1.489$ such that both players receive a payoff of zero. For all $\lambda > \lambda^\star$, Player 2 receives a positive payoff despite having strictly higher costs. The payoffs of both players are plotted in Figure 1.2.

$\triangle$

The reversal in Example 1.1 occurs because marginal costs are not ranked. While Player 1 has lower costs in absolute terms, Player 2 has a lower marginal cost for all scores above 1/2. This causes Player 2 to place comparatively more density on these bids. As can be seen in Figure 1.3, spillovers make the prize sharply less valuable when the opponent bids above 1/2. So, these higher bids from player 2 damage player 1's valuation enough to reduce her participation.[5]

Example 1.1 highlights a potential problem when giving one side an advantage in a contest. In the presence of spillovers, decreasing a player's costs can reduce their welfare

[5]The idea that underdogs may be harmed by policies that handicap stronger players appear in other settings in the literature. Kirkegaard (2013), for example, shows that in contests with three players and incomplete information, handicapping the strongest contestant may indeed harm the weakest one. Interestingly, the fact that marginal costs are not ranked is also key in their construction: if the weakest player has a marginal cost advantage over low, but not high, scores, she might find it worthwhile to participate with low bids in the absence of handicaps.

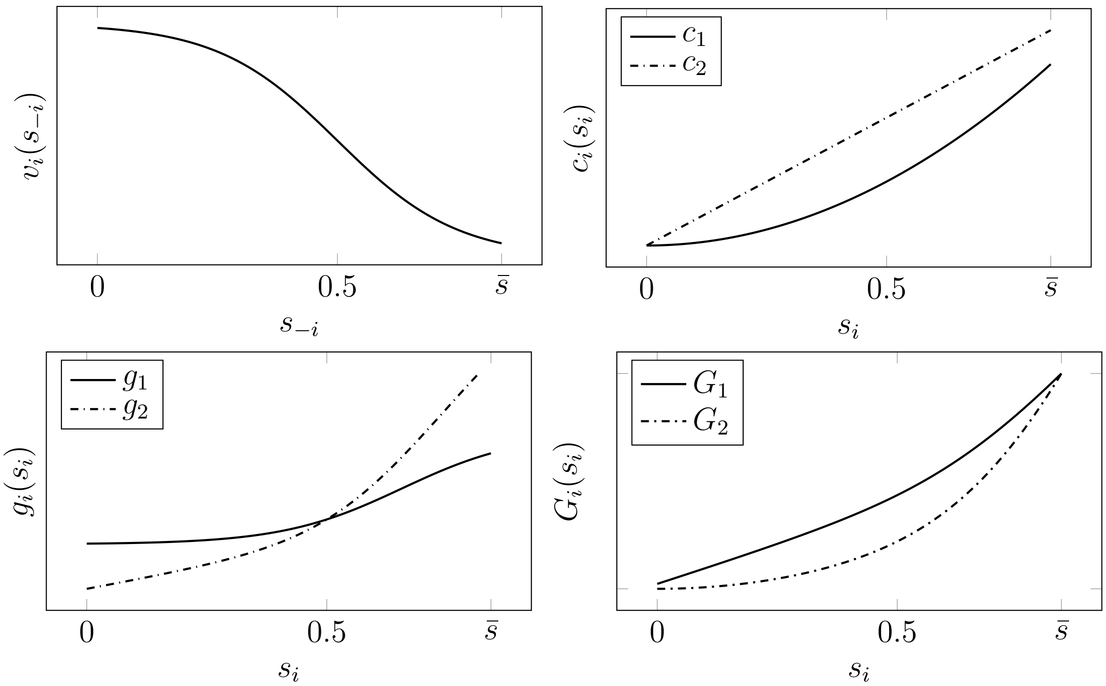


Figure 1.3. Value (upper left), costs (upper right), strategy densities (lower left), and distributions (lower right) for Example 1.1 with $\lambda = 4$. Note that Player 2 has a lower marginal cost for scores above 1/2 and, because of spillovers, these scores devalue the prize for Player 1.

in equilibrium. The example also implies that it's possible to have a contest where one or more players would prefer to ex-ante increase their own costs.[6]

Whenever marginal costs are ranked, the following proposition highlights necessary conditions for the high-marginal cost player to achieve a positive payoff. That is, the following conditions are necessary for a "reversal", where the higher cost player nonetheless obtains a positive payoff.

[6]Suppose both players are as in Example 1.1 except $c_1(s) = c_2(s) = s^2$. Then, the game is symmetric. So, both players have a payoff of zero. If player i increased her cost to $c_i(s) = s$, then she would receive a positive expected payoff, as in the example.

Proposition 1.1. *Consider a two-player all-pay auction. Suppose the players value for the prize is given by the function* $v(s;y) := v_1(s;y) = v_2(s;y)$*, and that* $c_2'(s) > c_1'(s)$ *for all* $s \in S_i \cap S_{-i}$*. Then Player 2 has a positive payoff only if all of the following apply*

(i) Costs are not scaled: there does not exist a $0 < \lambda < 1$ *such that* $c_1(s) = \lambda c_2(s)$ *for all* s*.*

(ii) There exist some $t, z \in S_i \cap S_{-i}$ *such that*

$$\frac{\partial v(t;z)}{\partial z} < 0.$$

(iii) There exist some $t, z \in S_i \cap S_{-i}$ *such that* $v'(t;z) > c_2'(t) - c_1'(t) > 0$*.*

(iv) There exists some $s \in S_i \cap S_{-i}$ *such that*

$$\max_{y \leq s} v'(s;y) - \min_{y \leq s} v'(s;y) > c_2'(s) - c_1'(s) > 0.$$

(v) There exist $t, z \in S_i \cap S_{-i}$ *such that*

$$\frac{\partial v(t;z)}{\partial t \partial z} < 0,$$

i.e. the common value function is not weakly supermodular.

Proposition 1.1, which is proven in the Appendix, is useful for contests where the opponent's actions are detrimental to the prize's value. It also pins down the circumstances under which the higher cost players can win: either the marginal costs are not ranked (as in Example 1.1) or all of the conditions 1-5 of Proposition 1.1 hold.

1.5. Closed forms

In some cases, it is possible to express the equilibrium strategies in closed form instead of as a series. We consider classes of prize value functions where this is possible. Section 1.6 contains applications of each of the propositions below.

Proposition 1.2. *Consider a two-player contest where $v'_{-i}(s_{-i}; y)$ does not depend on y. That is, for each $i \in I$, $v_{-i}(s_{-i}; y) = v_{-i}^{-i}(s_{-i}) + v_{-i}^{i}(y)$. Then,*

$$\tilde{G}_i(s) = \frac{1}{f(s)} \int_0^s \frac{c'_{-i}(y)}{v_{-i}(y; y)} f(y) dy, \tag{1.10}$$

where

$$f(y) := \exp\left(- \int_0^y \frac{(v_{-i}^{-i})'(u)}{v_{-i}(u; u)} du \right).$$

When spillovers are not linear, we might still be able to find closed form solutions to equilibrium strategies. We highlight two particular cases where the VIEs in Equations (1.3) and (1.6) can be solved using Laplace transforms.

Definition 1.4 (Laplace Transform)**.** [7] A function f defined on $\mathbb{R}_+$ admits a Laplace transform $F : \mathbb{C} \to \mathbb{C}$ given by

$$F(x) := \mathcal{L}\{f(s)\} = \int_0^\infty f(s) e^{-sx} ds$$

if and only if the above integral conditionally converges. That is, it does not need to converge absolutely.

[7] See Churchill (1972) for an exposition on Laplace transforms.

We require an extra technical assumption to ensure that the integral above converges. For simplicity, we will assume the relevant functions are of exponential order.

Definition 1.5 (Exponential order)**.** A function f is of exponential order if and only if there exist s', $q, M \in [0, \infty)$ such that, for all $s \geq s'$,

$$|f(s)| \leq Me^{qs}.$$

Proposition 1.3. *Assume a two-player contest such that (i) for some* i, $v_i'(s; y)/v_i(s; s) =: \nu_i(s-y)$ *depends only on the score differential* $s-y$, *and (ii)* ν_i *and* $c_{-i}'(s)/v_{-i}(s; s)$ *are of exponential order. Though, it is sufficient to assume that it admits a Laplace transform. Then, for all* $s \in (0, \overline{s}]$,

$$\tilde{g}_{-i}(s) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\left\{\frac{c_i'(s)}{v_i(s;s)}\right\}}{1+\mathcal{L}\{\nu_i(s)\}}\right\}$$

and

$$\tilde{G}_{-i}(s) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\left\{\frac{c_i'(s)}{v_i(s;s)}\right\}}{x+x\mathcal{L}\{\nu_i(s)\}}\right\},$$

where $\mathcal{L}$ *and* $\mathcal{L}^{-1}$ *denote the Laplace and inverse Laplace transforms, respectively.*

Proposition 1.3 can be used whenever the prize's value depends on the margin of victory, i.e. on the difference $(s - y)$ between the winning bid s and the losing bid y. We use Proposition 1.3 to solve the war of investment in Section 1.6.3.

Proposition 1.4. *Assume a two-player contest such (i) for some* i, $(v_i(s; s))^{-1}(\partial v_i(s; y)/\partial y) =: \psi_i(s-$ *depends only on the score differential* $s - y$, *and (ii)* $\partial v_i(s; y)/\partial y$ *and* ψ_i *and* $c_i(s)/v_i(s; s)$

are of exponential order. Then, for all $s \in (0, \overline{s}]$,

$$\tilde{g}_{-i}(s) = \mathcal{L}^{-1}\left\{ \frac{x\mathcal{L}\left\{\frac{c_i(s)}{v_i(s;s)}\right\}}{1-\mathcal{L}\left\{\psi_i(s)\right\}} \right\}$$

and

$$\tilde{G}_{-i}(s) = \mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\left\{\frac{c_i(s)}{v_i(s;s)}\right\}}{1-\mathcal{L}\left\{\psi_i(s)\right\}} \right\}$$

where $\mathcal{L}$ *and* $\mathcal{L}^{-1}$ *denote the Laplace and inverse Laplace transforms, respectively.*

Proposition 1.4 can be used whenever the prize value is of the form $v_i(s;y) = v_i^1(s)v_i^2(s-y)$. For an application where we solve an all-pay auction with winner's regret, see Section 1.6.4.

1.6. Applications

1.6.1. War of attrition with costly preparation

The canonical war of attrition is a game between two players $i = 1, 2$. Each picks a score, which represents an exit time, in $[0, \infty)$ and the player i to select the largest score s_i wins an amount that is decreasing in the loser's choice s_{-i} and constant in her own. A player's payoff function is thus given by:

$$u_i(s_i; s_{-i}) = \begin{cases} f_i(s_{-i}) & \text{if } s_i > s_{-i} \\ \ell_i(s_i) & \text{if } s_i < s_{-i} \\ \alpha_i f_i(s_{-i}) + (1-\alpha_i)\ell_i(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

where f_i, ℓ_i are strictly decreasing, continuously differentiable functions such that $f_i(s) > \ell_i(s)$, $\lim_{s\to\infty} \ell_i(s) = -\infty$, $\ell_i(0) = 0$, and $\alpha_i = 1 - \alpha_{-i} \in (0, 1)$.

The typical WoA admits multiple equilibria and therefore does not satisfy the assumptions in Section 1.2.1. In particular, it violates monotonicity (A1.2) and interiority (A1.3) because the payoff of the winner is constant (and therefore non-decreasing) in the Player's own score.

We propose a general perturbation that selects a unique equilibrium of the WoA, and show that such a perturbation is solvable under our framework.[8]

Suppose, the winner's outcome is decreasing in her own score – even if this dependence is minimal:

$$u_i(s_i; s_{-i}) = \begin{cases} f_i(s_{-i}) - \varepsilon_i(s_i) & \text{if } s_i > s_{-i} \\ \ell_i(s_i) - \varepsilon_i(s_i) & \text{if } s_i < s_{-i} \\ \alpha_i f_i(s_{-i}) + (1-\alpha_i)\ell_i(s_i) - \varepsilon_i(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

for any strictly increasing continuously differentiable function ε_i with $\varepsilon_i(0) = 0$ and $\lim_{s\to\infty} \varepsilon_i(s) > f_i(0)$ for all i.

We denominate this variant a *WoA with costly preparation*, as there is some small preparation cost $\varepsilon(s)$ incurred to set score s – i.e. the maximum amount of time s one wishes to participate for. For example, a company engaged in a price war might have to build up inventory in advance or secure a costly line of credit.

[8] The problem of equilibrium selection in WoAs has been widely studied in the literature (Georgiadis et al., 2022; Myatt, 2005). One way to select a unique equilibrium is to truncate the game, as in Ghemawat and Nalebuff (1985), so that at some point in finite time both players prefer to exit. A different way to select for an equilibrium, which we discuss in more detail, is to introduce a small probability that a player that never exits. See, for example, Abreu and Gul (2000), Kambe (2019), and Kornhauser et al. (1989).

A WoA with costly preparation fits the two-player all-pay auction with spillovers where

$$v_i(s_i; s_{-i}) := f_i(s_{-i}) - \ell_i(s_i) \text{ and}$$

$$c_i(s_i) := \varepsilon_i(s_i) - \ell_i(s_i),$$

which satisfy assumptions A1.1-1.4. Therefore, this game has a unique equilibrium, and there exists some $\bar{s}$ such that no player bids above $\bar{s}$. Theorem 1.1 further allows us to characterize the equilibrium and Proposition 1.2 gives a closed form expression for the equilibrium strategies.

As the preparation costs become small (with $\varepsilon_i'(s) \to 0$ uniformly for all s), the unique equilibrium of a WoA with costly preparation approaches the mixed-strategy equilibrium of the classic WoA. This is proven in the Appendix.

The WoA with costly preparation generalizes other perturbations that have a unique equilibrium. For example, Abreu and Gul (2000) and Kambe (2019) extend the WoA to let a rational player's opponent be of an uncompromising type with positive probability, where "uncompromising" describes someone who bids (or exits at) infinity. Let z_i denote the (known) probability that player i is of an uncompromising type. Against such an opponent, a rational or compromising player loses with certainty. This is a special case of the WoA with costly preparation where $\varepsilon_i(s) := -(z_{-i}/(1 - z_{-i}))\ell_i(s)$.

This relationship sheds light on the uniqueness of equilibrium found in the WoA with an uncompromising type. Indeed, by adding the possibility of a never-yielding opponent, we effectively introduce an unavoidable cost that depends on the player's own score. As was shown in the WoA with costly preparation, this characteristic is actually sufficient for a unique equilibrium.

1.6.2. Offensive/defensive balance

Military strategists generally agree that warfare is naturally asymmetric: the defending party can usually prevail with less expenditure of resources than the attacker (Clausewitz, 1982). More generally, scholars have tried to identify which factors influence the so-called offensive/defensive balance – that is, the many elements of military technology that generate either offensive or defensive advantages, and thus affect the probability of war (Levy, 1984). Our model is able to capture both the defensive advantage and the role of the prize-depleting nature of war in the offensive/defensive balance debate.

An attacker (a) invades a defender's (d) territory, which is worth V. Both combatants purchase costly scores in $[0, \infty)$, and the combatant with the higher score wins. A score of s_i costs $c_i s_i$, where $c_i > 0$ is a positive constant, for player $i \in I := \{a, d\}$. Furthermore, a's score inflicts $\delta_a s_a$ damage to the territory. Assuming the defender also inflicts a cost of $\delta_d s_d$ onto the attacker does not change the analysis. If the attacker wins, it internalizes all costs faced by the defender, as these costs effectively depleted the resources available from the territory. Consider the following payoff functions $u_a : [0, \infty) \to \mathbb{R}$ for the attacker:

$$u_a(s_a, s_d) = p_a(s_a, s_d)(V - \delta_a s_a) - c_a s_a,$$

and the following payoff function $u_d : [0, \infty) \to \mathbb{R}$ for the defender:

$$u_d(s_a, s_d) = (1 - p_a(s_a, s_d))(V - \delta_a s_a) - c_d s_d,$$

where $p_a(\cdot) : [0,\infty)^2 \to [0,1]$ denotes the probability that the attacker is victorious. Accordingly, we let $p_a(s_a, s_d) = 1$ whenever $s_a > s_d$, $p_a(s_a, s_d) = 0$ when $s_a < s_d$, and $p_a(s_a, s_d) = \lambda \in [0,1]$ whenever $s_a = s_d$.

When we transform this model into our framework, we get $c_i(s_i) := c_i s_i$ and

$$v_a(s_a; s_d) = v_d(s_d; s_a) := V - \delta_a s_a.$$

Assume it costs weakly more to attack than to defend (i.e., $c_a \geq c_d$). The attacker does not have any spillovers while the defender is harmed by her opponent.

We are able to leverage the linearity of payoffs in this case to obtain a closed-form solution to the problem using Proposition 1.2. The defender receives positive payoffs if, and only if,

$$\bar{s}_d = \frac{V}{c_a + \delta_a} < \frac{V}{\delta_a}\left[1 - \exp\left(-\frac{\delta_a}{c_d}\right)\right] = \bar{s}_a,$$

which holds whenever $\delta_a > 0$ and $c_a \geq c_d$. In this case,

$$G_a(s) = 1 + \frac{c_d}{\delta_a}\log\left[\frac{c_a V}{(c_a + \delta_a)(V - \delta_a s)}\right] \quad \text{and} \quad G_d(s) = \frac{c_a s}{V - \delta_a s}.$$

The probability $P(s_a > s_d|\delta_a, c_a, c_d)$ that the attacker succeeds, in equilibrium, is given by

$$P(s_a > s_d|\delta_a, c_a, c_d) = \frac{c_d}{\delta_a^2}\left(\delta_a + c_a \log\left[\frac{c_a}{c_a + \delta_a}\right]\right) < \frac{c_d}{2c_a} \leq \frac{1}{2},$$

where the supremum is reached as $\delta_a \to 0$. If the war damages the territory at least as much as it costs the attacker to inflict such damage, ($\delta_a \geq c_a$), a tighter bound is obtained:

$$P(s_a > s_d|\delta_a, c_a, c_d) < 1 - \log(2) < \frac{1}{3}.$$

Even if $c_a = c_d$, the defender is more than twice as likely to win than the attacker is. In our model, the stronger position of the defensive party comes as a byproduct of the inverse relationship between the attacker's strength and the erosion of the prize's value. This provides an alternate explanation on why it is typically easier to defend than to attack, something usually attributed to the high costs of maintaining long supply lines and of keeping seized territories (Glaser and Kaufmann, 1998). The defender's stronger position also suggests that any positive participation cost in a war contest imposed on the aggressor would be effective in discouraging aggression. [9]

1.6.3. War of investment

Investment has long been considered as a method of committing to entry deterrence (Dixit, 1980), while the war of attrition is a popular model of exit (Fudenberg and Tirole, 1986). Our model can combine the two attributes into a single model of competition in continuous time, where players invest to stay in the game, but are able to recoup part of that investment if their opponent invests less. Wars of investment can also be used to model Cold-War style defense spending and competition between technology companies and R&D races.

Assume two competitors, 1 and 2, invest in capital s_i at cost $c_i(s_i)$. The capital is necessary to engage in competition and depreciates at a constant rate. Competition results in zero profits. However, the winner is able to extract monopoly profits and benefits from

[9] In the more general nonlinear model, where the value of the territory after invasion is given by $v_\delta(s_a)$ and the cost of choosing score s_i to player i is given by a continuously differentiable function $c_i : [0, \infty) \to \mathbb{R}_+$ satisfying the required assumptions A1.1 to A1.4, $c_d(s) \leq c_a(s)$ is sufficient to ensure that $\tilde{G}_a(s) < \tilde{G}_d(s)$ for all $s > 0$. This guarantees the defender's payoff remains positive, with $G_d(s) = c_a(s)/v_\delta(s)$.

the remaining capital according to an increasing function $v_i(s_i - s_{-i})$. More concretely, assume payoffs are

$$u(s_i; s_{-i}) = \begin{cases} v_i(s_i - s_{-i}) - c_i(s_i) & \text{if } s_i > s_{-i} \\ -c_i(s_i) & \text{if } s_i < s_{-i} \\ \alpha_i v_i(0) - c_i(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

for any $\alpha_i \in [0, 1)$.

If assumptions A1-4 are met, there is a unique equilibrium of capital investments in mixed strategies on finite support. Moreover, the equilibrium admits a closed-form solution by Proposition 1.3.

Example 1.2. Let $v_i(s; y) := e^{\rho_i(s-y)}\omega_i$ and $c_i(s) = e^{\rho_i s} - 1$, where $\omega_i, \rho_i \in (0, 1)$ for each $i \in I := \{1, 2\}$. Then,

$$\tilde{g}_i(s) = \frac{\rho_{-i}}{\omega_{-i}}$$

so the equilibrium strategies, excluding the possible mass point at zero, will be uniform with $\tilde{G}_i(s) = (\rho_{-i}/\omega_{-i})\, s$.

The pair of ratios ω_i/ρ_i is therefore a sufficient statistic for the equilibrium of this game. Assume, without loss of generality that this ratio is weakly larger for Player 1. Then, the maximum duration of the game is Player 2's ratio $\bar{s} = \omega_2/\rho_2$.

The equilibrium is fully characterized by the overall *strength* of the players $\bar{s}$ and the *competitive balance* $\delta := (\omega_2/\rho_2)/(\omega_1/\rho_1) \in (0, 1]$.

Because the strategies are uniform, Player 1's average commitment duration is half of the strength. Player 2 on the other hand has a mass point of size

$$G_2(0) = 1 - \delta$$

which decreases as the competition becomes more balanced.

Overall, the conflict is expected to last for

$$\mathbb{E}[\min(s_1, s_2)] = \int_0^{\bar{s}} (1 - G_1(y))(1 - G_2(y))dy = \frac{\delta \bar{s}}{3}$$

total periods. The relationship between overall power and war duration is one to one. The duration is also increasing in the competitive balance. So, a large strength differential implies the conflict will typically be short-lived, whereas close contests can have delayed resolutions.

△

1.6.4. All-pay auction with winner's regret

Winner's regret is the remorse that the winner has from spending more than is necessary to win a contest or auction. This phenomenon has mostly been studied in the context of winner-pay first-price, auctions (Engelbrecht-Wiggans, 1989; Filiz-Ozbay and Ozbay, 2007). We instead apply our framework to model winner's regret in an all-pay auction.

Let each Player $i \in I := \{1, 2\}$ choose a score in $[0, \infty)$. Suppose i values the prize at $\mu_i(s_i)[1 - h_i(s_i - s_{-i})]$, where $\mu_i(s_i)$ is the player's objective value of the prize and $h_i(s_i - s_{-i})$ is the share of the winnings that is unappreciated due to regret. Each player pays the cost $c_i(s_i)$ whether they win or lose. So payoffs are

$$u(s_i; s_{-i}) = \begin{cases} \mu_i(s_i)[1 - h_i(s_i - s_{-i})] - c_i(s_i) & \text{if } s_i > s_{-i} \\ -c_i(s_i) & \text{if } s_i < s_{-i}, \\ \alpha_i \mu_i(s_i) - c_i(s_i) & \text{if } s_i = s_{-i}, \end{cases}$$

for any $\alpha_i \in [0,1)$. We assume all functions are continuously differentiable with $c'(s) > 0$ and $h_i'(s) \geq 0$. Moreover, $\mu(0) > h(0) = c(0) = 0$ and $c'(s) > \mu'(s)$ for each s, so that lower bids are preferable even with no regret. Intuitively, the regret function, h, should not exceed one. Though, this is not a technical requirement. We can solve for the equilibrium with two Players using Proposition 1.4 and we can extend this equilibrium the game with any number of identical players or prizes using Theorem 1.4.

Example 1.3. Let $\mu_i(s) := \omega_i \in \left(0, \frac{1}{2}\right)$, $h_i(s) = s^2/2$ and $c_i(s) := s - s^2/2$ for $s \in [0,1]$. Then,

$$\tilde{g}_i(s) = \frac{e^{-s}}{\omega_{-i}}.$$

Without loss of generality, let $\omega_1 \geq \omega_2$, implying Player 1 receives a non-negative payoff. Player 1 will thus play a truncated exponential distribution with parameter 1 and support $[0, -\log(1-\omega_2)]$. Her expected score will be:

$$\mathbb{E}[s_1|\omega_1, \omega_2] = 1 + \left(\frac{1-\omega_2}{\omega_2}\right)\log(1-\omega_2).$$

which depends negatively on her opponent's payoff scaling factor ω_2. This is lower than in the same game without regret.

The player with zero expected payoffs will place a mass point at zero of size:

$$G_2(0) = 1 - \frac{\omega_2}{\omega_1}$$

which is exactly the same size as if there were no regret. Player 2 will have expected score

$$\mathbb{E}[s_2|\omega_1, \omega_2] = \frac{\omega_2}{\omega_1}\left[1 + \left(\frac{1-\omega_2}{\omega_2}\right)\log(1-\omega_2)\right]$$

which is also less than in the same game without regret. The expected sum of the two scores score is:

$$\mathbb{E}[s_1 + s_2|\omega_1, \omega_2] = \left(1 + \frac{\omega_2}{\omega_1}\right)\left[1 + \left(\frac{1-\omega_2}{\omega_2}\right)\log(1-\omega_2)\right]$$

which is decreasing in ω_1 and increasing in ω_2. In contests such as a labor tournaments, a large productivity differential between participants in the form of a high ω_1 and low ω_2 depresses aggregate effort. This is true in a contest with no spillovers, but the partial derivative of ω_1 is larger in absolute value when there is regret. That is, the effect is exacerbated by the fact that the stronger player is penalized for winning by a large margin. △

1.7. More players

In contests with spillovers and more than two players, many of the results considered here are violated. Existence still holds (see Olszewski and Siegel, 2022), but uniqueness does not. Moreover, expected payoffs will now depend on which equilibrium is played.[10]

[10] This is also true of contests with no spillovers if monotonicity does not hold (Siegel, 2009, Example 2).

When the normalized costs are ranked, Theorem 2 in Siegel (2010) and Theorem 2 in Siegel (2009) show that only two players ever participate in the equilibrium of a contest for a single prize. This effectively collapses the problem into a two-player contest.

A version of this condition holds in our setting. We still require normalized costs to be ranked in some sense, but in a way that takes the spillovers into account.

Theorem 1.3. *Assume i, j, $i \neq j$, are two of the $n > 2$ players in a contest satisfying assumptions A1.1 to A1.4 Assume i, j, $i \neq j$, are two of the $n > 2$ players in a contest satisfying assumptions A1.1 to A1.4. Suppose that Player i has a positive payoff in the two-player contest where i and j are the participants, and that the following "ranked costs" condition holds for all $k \notin \{i, j\}$, $s \in \tilde{S}_k$,$s_i \in \tilde{S}_i$ and $s_j \in \tilde{S}_j$*

$$\frac{c_k(s)}{v_k\left(s; \mathbf{s}_{\{i,j\}}\right)} \geq \frac{c_j(s)}{v_j\left(s; \mathbf{s}_{\{i\}}\right)}, \tag{1.11}$$

where $\mathbf{s}_H$ is a vector of opponent scores that is zero for all players not in set H. Then, there exists an equilibrium where only Players i and j participate.

To understand condition (1.11), consider the candidate equilibrium where Players i and j compete using their two-player strategies and Player k does not participate. By not participating, Player k earns a payoff of zero – the same payoff as Player j. Condition (1.11) says that if she enters, Player k's normalized cost will be higher at every point than Player j's already is. Therefore, her payoff from participating is at most zero (Player j's payoff). So, there is no profitable deviation for any player.

Note that it is possible for $k \succ j$ and $j \succ k$ in the sense of (1.11) when spillovers decrease the value of the prize. In this case, there are multiple equilibria where different pairs of players participate.

In the absence of spillovers, multiple equilibria also arise with three or more players. However, if payoffs are asymmetric, there can be at most one equilibrium where the support of each player's strategy is a union of intervals. Additionally, the payoffs of each player are consistent across all equilibria. Neither of these properties hold in contests with spillovers. Payoffs generally vary across equilibria in which different Players participate.

1.7.1. Symmetric equilibria

The same method used to find the equilibrium of two-player auctions with spillovers can be applied more generally to find symmetric equilibria of all-pay auctions with $n > 2$ identical players and $m < n$ prizes, where the value of the prize for any given participant depends on their own score and on the score of the first runner-up (the player with the $m + 1$-th highest bid). More specifically, each prize has value $v(s; y)$ where s is the player's own score and y is the score of the first runner-up. When there is only one prize, this amounts to saying that its value depends only on the two highest bids. Spillovers depend only on the score of the runner up in many games such as the all-pay auction with winner's regret and any game with a structure that resembles a war of attrition. For example, bargaining games and free riding games frequently have this structure where the last holdout to comply delays the prize for the winners. In the case where there is one prize, spillovers that depend on the first runner up capture the margin of victory which is relevant in many applications including elections and R&D races.

Formally, we define a symmetric auction with runner-up spillovers as a family $\{I, P, \{\tilde{S}_i\}_{i\in I}, \{u_i\}_{i\in I}\}$ where

(1) $I := \{1, 2, \ldots, n\}$ is the index set of players, with $n \geq 2$.

(2) $P := \{1, 2, \ldots, m\}$ is the index set of prizes, with $m < n$.

(3) For each $i \in I$, $\tilde{S}_i := [0, \infty)$ is Player i's action space. We let $\mathbf{s}_{-i}$ denote an arbitrary element of $\tilde{S}_{-i} := \prod_{j\neq i} \tilde{S}_j$. We further let $s_{(j)}$ denote the j-th highest score.

(4) For each $i \in I$, $u_i : \tilde{S} \to \mathbb{R}$, where $\tilde{S} = \prod_{i\in I} \tilde{S}_i$.

For each $\mathbf{s} := (s_i; \mathbf{s}_{-i}) \in \tilde{S}$, we further define:

$$u_i(\mathbf{s}) := p_i(\mathbf{s})v(s_i, s_{(m+1)}) - c(s_i)$$

where (i) $p_i(\mathbf{s})$ denotes the probability that i wins a prize given the score profile $\mathbf{s}$, with $\sum_{i\in I} p_i(\mathbf{s}) = m$ and

$$p_i(\mathbf{s}) = \begin{cases} 1 & \text{if } s_i \geq s_{(m)} > s_{(m+1)}, \\ \frac{1}{|\{k\in I: s_k = s_{(m)}\}|} & \text{if } s_i = s_{(m)} = s_{(m+1)}, \\ 0 & \text{if } s_i \leq s_{(m+1)} < s_{(m)}; \end{cases}$$

(ii) $v : [0, \infty)^2 \to \mathbb{R}_+$ maps each pair of scores $(s_i, s_{(m+1)})$ to Player i's value $v(s_i; s_{(m+1)})$ from winning the prize, and (iii) $c : [0, \infty) \to \mathbb{R}_+$ outputs Player i's private cost $c(s_i)$ given her submitted score s_i.

In contrast with the two-player case introduced in Section 1.2, here we assume all players are symmetric in the sense that they have identical value (v) and cost (c) functions. Moreover, all prizes are equally valuable to each player i conditional on $(s_i, s_{(m+1)})$.

In this context, we are able to use the two-player, one prize equilibrium characterized in Theorem 1.1 to construct the symmetric equilibria of a symmetric n-player, m-prize all-pay auction with spillovers

Theorem 1.4. *Consider a symmetric n-player, m-prize all-pay auction with runner-up spillovers. Assume v, c satisfy assumptions A1.1 to A1.4. Let $\hat{G}$ be defined as in Corollary 1.1.1. That is, let $\hat{G}$ be the equilibrium cumulative distribution function of a two-player all-pay auction with spillovers:*

$$\hat{G}(s) = \frac{c(s)}{v(s,s)} + \int_0^s \frac{c(y)}{v(y,y)} \frac{R(s,y)}{v(s,s)} \, dy,$$

with

$$R(s,y) = K^0(s,y) + K^1(s,y) + K^2(s,y) + \dots$$

for $K^0(s,y) = \partial v(s,y)/\partial y$ and $K^t(s,y)$, $t = 1, 2, \dots,$ defined recursively by $K^t(s,y) := \int_y^s (\partial v(s,z)/\partial z)(K^t(z,y)/v(z,z))\, dz$.

Then, the symmetric equilibrium of the n-player, m-prize all-pay auction with runner-up spillovers is given by the unique G that solves:

$$\hat{G}(s) = \sum_{j=n-m}^{n-1} \binom{n-1}{j} [G(s)]^j [1 - G(s)]^{n-j-1}. \tag{1.12}$$

To see why Theorem 1.4 holds, consider the expected payoff of a Player i who bids s:

$$\int_0^s v(s;y) \, d\hat{G}(y) - c(s),$$

where $\hat{G}$ is the probability measure of the m-th largest score out of the $n-1$ players in $I \setminus \{i\}$. In a symmetric equilibrium, $\hat{G}$ is the $n-m$-th order statistic of a sample of $n-1$

draws from the equilibrium distribution G, giving us (1.12). The right-hand side of (1.12) is increasing in $G(s)$, and thus may be inverted to obtain G given $\hat{G}$.

At the same time, $\hat{G}$ is the equilibrium of a symmetric two-player auction, since each Player's indifference condition is identical to (1.3). This allows us to use Theorem 1.1 to find $\hat{G}$.

Theorem 1.4 shows that the equilibrium in the two-player case is also the symmetric equilibrium of the game with any number of players and prizes, subject to a particular monotone transformation. Intuitively, increasing the number of players (or decreasing the number of prizes) reduces the scores of each player.

Asymmetric equilibria are more difficult to characterize when either spillovers or incomplete information are present. In a world without spillovers but complete information, the equilibrium of an n-player, m-prize contest is unique under mild conditions, and Siegel (2010) was able to provide an algorithm for its construction. Under incomplete-information, asymmetric equilibria with more than two players are notoriously difficult to analyze. A general characterization is still an open question, as far as we are aware.[11]

1.8. Related literature and conclusions

Throughout this paper, we characterized and established uniqueness for the equilibrium of two-player contests using techniques from the theory of integral equations. We then extended these results to symmetric contests with more players and prizes to characterize the symmetric equilibrium. This allowed us to derive insights on equilibrium payoffs, winners and losers, and on the importance of spillovers for applications. This

[11] See Kirkegaard (2013), Parreiras and Rubinchik (2010) for analyses on particular equilibria of $N \geq 3$-players all-pay, incomplete-information auctions.

model does not require or imply that the results of a contest are known in advance. In fact, players are always uncertain of their own victory. However, this uncertainty stems from not knowing the resources that your opponent dedicated to the contest. The fact that ranked normalized costs are not enough to establish dominance demonstrates how spillovers can favor high-cost, low-value players that nevertheless have a marginal cost advantage over their opponent when bids are high. In particular, the results in this paper suggest several potential consequences of legal structures, conflicts and competition.

This paper is most closely related to two others. Baye et al. (2012), also considers spillovers in two-player contests, but focuses on symmetric equilibria and linear symmetric costs and valuations. We show that there are no asymmetric equilibria in this two player case and extend the analysis to include asymmetric players and general functional forms for the prize values. This allows us to establish equilibrium uniqueness, express novel results about payoffs, and characterize the equilibrium in different applications (Sections 1.6.1 and 1.6).

The second paper that approaches a similar question to our own is Xiao (2018). The author, however, focuses on constant prize values and separable spillovers in the cost functions, which are independent of winning or losing. This independence significantly restricts the equilibrium effects of the spillovers. Linearly separable spillovers on the cost have no effect on the equilibrium, while multiplicatively separable spillovers scale the cost of bids by an endogenous constant. This is not true when spillovers are in the prize value.

Fu and Lu (2013) also analyze two-player all-pay auctions with linear spillovers in the costs. They assume that each contestant is a firm with a minority stake in their opponent's profits. As such, even when a firm loses the auction, they still get to keep a share of the prize. On the other hand, regardless of winning or losing, they must also share in on the cost of effort incurred by their opponent – hence, the existence of cost spillovers. Because these spillovers are linear and do not affect the prize of the winner, they have no effect on the equilibrium distributions – as was also noted in Xiao (2018).

Our paper is also connected more broadly to the literature of spillovers in other auction and auction-like frameworks. Hodler and Yektaş (2012), for example, use a linear first-price auction with spillovers to model war. The authors refer to this as an all-pay contest, but only the winner actually pays because of the way funds are handled.

Notably, spillovers have been given comparatively more attention in the Tullock contest framework. In these contests, each participants' probability of winning is given by $p_i(s_i; s_{-i}) = s_i^r/(s_i^r + s_{-i}^r)$, if $(s_1, s_2) \neq (0,0)$, and $p_i(s_i; s_{-i}) = 1/2$ if $(s_1, s_2) = (0,0)$. Here, $r \in (0, \infty)$ is a parameter that controls how much one's probability of winning responds to an increase in scores. The all-pay auction is a Tullock contest where $r = \infty$; when $r = 1$ we have a Tullock lottery instead.

Chowdhury and Sheremeta (2011a) study a generalized Tullock lottery in which payoffs linearly incorporate one's own effort and the effort of the rival. The paper studies symmetric payoff and cost structures and, as is usual in Tullock-type contests, both players are able to extract positive payoffs. The authors obtain asymmetric, pure-strategy equilibria – even when players are identical (Chowdhury and Sheremeta, 2011b). In contrast, our all-pay framework yields a unique symmetric mixed-strategy equilibrium when players are identical, and expected payoffs are zero.

Damianov et al. (2018) allows for players' efforts to produce either positive (productive) or negative (destructive) externalities in a two-player Tullock lottery. The author finds that spillovers can either accentuate or reduce the competitive balance between participants, when contrasted to a comparable fixed-prize contest. However, unlike our results, no reversal can ever occur: the “favored” player is always more likely to win and has a higher expected payoff.[12]

We identify several avenues for future work. The class of contests that include spillovers is very large and fits many applications. The fact that we are able to construct very different contests with the same equilibrium strategies (e.g. the all-pay auction with winner’s regret in Section 1.6.4 has the same equilibrium as a war of attrition with costly preparation of Section 1.6.1) suggest that it might be possible for a contest designer to induce behavior more cheaply through spillovers.

Other contest design problems where spillovers are available are also of interest. A.2 contains a brief exposition that shows that when a constrained designer that cares about aggregate effort can reward contestants with prizes that may include spillovers, no contestant will be allowed positive rents. This in particular would make computing equilibrium strategies straightforward. Under what circumstances introducing spillovers is desirable to a contest designer is however still an open question.

[12] There are many other examples of spillovers in Tullock-type contests, see e.g. Chung (1996), Hirai and Szidarovszky (2013).

CHAPTER 2

Regulation of Wages and Hours

2.1. Introduction

Most workers cannot freely set their hours.[1] This restriction is not studied in the theoretical literature on labor regulation, where hours are assumed to be either fixed or chosen by workers.[2]

However the practice of regulation recognizes this issue. Many policies are designed to reduce the hours of individual workers. For example, the European Union imposes a sharp cap of 48 hours per week. In the United States, overtime pay requires companies to pay "time and a half" (1.5 times the wage) for each hour worked above 40 hours per week. Japan uses a lesser amount of overtime, "time and a quarter", combined with a sharp cap of 55 hours per week. These policies are depicted in Figure 2.1.

This paper addresses this gap in the theoretical literature by analyzing the effects of regulation in a model in which workers and firms bargain over both wages and hours of work.

Consider a fully-informed monopsonist firm that submits a "take it or leave it" offer of compensation and hours to a single risk-neutral worker with convex disutility from labor. We refer to this as the "ultimatum model" of monopsony to distinguish it from

[1]For example, less than 5% of hourly paid respondents to the 2016 United States General Social Survey said that they had full control of their hours compared to 47% who responded that their employer decides unilaterally.

[2]It is, however, considered outside the context of regulation. For example Manning (2005) shows differences between the canonical model of monopsony and one where the employer chooses hours.

the traditional model.[3] In the absence of regulation, the proposed contract will maximize total surplus, which will be entirely appropriated by the firm, and the worker's hours will be longer than she would like to work at the implied wage (i.e., total compensation divided by hours of work). Suppose that a regulator maximizes a weighted sum of worker utility and firm profits with more than half of the weight placed on the utility of the worker. The regulator would thus implement a regulation that (1) benefits the worker and (2) is not Pareto dominated.

We show that, when the regulator has complete information, minimum wage policies dominate overtime pay and caps on hours. This is because workers only want to reduce

[3]Ultimatum bargaining is just a special case of the bargaining studied in this paper.

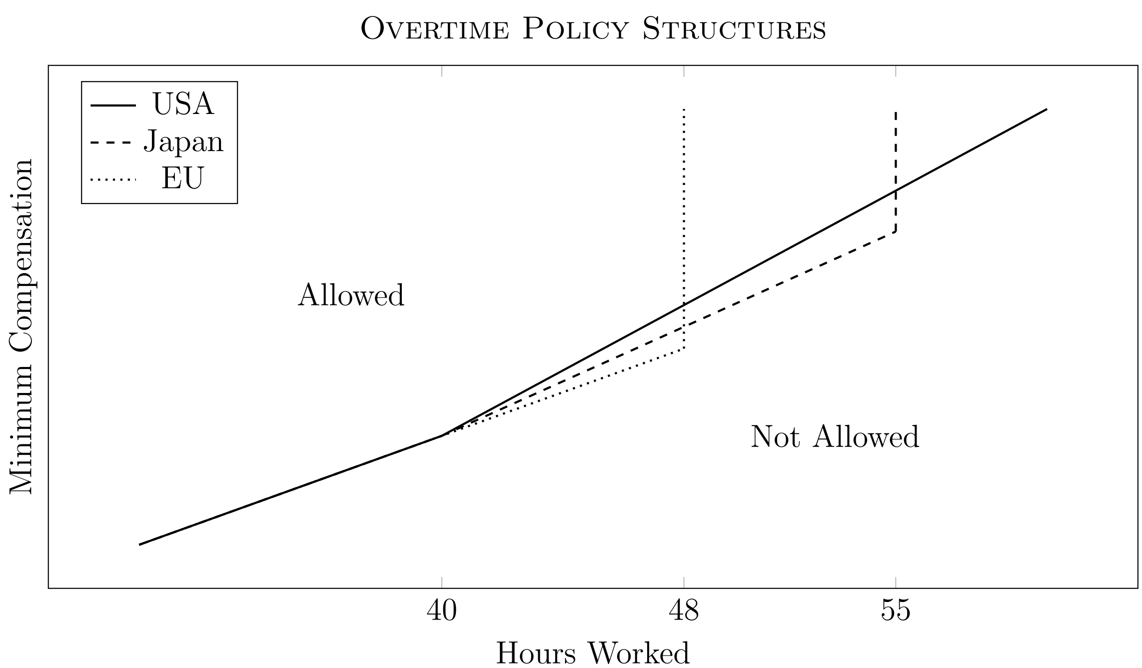


Figure 2.1. Regulations for minimum total compensation for each number of hours in the United States, Japan, and the European Union with a normalized minimum wage. Contracts with hours and compensation above the line are allowed. Contracts below the line are not.

their labor if their implied wage is too low; if the wage is increased sufficiently, the problem vanishes. Moreover, an efficient minimum wage exists where both the firm and worker receive their preferred hours.

If the regulator has limited information about the market, he may be unable to implement the efficient minimum wage. In the standard model of monopsony, which we will refer to as the "flexible-hours" model, the worker and firm agree to each hour worked at a wage set by the firm. This flexible-hours model yields a strictly increasing relationship between agreed-upon hours and total surplus. As a result, any minimum wage below the efficient minimum wage is "better than nothing" in the sense that it benefits workers and increases total surplus above the market level. Thus, even if the regulator is unable to implement the first-best regulation, he still prefers any reasonable lower bound on the minimum wage to the free-market outcome.

This is not the case in the ultimatum model. If the minimum wage is binding but below the efficient minimum wage, the firm will select more hours than is efficient in an attempt to "claw back" the additional surplus the worker derives from higher wages. Consequently, the minimum wage that maximizes hours actually *minimizes* total surplus locally.[4] This implies a suboptimal minimum wage is not necessarily "better than nothing": total surplus is lower, worker surplus may be lower, and the overall allocation may be strictly Pareto dominated by that of the unregulated market.

This fragility of the minimum wage as a welfare improving policy motivates our interest in robust regulation that always increases the worker's utility even when the regulator's information is quite limited.

[4] It is not the global minimum. Total surplus is positive at this point, yet a sufficiently large minimum wage will result in zero labor (and therefore zero surplus).

We consider a framework where the regulator has no prior knowledge of production and costs, but: (1) observes the contract that prevails in the market before regulating, and (2) knows a specific reduction in hours at the implied wage which benefits the worker. Given this information, it is clear that reducing hours to this specific quantity at the current wage would make the worker better off, regardless of the true production and disutility from labor.

However, this reduction in hours is actually weakly dominated: that is, alternative policies exist which provide weakly greater payoffs for both the worker and firm for every possible combination of production and cost functions and strictly larger payoffs for some case. Two insights demonstrate this observation. First, observing the market contract enables the regulator to bound the worker's disutility from labor. Second, an inflexible reduction in hours is weakly dominated by any policy that guarantees the worker enough additional pay to compensate for the additional hours. Intuitively, if the firm is willing to pay the worker sufficiently in exchange for an additional hour, blocking the transaction is Pareto dominated.

The regulator is then able to use this bound on the worker's disutility in order to ensure that the worker is compensated enough to weakly benefit from increased hours. The optimal policy is to combine a minimum wage, overtime pay, and hours cap calibrated such that worker surplus will not be harmed. The minimum wage is set at the current implied wage, overtime pay begins at a preferred reduced number of hours, and the hours cap is set at the hours that prevailed in the market before regulation. We show that this policy uniquely maximizes total surplus for any combination of production and worker

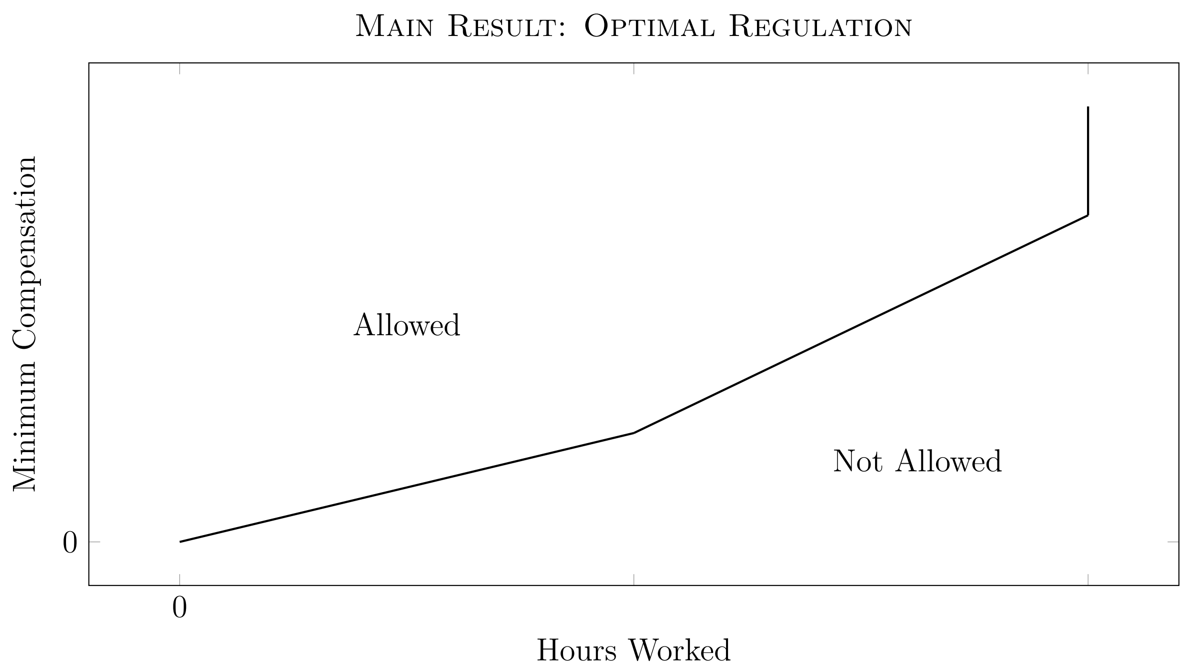


Figure 2.2. The never Pareto dominated satisficing policy combines a minimum wage, overtime pay, and a cap on hours. This policy guarantees the worker at least as much utility as the contract at the "kink".

disutility functions within the class of regulations that dominate the inflexible reduction in hours.

Policies with this described shape are common. For example, Japan and France both combine overtime pay with a cap on hours. Overtime pay in countries with no caps on hours, such as the United States are also fundamentally similar because there is a natural cap on the number of hours that one can work in a week. Some real-world regulations are plotted in Figure 2.1 and can be compared to our optimal regulation in Figure 2.2.

This paper is the first to theoretically study the effects of regulation in a setting where hours and compensation are jointly contracted. Our analysis contributes to a large theoretical and empirical literature on labor regulation. We additionally build on a small literature on the joint determination of hours and compensation. In particular, Manning (2005) analyzes the ultimatum model of monopsony without regulation.

This study also joins a growing literature on robust contracting. Our informational setting shares key elements with Carroll (2015), which considers a robust moral hazard problem where the principal: (1) knows at least one of the agent's available actions and (2) obtains an optimal contract which exploits the alignment of incentives between the principal and agent. In our setting, the regulator instead: (1) knows one *regulation* which benefits the worker, instead of an action, and (2) obtains an optimal regulation which exploits efficiency,[5] which is analogous to aligned incentives in Carroll's context.[6]

The paper is organized as follows. In Section 2.2 we restate some standard results about monopsony to frame our results. We present the formal model in Section 2.3. In Section 2.4, we study the problem of regulation with complete information. The case where the regulator lacks information about production and disutility is presented in Section 2.5. We explore extensions in Sections 2.6 (more general bargaining), 2.7 (more workers), and 2.8 (information manipulation). In Section 2.9, we review the related literature and discuss the results.

2.2. The standard, "flexible-hours" model of monopsony

To illustrate our approach and the significance of our results for regulation, we present the model of monopsony that is common in the literature. We then show how our model differs from this standard model and how this difference affects the optimal regulation.

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[5]That is, bargaining between the worker and firm is Pareto optimal under the constraints of the regulation.

[6]Our framework can be applied to any robust delegation context. See Chapter 3.

The canonical Stigler (1946), henceforth flexible-hours, model of monopsony under a minimum wage is the standard model used in the literature. A monopsonist firm (it) contracts with a worker (she) to obtain hours of labor, ℓ, in exchange for a transfer, τ. From any given contract (ℓ, τ), the firm receives profits, $f(\ell) - \tau$, where f is a strictly concave, differentiable production function. The worker receives payoff $\tau - c(\ell)$, where c is a strictly convex, differentiable, and increasing labor cost function. Without loss, $f(0) = c(0) = 0$. We additionally make the standard assumption that $c'(x)x$ is convex (i.e., marginal expenditure is increasing).[7] The firm makes some quota of employment hours, $\bar{\ell}$, available to the worker at an hourly wage, $w \equiv \tau/\ell$, and the worker then chooses to provide $\ell \leq \bar{\ell}$ hours of labor.

In the absence of a minimum wage, the firm equates marginal productivity and marginal expenditure, yielding ℓ^m work hours – a quantity notably below the efficient, surplus maximizing, amount (see Figure 2.3). The introduction of a minimum wage $\bar{w}$ then improves total welfare by helping increase labor, as the higher compensation incentivizes workers to supply longer hours. This movement happens up until the worker is paid her marginal productivity, or $\bar{w}^* \equiv c'(\ell^*) = f'(\ell^*)$. Only beyond this level do we have that further increases in $\bar{w}$ become counterproductive. This relationship between the minimum wage and labor (i.e., the labor response curve, L) is plotted in Figure 2.4.

Note that, in this model, the minimum wage that maximizes labor among any set of minimum wages also maximizes total surplus in that set. This suggests that, if the introduction of a minimum wage has increased the number of hours worked, then the total

[7] This assumption is important for the standard model of monopsony. Without this assumption, Loertscher and Muir (2021) show that a menu of stochastic wages may be optimal for the firm. The assumption is not relevant to the ultimatum model presented in Section 2.3 or the more general model in Section 2.6.1.

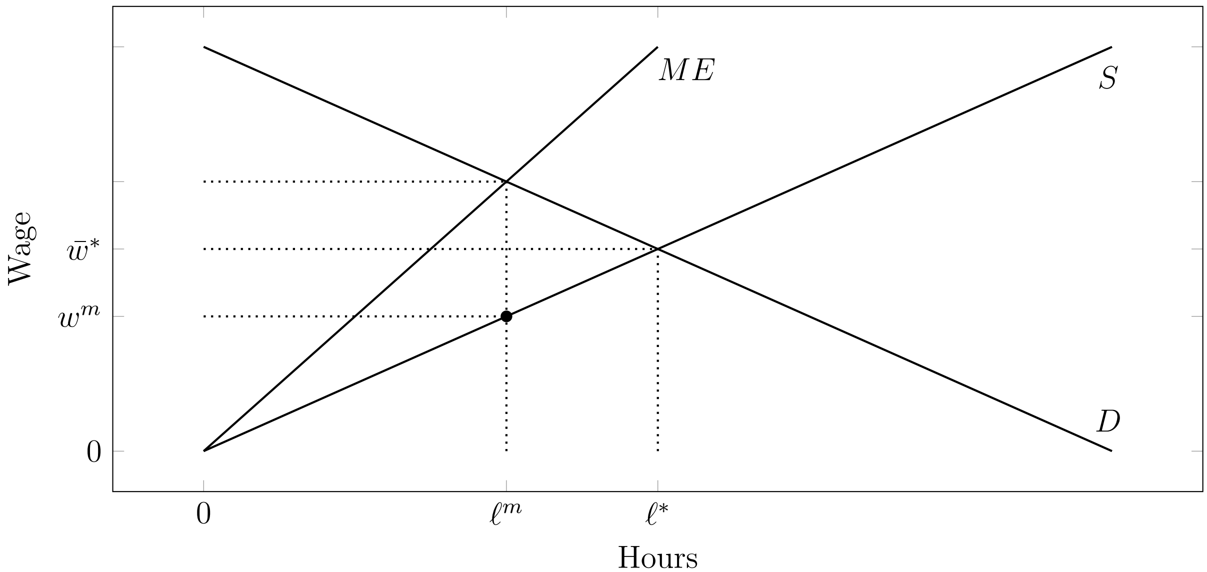

Figure 2.3. Labor supply, demand, and equilibrium under flexible-hours monopsony. The equilibrium labor hours occur at the intersection between marginal expenditure and labor demand.

surplus has inevitably improved – even if no other information about costs and production processes is available. This is a powerful relationship in terms of policy evaluation, and offers a clear guide to regulators considering introducing or altering the minimum wage. However, as we will show next, this intuition depends crucially on accepting that the worker has total flexibility in reducing her own hours. Without this supposition, the conclusions may be reversed.

2.3. Model

We adapt the model in Section 2.2 to the setting where the firm chooses hours. The players, payoffs, and definitions remain the same.

Contracts take the form $(\ell, \tau) \in \mathbb{R}^2_+$ where ℓ is hours of labor and τ is a gross payment to the worker. The worker's wage is defined as the average payment per hour: $w \equiv \tau/\ell$.

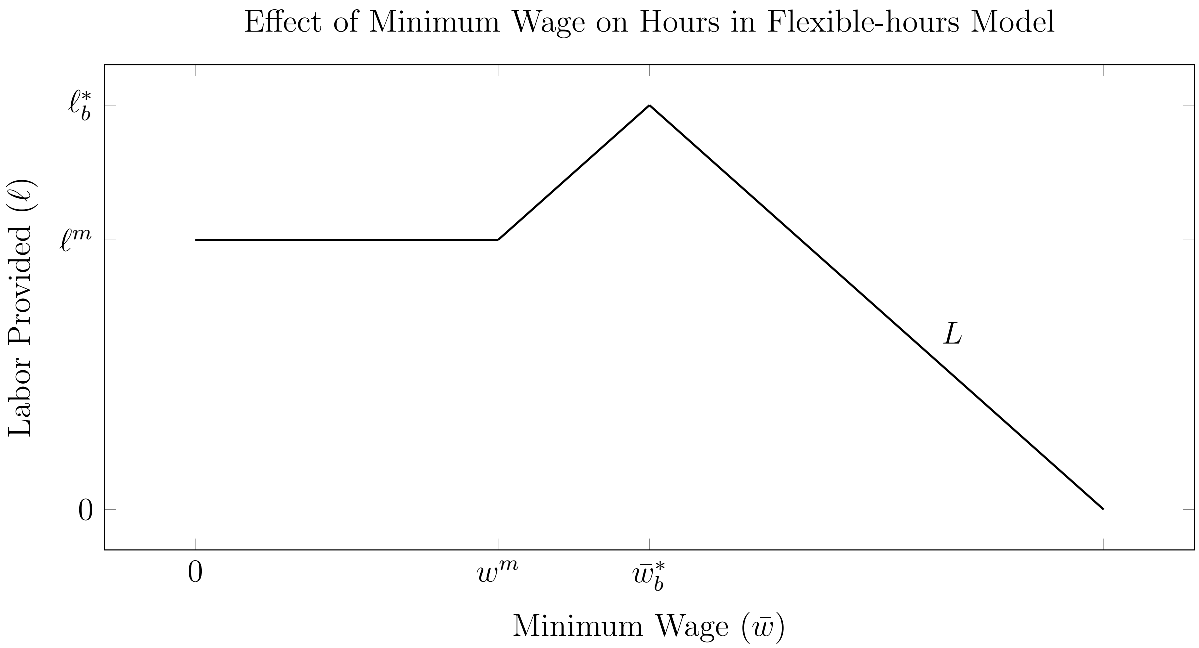

Figure 2.4. Plot of labor response curve, L for the flexible-hours model. For any minimum wage $\bar{w} \in (c'(\ell^m), c'(\ell_b^*))$, the worker is employed for a quantity of hours between the market employment and the efficient level of employment (the inverse of the marginal cost). The minimum wage $\bar{w}_b^* = c'(\ell_b^*) = f'(\ell_b^*)$ implements efficient employment. Labor is decreasing in the minimum wage for wages above $\bar{w}_b^*$ because the firm's limit on work hours is binding.

Players. The firm's payoff from a contract (ℓ, τ) is

$$\pi(\ell, \tau) \equiv f(\ell) - \tau.$$

Failing to hire the worker results in no labor or payment.

The worker's utility function is $u(\ell, \tau) \equiv \tau - c(\ell)$ where τ is the gross payment from the firm for the worker's labor and c is the worker's disutility from labor. The worker's outside option is no labor or payment.

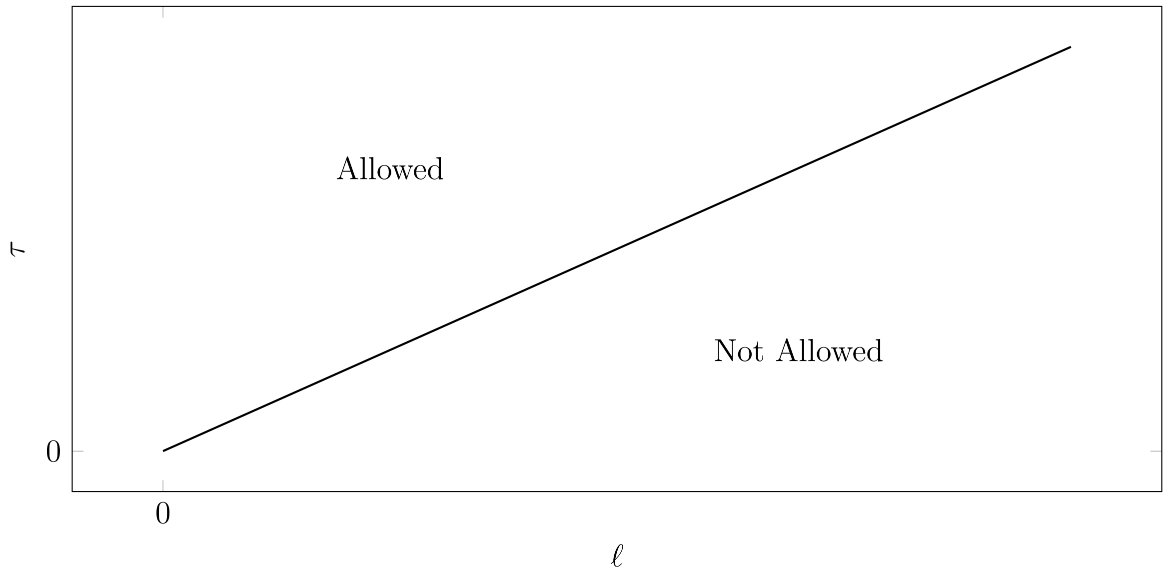

Figure 2.5. An example of a minimum wage policy. Only contracts above the lines are allowed.

Without loss of generality, $f(0) = c(0) = 0$. As in Section 2.2, we assume f, c are differentiable, c is weakly increasing and strictly convex, f is strictly concave, and there exists a $t > 0$ such that $f'(t) < c'(t)$.

Regulation. A regulator imposes a regulation to constrain the contracting space. A regulation a function, $\phi : \mathbb{R}_+ \to [0, \infty]$, such that the worker and firm are restricted to contracts with $\tau \geq \phi(\ell)$. We require that ϕ is weakly convex and $\phi(0) = 0$. In other words, the contracting space is a convex set which contains the disagreement point. This allows for policies such as overtime pay and caps on hours. However, it does meaningfully constrain the regulator and prevents full surplus extraction. Convexity of the bargaining space is a standard assumption in the bargaining literature. If the bargaining set were not convex, then there would exist two permitted contracts such that some convex combination of them is not allowed. In this case, a real-world firm might find it optimal to switch between the two permitted contracts in alternating weeks to approximate the disallowed contract. That is, the firm may rest workers in alternating weeks. We prevent this perverse behavior by requiring convexity.

If $\phi(\ell) \equiv 0$ for all ℓ, there is no regulation. A linear regulation, $\phi(\ell) \equiv \bar{w}\ell$, is a *minimum wage regulation* with *minimum wage*, $\bar{w}$.

Bargaining. The firm makes an ultimatum offer to the worker in order to maximize profits. The worker can accept or reject the offer. Rejecting the offer yields disagreement point $(0, 0)$.

Timing. The game has three stages.

(1) The regulator announces regulation ϕ.
(2) The firm offers a contract, (ℓ, τ).
(3) The worker can accept or reject the contract to obtain $(0, 0)$.

2.4. Results with complete information

In this section, we find the equilibrium, identify the optimal regulation, and show how it differs from the flexible-hours model.

This ultimatum employment game has a unique subgame perfect Nash equilibrium. The equilibrium contract solves the firm's profit maximization problem:

$$\max_{\ell,\tau} f(\ell) - \tau \text{ s.t. } \tau \geq c(\ell) \text{ and } \tau \geq \phi(\ell). \tag{2.1}$$

If ϕ is identically zero (i.e., there is no regulation) the firm extracts all surplus from the worker and labor hours are efficient. The wage is equal to the worker's average cost,

$w^m = {c(\ell^*)}/{\ell^*}$, implying the worker would prefer fewer hours (since c is convex, average costs are lower than marginal costs). In this case, we say that the worker is *overworked*.[8]

We say a regulation is *redistributive* if it is better for the worker than no regulation. The following proposition demonstrates that for every redistributive regulation, there is a minimum wage that gives the worker at least as much utility while maintaining the same total surplus.

Proposition 2.1. *Let ϕ be a redistributive regulation that implements ℓ. There exists a minimum wage, $\bar{w}$, that implements ℓ such that $\bar{w}\ell \geq \phi(\ell)$.*

Because of this, minimum wages are without loss of optimality for any fully informed regulator with an objective that is increasing in τ. As a result, we can restrict attention to minimum wage regulations and compare the effects of the minimum wage in this model to those in the flexible-hours model in Section 2.2.

The labor response curve, L, for the ultimatum model is shown in Figure 2.6. The curve for the flexible-hours model (plotted in Figure 2.4) has the same shape. In fact, the two curves are indistinguishable, as the following proposition demonstrates.

Proposition 2.2. *A flexible-hours model with cost, c_b, generates the same labor response curve as an ultimatum model with the same production function and cost, c_u, where*

$$c_u(x) \equiv c_b'(x)x.$$

[8]In the more general model presented in Section 2.6, overwork in the absence of regulation is a necessary and sufficient condition for the market to behave similarly to the ultimatum model in response to regulation.

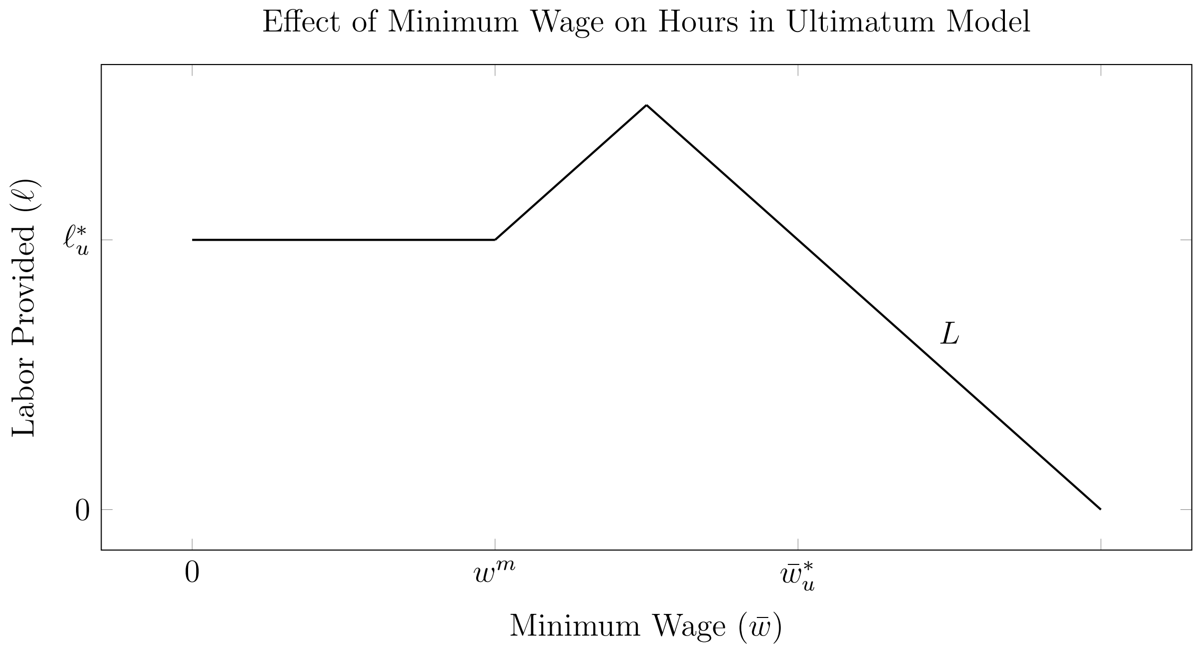

Figure 2.6. Plot of labor response curve, L for the ultimatum model. It has the same shape as the flexible-hours model in Figure 2.4. However, for any minimum wage $\bar{w} \in (c(\ell_u^*)/\ell_u^*, c'(\ell_u^*))$, the worker is employed for a number of hours that exceeds the efficient quantity. Therefore, the minimum wage that maximizes hours does not implement an efficient allocation.

This is significant because empirical studies on labor regulation generally focus on measuring some part of the labor response curve – often the change in labor after an increase in the minimum wage. Suppose that the regulator has access to the *entire* labor response curve but not f, c, or even whether this labor response comes from the flexible-hours or ultimatum model. Due to Proposition 2.2, the regulator cannot identify the true model without knowledge of the worker's disutility from labor, c. For any labor response curve, L, and production function, f, there is one disutility, c_b, that is consistent with the flexible-hours model and another disutility, c_u, that is consistent with the ultimatum model.

Of course, the two models are *not* equivalent for the worker, and the efficient minimum wage is markedly not the same. The unregulated market is already efficient in the ultimatum model: thus, a minimum wage that increases hours increases them *above* the efficient level. This is undesirable, an assertion we emphasize in the below claim.

Proposition 2.3. *The efficient minimum wage for the flexible-hours model with cost c_b (i.e., $\bar{w}_b^*$) locally minimizes welfare in the ultimatum model with cost c_u.*

Proposition 2.3 implies that a regulator who assumes the flexible-hours model and has sufficient knowledge of L to implement the efficient minimum wage under this model will locally minimize welfare in the case that the ultimatum model holds.

The next claim further demonstrates why knowledge of the labor response curve is not sufficient for optimal regulation.

Proposition 2.4. *For any labor response curve, L:*

- $c_b(x) < c_u(x) \; \forall x > 0$;
- $\bar{w}_b^* < \bar{w}_u^*$;
- $\ell_b^* > \ell_u^*$; *and*
- *for* $i \in \{b, u\}$ *and all* $\bar{w} \geq 0$,

$$\left.\frac{d\,[f(L(w)) - c_i(L(w))]}{dw}\right|_{w=\bar{w}} > 0 \implies \left.\frac{d\,[f(L(w)) - c_{-i}(L(w))]}{dw}\right|_{w=\bar{w}} < 0.$$

The last point of Proposition 2.4 implies that total surplus is always weakly decreasing in the minimum wage in at least one of the two models. This poses an impossibility for robust regulation using L. However, there is information other than L that can be used to regulate. For example, if the regulator knew that the worker was overworked, he could reject the flexible-hours model.

Overwork information is not only useful for determining the true model, but also for finding whether the current minimum wage is above or below the efficient one.

Proposition 2.5. *For $i \in \{b, u\}$, $\bar{w}_i^*$ is the largest minimum wage such that the worker is not underworked in model i.*

Proposition 2.5 demonstrates the importance of worker preferences in regulating labor. Knowing the actual level of labor that is achieved at each minimum wage is neither necessary nor sufficient for determining the efficient minimum wage. However, knowing whether the worker wants to work more or fewer hours at each minimum wage is sufficient.

2.5. Robust regulation

Suppose the regulator does not observe production or costs. Moreover, he has no prior over these objects. Instead, the regulator observes: (1) the equilibrium contract, (ℓ_0, τ_0), that prevails in ultimatum bargaining prior to the regulation and (2) reduced hours $\hat{\ell} < \ell_0$ that the worker would prefer at the average wage, $w_0 \equiv \tau_0/\ell_0$.

Therefore, the regulator is aware of a contract, $(\hat{\ell}, w_0\hat{\ell})$, which the worker prefers to the status quo. Because the contacting space is convex, the regulator knows that both parties prefer this contract to the disagreement point (i.e., the point is feasible). We are interested in regulations that are *satisficing* in that they guarantee at least as much utility for the worker as this contract.

Definition 2.1 (Satisficing)**.** A regulation is satisficing if

$$\inf_{(f,c)\in I(\ell_0,\tau_0,\hat{\ell})} \quad (\tau - c(\ell)) - (w_0\hat{\ell} - c(\hat{\ell})) \geq 0$$

$$\text{s.t.} \quad (\ell,\tau) = \operatorname*{arg\,max}_{l,t\geq\max\{\phi(l),c(l)\}} f(l) - t$$

where $I(\ell_0, \tau_0, \hat{\ell})$ is the set of possible M, f, c that are consistent with $\ell_0, \tau_0, \hat{\ell}$.

The satisficing property ensures that the worker weakly benefits from the market over the outcome where $(\hat{\ell}, w_0\hat{\ell})$ is the only contract allowed. A cap of $\hat{\ell}$ hours combined with a minimum wage of w_0 is clearly satisficing, but it is not the only such policy. There is an opportunity for improvement through refinement.

Intuitively, because the worker wants to work fewer hours at her current wage, our regulation will reduce the hours that the worker works. However, because ultimatum bargaining is efficient in the absence of regulation, this reduction in hours is inefficient. Therefore, we want to minimize the reduction in hours while still satisfying the worker. This is formalized in the following definition.

Definition 2.2 (TS maximizing)**.** Policy ϕ is TS maximizing if for all f, c such that $f'(\ell_0) = c'(\ell_0)$ and $c(\ell_0) = \tau_0$ and all satisficing ψ,

$$f(\mathcal{L}[\phi]) - c(\mathcal{L}[\phi]) \geq f(\mathcal{L}[\psi]) - c(\mathcal{L}[\psi])$$

We are interested in finding an element within the class of satisficing policies which has the greatest total surplus in every state. It is not obvious that such a policy exists.

Indeed, in many settings, such a regulation would not exist. The following theorem shows that this policy does exist in our setting and guarantees that it is unique.

Theorem 2.1. *There is a unique TS-maximizing satisficing policy. It is*

$$\phi^*(x) \equiv \begin{cases} w_0 x & \text{if } x \leq \hat{\ell} \\ w_0 \hat{\ell} + \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}}(x - \hat{\ell}) & \text{if } \hat{\ell} < x \leq \ell_0 \\ \infty & \text{if } x > \ell_0. \end{cases}$$

Theorem 2.1 establishes that the unique TS maximizing, satisficing regulation, ϕ^*, is a combination of a minimum wage, overtime-pay, and a cap on hours. This regulation is plotted in Figure 2.2. Each of the three policies are very common and a similar combination of the three exists in Japan and France.

2.6. More general bargaining

2.6.1. Model

We now assume that the contract (ℓ, τ) solves the following optimization problem:

$$\max_{\ell,\tau} M(f(\ell) - \tau, \tau - c(\ell)) \text{ s.t. } \tau \geq \phi(\ell)$$

where $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the *bargaining objective*. The ultimatum bargaining model studied in 2.4 is the special case where $M(x, y) \equiv x$. We make the following assumptions about M.

Assumption 2.1 (Weak monotonicity). *For all $x, y, x', y' \in \mathbb{R}_+$ such that $x' > x$ and $y' > y$,*

$$M(x', y') > M(x, y).$$

Assumption 2.2 (Strict quasiconcavity). *For all $x, y, x', y' \in \mathbb{R}_+$ such that $x' \neq x$ and $y' \neq y$ and for all $\lambda \in (0, 1)$,*

$$M(\lambda x' + (1 - \lambda)x, \lambda y' + (1 - \lambda)y) > \min\{M(x, y), M(x', y')\}.$$

Assumption 2.3 (Continuity). *The function, M, is continuous in both arguments.*

These assumptions admit the most popular bargaining models including ultimatum bargaining, asymmetric Nash bargaining, egalitarian bargaining, and the more general proportional bargaining of Kalai (1977).[9]

2.6.2. Comparative statics results

In this section, we show that ultimatum bargaining typifies the properties of other bargaining protocols. Because of this, the claims established in Section 2.4 for minimum wage regulation in the ultimatum framework extend to this more general setting.

Proposition 2.1 extends to this setting without modification. So, minimum wage regulations are still without loss of optimality for a regulator with complete information and an objective function that is strictly increasing in τ.

[9] Our results also extend to the choice theoretic bargaining of Peters and Wakker (1991).

In the analysis of ultimatum bargaining in Section 2.4, a minimum wage of zero and the minimum wage $\bar{w}^* \equiv f'(\ell^*)$ are both *efficient* regulations in that they both implement the efficient quantity of hours, ℓ^*. This remains true in the general case.

However, in the ultimatum model: (1) the worker is overworked without regulation and (2) the efficient minimum wage, $\bar{w}^*$, is always redistributive. For a general bargaining protocol, it may be that neither holds. The following proposition demonstrates that one cannot hold without the other.

Theorem 2.2. *Let ϕ be an efficient regulation that implements τ. The worker is overworked under ϕ if and only if there exists another efficient regulation ψ such that $\psi(\ell^*) > \tau$.*

In other words, the worker is overworked under an efficient regulation if and only if there is another efficient regulation which redistributes more to the worker. Overwork, which is the setting under which a regulation such as overtime pay is appealing, is also a necessary and sufficient condition for there to exist some "costless" redistribution in the sense that we can redistribute to the worker without sacrificing any total surplus.

For the forward direction, if the worker is overworked, her wage is less than her marginal cost of labor, which is precisely the efficient minimum wage, $\bar{w}^* \equiv f'(\ell^*) = c'(\ell^*)$.

Intuitively, the proof of the converse uses the fact that any bargaining protocol responds to regulation in a manner similar to the ultimatum model when labor hours are efficient. Because marginal productivity and marginal disutility are equal at the efficient labor, hours are an almost perfect substitute for the constrained transfers. Because of this, if a regulation ψ is redistributive and $\psi'(\ell^*) \neq f'(\ell^*)$, some total surplus will always be traded off to increase firm profits. By convexity, a regulation with $\psi'(\ell^*) = f'(\ell^*)$ cannot result in a wage that exceeds the workers marginal productivity, $f'(\ell^*)$.

Two convenient choices of ϕ in Theorem 2.2 allow us to obtain new results and provide intuition for the theorem. First, if we set ϕ to zero, we obtain the following corollary.

Corollary 2.2.1. *There exists an efficient, redistributive regulation if and only if the worker is overworked under the free-market contract.*

Corollary 2.2.1 is a special case of Theorem 2.2 which uses the fact that the free-market is efficient. Using the efficient minimum wage $\bar{w}^*$ as ϕ yields a second corollary.

Corollary 2.2.2. *If ψ is an efficient, redistributive regulation, then $\psi(\ell^*) \leq f'(\ell^*)\ell^*$.*

Corollary 2.2.2 implies that $\bar{w}^*$ maximizes worker utility in the class of efficient policies. Corollary 2.2.2 follows from the fact that $f'(\ell^*)$ is equal to (and thus not less than) $c'(\ell^*)$. This characterizes the maximal efficient, redistributive minimum wage. However, we have yet to describe the effects of any other minimum wages.

In Section 2.4, any minimum wage between the free-market wage and $f'(\ell^*)$ implement labor that exceeds the efficient quantity, ℓ^*. This holds true in the general model when the worker is overworked in the absence of regulation. Otherwise, these minimum wages are below the free-market wage and thus do not bind.

In the case of overwork, the shape of the labor response curve is similar to that of the ultimatum model. That is, overwork is necessary and sufficient for the market to behave "like ultimatum bargaining" in response to regulation.

Corollary 2.2.3. *Let $L : \mathbb{R}_+ \to \mathbb{R}_+$ define the level of labor at each minimum wage and $w_0 \in [c(\ell^*)/\ell^*, c'(\ell^*))$ be the wage in the absence of regulation. Then, the labor response function, L, is continuous and*

$$L(x) \begin{cases} = \ell^* & \text{if } x \in [0, w_0] \\ > \ell^* & \text{if } x \in (w_0, f'(\ell^*)) \\ = \ell^* & \text{if } x = f'(\ell^*) \\ < \ell^* & \text{if } x \in (f'(\ell^*), f'(0)) \\ = 0 & \text{if } x \geq f'(0). \end{cases}$$

First, labor is constant when the minimum wage is too low to bind. Second, hours exceed the efficient quantity when the minimum wage is between the market wage and $\bar{w}^*$. Therefore, the minimum wage that maximizes labor is inefficient. Finally, $\bar{w}^*$ is the unique efficient, redistributive minimum wage. So, any larger minimum wage implements fewer hours than is efficient. Therefore, the analysis in Section 2.4 is relevant for any bargaining protocol that results in overwork.

The general setting allows for new effects on the surplus of workers and firms. For example, it is possible for a minimum wage below $\bar{w}^*$ to *strictly* decrease the surplus of both the worker and the firm.

Example 2.1 (Egalitarian bargaining)**.** The worker and firm split the market surplus evenly. So, the market is described by

$$\max_{\ell,\tau} \min\{f(\ell) - \tau, \tau - c(\ell)\} \text{ s.t. } \tau \geq \phi(\ell).$$

Let $-c$ be "more concave" than f on $[0, \ell^*]$ in the sense that $f(\ell^*) - f'(\ell^*)\ell^* < c'(\ell^*)\ell^* - c(\ell^*)$. This condition is necessary and sufficient for overwork.

In the absence of regulation, $\tau_0 = \frac{f(\ell^*)+c(\ell^*)}{2}$ and profits and worker utility are both equal to half the maximum total surplus, $\frac{f(\ell^*)-c(\ell^*)}{2}$.

By assumption, the worker is overworked in equilibrium. As the minimum wage increases above the free-market wage, $w_0 = \frac{f(\ell^*)+c(\ell^*)}{2\ell^*}$, labor will increase to keep profits and worker utility equal. This will occur until the minimum wage reaches $f'(z)$ where z is the smallest solution to $\frac{f(z)+c(z)}{2z} = f'(z)$. For all minimum wages between w_0 and $f'(z)$, the worker's welfare and the profits of the firm are lower than in the unregulated state. This is because each is taking the same share of a smaller pie.

For any minimum wage above $f'(z)$, the worker obtains more than half of the total surplus. So, labor is set to maximize profits. As a result, the equilibrium contract at each minimum wage is the same as in the ultimatum model. △

Example 2.1 shows the effect that minimum wages below the efficient minimum wage can have on the market. If a binding minimum wage benefits the worker, it is because it increases the worker's share of total surplus enough to compensate for the weak reduction in total surplus imposed by the policy.[10] In Example 2.1, any minimum wage in $(w_0, f'(z)]$ reduces total surplus without affecting the worker's share. As a result, both the worker and firm are strictly worse off under these regulations.

[10]It is a strict reduction for any minimum wage other than $f'(\ell^*)$.

2.6.3. Optimal regulation

We now apply the results of Section 2.6.2 to the problem of a regulator. We restrict attention to settings where the worker is overworked in the absence of regulation. By Theorem 2.2, this is the same as restricting attention to the setting where the worker's wage is below her marginal cost.

2.6.3.1. Regulation with complete information. Suppose that the regulator knows f, c, and M and maximizes a weighted sum of worker utility and firm profits with more weight on workers. That is, the regulator's objective is to choose the ϕ that maximizes

$$\alpha u(\ell,\tau)+(1-\alpha)\pi(\ell,\tau)=\alpha(\tau-c(\ell))+(1-\alpha)(f(\ell)-\tau)$$

for $\alpha\in(0.5,1]$. The case where $\alpha\to 0.5$ is of special interest. In this case, the regulator is not willing to sacrifice any total surplus to improve the welfare of the worker.

In this setting, Proposition 2.1 guarantees that any regulation can be weakly improved upon with a minimum wage. With this, it is straightforward to find an optimal policy.

Theorem 2.3. *Any minimum wage in $w^*(\alpha)$ where*

$$w^*(\alpha)\equiv \underset{w\geq f'(\ell^*)}{\arg\max}\quad \alpha(w\ell-c(\ell))+(1-\alpha)(f(\ell)-w\ell)$$

$$s.t.\quad \ell=\arg\max_{l} M(f(l)-wl,wl-c(l))$$

is a redistributive optimal regulation for $\alpha\in(0.5,1]$.

The assumption that the worker is overworked ensures that the optimal minimum wage is redistributive. If the worker is not overworked, minimum wage regulation is still without loss of optimality. However, it may be optimal not to regulate.[11]

Note that Theorem 2.3 does not imply that the optimal minimum wage is unique. This is further explored in Example 2.2.[12]

Example 2.2. Suppose that bargaining is proportional as in Kalai (1977) with proportion $\beta \in [0,1]$ of total surplus going to the worker. The equilibrium contract solves:

$$\max_{\ell,\tau} \min\{(1-\beta)(f(\ell)-\tau), \beta(\tau - c(\ell))\} \text{ s.t. } \tau \geq \phi(\ell).$$

Note that this admits ultimatum bargaining ($\beta = 0$) and egalitarian bargaining ($\beta = 0.5$) as special cases.

Let $-c$ be sufficiently "more concave" than f on $[0, \ell^*]$ in the sense that $(1-\beta)(f(\ell^*) - f'(\ell^*)\ell^*) < \beta(c'(\ell^*)\ell^* - c(\ell^*))$. This condition is necessary and sufficient for overwork.

Because the free-market is efficient, regulation cannot increase total surplus. So, any redistributive regulation must give the worker more than β of the total surplus. Therefore, as in Example 2.1, labor is chosen to maximize profits in any redistributive regulation.

Because the worker is overworked in the absence of regulation, the minimum wage $f'(\ell^*)$ is redistributive. Therefore, labor is chosen to maximize profits for all minimum wages greater than or equal to $f'(\ell^*)$.

[11]For example, if the worker chooses both hours and pay, the worker will extract all of the surplus in the market. The free-market outcome is clearly optimal in this case.

[12]Example B.1 (in the Appendix) demonstrates another sort of multiplicity where there are multiple workers and different wages benefit different workers.

Therefore, the equation in Theorem 2.3 is

$$\begin{aligned} \underset{w \geq f'(\ell^*)}{\arg\max} \quad & \alpha(w\ell - c(\ell)) + (1-\alpha)(f(\ell) - w\ell) \\ \text{s.t.} \quad & \ell = \arg\max_l f(l) - wl. \end{aligned}$$

The constraint simplifies to $w = f'(\ell)$. So, any optimal minimum wage satisfies $w^*(\alpha) = f'(\ell_\alpha)$ where

$$\ell_\alpha \in \underset{l<\ell^*}{\arg\max}\,(2\alpha - 1)f'(l)l + (1-\alpha)f(l) - \alpha c(l). \tag{2.2}$$

△

There may be multiple maxima that satisfy (2.2). Consequently, more assumptions are required to ensure that the optimal minimum wage for a given α is unique. In the case of Example 2.2, it is sufficient to assume that $f'(x)x$ is concave.

2.6.3.2. Robust regulation. The TS-maximizing, satisficing policy introduced in Section 2.5 does not necessarily exist in a general bargaining environment. However, it is possible to find a policy that is both satisficing and never Pareto dominated by any other satisficing policy. That is, we replace TS-maximizing with the weaker notion of never Pareto dominated. We show in Chapter 3 that this rule works in a general delegation environment. In this section, we show that it works in the specific environment of this paper.

Definition 2.3 (Satisficing)**.** A regulation is satisficing if

$$\inf_{(M,f,c)\in I(\ell_0,\tau_0,\hat{\ell})} \quad (\tau - c(\ell)) - (w_0\hat{\ell} - c(\hat{\ell})) \geq 0$$

$$\text{s.t.} \quad (\ell,\tau) = \underset{l,t\geq\phi(l)}{\arg\max}\, M(f(l)-t, t-c(l))$$

where $I(\ell_0,\tau_0,\hat{\ell})$ is the set of possible M, f, c that are consistent with ℓ_0, τ_0.

Definition 2.4 (Never Pareto Dominated)**.** A regulation, ϕ, is Never Pareto Dominated (NPD) by ψ if there does not exist an $(M, f, c) \in I(\ell_0,\tau_0,\hat{\ell})$ such that the outcome of ϕ Pareto dominates the outcome of ψ.

Because bargaining satisfies weak Pareto, a more flexible regulation allowing more hours in exchange for compensation is NPD by the sharp cap on hours in any state of the world. However, there are many states of the world under which the more flexible policy Pareto dominates the cap.

We are interested in finding a policy that is both satisficing and NPD by any other satisficing policy. The following theorem identifies such a policy and guarantees that it is unique.

Theorem 2.4. *There is a unique policy which is satisficing and NPD by any satisficing regulation. It is*

$$\phi^*(x) \equiv \begin{cases} w_0 x & \textit{if } x \leq \hat{\ell} \\ w_0\hat{\ell} + \frac{w_0\ell_0}{\ell_0-\hat{\ell}}(x-\hat{\ell}) & \textit{if } \hat{\ell} < x \leq \ell_0 \\ \infty & \textit{if } x > \ell_0. \end{cases}$$

Theorem 2.4 establishes that the unique NPD satisficing regulation, ϕ^*, is a combination of a minimum wage, overtime-pay, and a cap on hours. The regulation is identical to that of Theorem 2.1. This regulation is plotted in Figure 2.2. Each of the three policies are very common and a similar combination of the three exists in Japan and France.

The NPD satisficing regulation is the most flexible satisficing policy as it is the pointwise minimum of all satisficing policies.[13] Intuitively, the NPD satisficing regulation allows $(\hat{\ell}, w_0\hat{\ell})$ as well as all contracts which the regulator *knows* that the worker prefers. This knowledge comes from the fact that the regulator can bound the worker's disutility from labor. The bound is obtained from the worker's individual rationality constraint at the initial contract, (ℓ_0, τ_0). That is, the regulator can bound the amount of additional income that the worker needs to work additional hours because he saw the worker work these hours in exchange for pay.

We now give a sketch of the main technique used in the proof to construct the upper bound on the worker's disutility from labor. Any violation of satisficing would come from contracts with labor in $(\hat{\ell}, \ell_0]$ that the worker likes less than $(\hat{\ell}, w_0\hat{\ell})$. We will show that any such contract is forbidden.

The worker weakly prefers a contract (x, y) to $(\hat{\ell}, w_0\hat{\ell})$ where $x \in (\hat{\ell}, \ell_0]$ if and only if the additional payment she receives offsets the increase in work hours,

$$y - w_0\hat{\ell} \geq c(x) - c(\hat{\ell}). \tag{2.3}$$

[13] Put another way, the bargaining set is the union of all satisficing bargaining sets. For a general delegation setting, we demonstrate that the union of satisficing sets is satisficing in Chapter 3.

We are able to bound the right hand side (the increase in labor costs) using convexity and individual rationality of (ℓ_0, τ_0).

$$c(x) - c(\hat{\ell}) < \frac{x - \hat{\ell}}{\ell_0 - \hat{\ell}} \left[c(\ell_0) - c(\hat{\ell}) \right] < \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}} (x - \hat{\ell})$$

Where the last step uses $c(\hat{\ell}) > 0$, and the fact that individual rationality implies $c(\ell_0) \leq \tau_0 = w_0 \ell_0$. Combining the above with (2.3) yields that the worker prefers (x, y) to $(\hat{\ell}, w_0 \hat{\ell})$ if

$$y \geq w_0 \hat{\ell} + \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}} (x - \hat{\ell}) = \phi^*(x).$$

The fact that this regulation is never Pareto dominated comes from the fact that it is the minimal satisficing regulation. Because bargaining satisfies weak Pareto, the least restrictive regulation is never Pareto dominated by more restrictive policies. We show in Chapter 3 that the set of satisficing delegation sets in general delegation problems forms an upper semi-lattice. As a result, the least restrictive regulation exists and is unique.

Note that in our complete information setting in Section 2.6.3.1, we found NPD satisficing regulations. Under complete information, maximizing a weighted sum of payoffs yields all of the policies that are NPD by *any* policy. Moreover, any policy that maximizes the objective for $\alpha > 0.5$ is satisficing. To see this, note that the efficient minimum wage, $\bar{w}^*$ is necessarily larger than w_0 because the worker is overworked. Recall that the worker obtains her preferred hours at $\bar{w}^*$. Therefore, $\bar{w}^*$ is satisficing because the utility that the worker obtains at this higher wage with her most preferred hours exceeds that of any hours, $\hat{\ell}$, at the lower wage. Recall that $\bar{w}^*$ is optimal for $\alpha \to 0.5$. Clearly, if $\alpha > 0.5$, more weight is placed on the worker. So, the minimum wages associated with these larger weights are also satisficing.[14]

2.7. Heterogeneous workers

Thus far, we have assumed that a firm acquires the services of a single worker. This makes sense if: (1) regulation can be customized to each worker or (2) workers have similar labor preferences. The first is unusual. While the second is not necessarily true, it is nevertheless common to use a representative agent to represent players with heterogeneous preferences.

2.7.1. Complete information regulation under heterogeneity

Suppose that there are $N \geq 2$ types of workers employed by a firm. Let c_i denote the cost of the i-th worker type. The individual costs and the production functions, f, satisfy A1-3. For convenience, order the types in terms of efficient labor hours. That is, for $j > k$, $\ell_j^* \geq \ell_k^*$ where $\ell_i^* \equiv \arg\max_z \{f(z) - c_i(z)\}$. The workers negotiate individual contracts in accordance with the ultimatum model. However, regulations cannot be customized to each worker. Instead, the regulator chooses a single regulation that applies to all workers. Absent this restriction, all of the results from the single worker case immediately apply to the general case.

[14] The only other never Pareto dominated, satisficing regulations are some that maximize the objective for $\alpha = 0.5$. These are efficient regulations that redistribute less than $\bar{w}^*$.

Assume that the regulator treats all workers equally. That is, the objective of the regulator is

$$\alpha \left(\sum_{i=1}^{N} \tau_i - c_i(\ell_i) \right) + (1-\alpha) \left(\sum_{i=1}^{N} f(\ell_i) - \tau_i \right) = \sum_{i=1}^{N} (2\alpha - 1)\tau_i + (1-\alpha) f(\ell_i) - \alpha c_i(\ell_i)$$

where $\alpha \in (0.5, 1]$.

We analyze this case of heterogeneous workers under complete information in depth in Appendix B.3. Here we highlight and discuss two significant results.

The first result is that if only the utility of the worker matters to the regulator, heterogeneity essentially has no effect on the problem.

Proposition 2.6. *For $\alpha = 1$, there is at least one optimal regulation that is a minimum wage. Any optimal minimum wage is the same as the optimal minimum wage in a single worker problem. This single worker has the average cost of all workers with positive utility under the regulation.*

Proposition 2.6 means that the optimal minimum wage in the heterogeneous case is the optimal minimum wage for a worker with costs that are averaged across some subset of the workers. We can interpret this virtual worker with averaged costs as a representative agent.

The proof contains an algorithm to find all optimum minimum wages by checking all possible subsets of workers. Example B.1 in the Appendix shows how to use the algorithm in practice. In the example, there are two optimal minimum wages which benefit different subsets of workers.

Intuitively, Proposition 2.6 holds because a redistributive regulation implements a contract that does not depend on workers' costs. As a result, all workers who benefit from a regulation receive the same contract.

Of course, there may be some workers who receive a different contract because the regulation is not redistributive for them. However, because of the ultimatum bargaining, these workers are not negatively impacted by the policy.

The firm, on the other hand, may be negatively impacted. Because of this, if the regulator cares about firms or efficiency, the optimal policy may not be a minimum wage.

Proposition 2.7. *Let $\alpha \to 0.5$. Suppose $\ell_N^* \neq \ell_{N-1}^*$. If $\min_{i=1}^{N-1} \{c_i(\ell_i^*)/\ell_i^*\} > f'(\ell_N^*)$, the optimal regulation is a minimum wage of $f'(\ell_N^*)$. Otherwise, consider*

$$\phi(x) \equiv \begin{cases} Conv(x) & \text{if } x \leq \ell_{N-1}^* \\ Conv(\ell_{N-1}^*) + f'(\ell_N^*)(x - \ell_{N-1}^*) & \text{if } x > \ell_{N-1}^*, \end{cases}$$

and Conv is the largest convex function to fit under $\{(\ell_i^*, c_i(\ell_i^*))\}_{i=1}^{N-1}$ with the restriction that the slope is capped at $f'(\ell_N^*)$. If $\phi(\ell_N^*) > c_N(\ell_N^*)$, then ϕ is an optimal regulation.

Proposition 2.7 demonstrates two important properties of these problems. First, the optimal regulation may not be a minimum wage. The reason that a piecewise linear regulation may be optimal in the heterogeneous worker setting is that it reduces the effects of regulation the contracts of workers with zero payoff. Such regulation only affects firms. So, the issue becomes less relevant as α increases. The second property is that minimum wage regulation is also optimal in settings where heterogeneity is sufficiently large. In this case, the problem is separable and the lowest cost worker can be regulated alone.

In Appendix B.3, Proposition B.2 shows that a minimum wage is also optimal for all $\alpha \in (0.5, 1]$ (but not for $\alpha \to 0.5$) when heterogeneity is sufficiently small.

2.7.2. Robust regulation under heterogeneity

The previous analysis uses the market state from just one worker to regulate. As a result, the regulation is only necessarily satisficing for this one worker. Extending the analysis to multiple workers is simple. One can construct the never Pareto dominated satisficing regulation for each and take the maximum pointwise.

Proposition 2.8. *Let ϕ_i^* be the never Pareto dominated satisficing regulation for worker i. Suppose that the request, $\hat{\ell}$, is below the labor of each worker and that all workers are overworked. The unique never Pareto dominated regulation which is satisficing for all workers is*

$$\phi^*(x) = \max_i \phi_i^*(x).$$

The proof is in Appendix B.1.12. For $\ell > \hat{\ell}$, each regulation only allows contracts which the worker prefers to the request. If we take the maximum, all of the points which are allowed are preferred to the request. The only trick is to ensure that $\ell < \hat{\ell}$ is satisficing. This comes from weak Pareto and overwork combined with Theorem 2.2.

The same proof also demonstrates that an hours cap at τ which uses the maximum of all workers wages is also satisficing. The assumption that all workers are overworked is important to establish that ϕ^* is satisficing. If some workers are not overworked, then ϕ^* may not be satisficing. However, for the same reason, an hours cap would also not be satisficing. The argument that ϕ^* is a Pareto improvement over an hours cap remains true.

2.8. Manipulation

In Section 2.6.3.2, we assumed that the firm and worker do not foresee the regulation that will be imposed. Moreover, we assume that the regulator knows some lower level of labor, $\hat{\ell}$, which the worker prefers to the current level of labor. This level could be arbitrary or be an internal belief of the regulator. However, it may be appealing to elicit this value.

In practice, manipulation of regulation is typically prevented through *grandfathering*, the use of information which predates the discussion of regulation. For example, in most cap-and-trade systems, permits are allocated based on historical energy usage that predates the discussion of these environmental policies.[15] This section considers what can happen when this practice is infeasible.

2.8.1. Manipulation by workers

Suppose that workers have complete information and interact with firms according to the ultimatum model. If the regulator asks the worker how many hours she wants to work, the worker wants to solve

$$\begin{aligned} \max_{\hat{\ell} \leq \ell_0} \quad & w_0 \hat{\ell} + \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}} (\ell - \hat{\ell}) - c(\ell) \\ \text{s.t.} \quad & \ell = \arg\max_{l \in [0, \hat{\ell}]} f(l) - \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}} (l - \hat{\ell}). \end{aligned}$$

[15] This also applies to regulations considered for individuals. For example, the 2022 student loan forgiveness policy in the U.S. does not apply to any loans taken out less than two months before its announcement.

The solution is interior because setting the request, $\hat{\ell} = 0$ or $\hat{\ell} = \ell_0$ results in no binding regulation being implemented. This can be solved using standard envelope theorem arguments. We instead consider two extreme cases for intuition: (1) when total surplus is small and (2) when total surplus is large.

If total surplus is small, both marginal cost and marginal productivity are low. The firm will not pay overtime unless the overtime pay multiplier is sufficiently small. Because the worker prefers all of the overtime points to the preferred point by construction, the worker wants to make a report such that she can earn overtime. To make the overtime pay multiplier sufficiently small, the worker will request a low $\hat{\ell}$. As a result, the regulator imposes little regulation when there is not much surplus to redistribute.

If total surplus is sufficiently large, both marginal cost and marginal productivity are high. The firm will pay overtime even when the overtime pay multiplier is large. To extract a larger payment, the worker will request a large $\hat{\ell}$. As a result, the regulator imposes a large overtime payment multiplier when there is a lot of surplus to redistribute, but most hours are worked without overtime.

The regulation does not require the worker to be strategic or have information about production. However, if the worker does have access to these inputs, they can be used to make the regulation better. In both cases, the worker makes a strategic decision which makes the regulator's bound on the disutility of labor more accurately reflect the worker's marginal costs.

2.8.2. Manipulation by firms

2.8.2.1. Manipulation before regulation. Suppose that the firm predicts the regulation and adjusts the contract in the preexisting regulation to interfere with the mechanism. For simplicity, suppose that $\hat{\ell}$ is fixed.

The firm cannot benefit from paying the worker more than her costs. This makes the regulation more restrictive. As a result, the firm can only manipulate by adjusting labor. The firm solves

$$\max_{z \geq \hat{\ell}, \ell \leq z} f(\ell) - \frac{c(z)}{z}\hat{\ell} - \frac{c(z)}{z - \hat{\ell}}(\ell - \hat{\ell}).$$

We consider the same two extreme cases. If total surplus is small, both marginal productivity and marginal cost are low. The firm cannot make the overtime payment multiplier arbitrarily small without making the wage arbitrarily large. Therefore, the firm will not pay overtime. This implies $\ell = \hat{\ell}$. In this case, the firm wants to set z as small as possible to reduce the hourly wage that must be paid. In equilibrium, $z = \ell = \hat{\ell}$.

If total surplus is large, both marginal productivity and marginal cost are large. In this case, a large choice of labor in the preexisting market is costly. The firm will end up paying overtime for all available hours.[16] In this case, the firm will set $\ell = z$. The firm's problem can then be rewritten as

$$\max_{z} f(z) - \frac{c(z)}{z}\hat{\ell} - c(z).$$

[16]Note that this case also applies when $\hat{\ell}$ is sufficiently small relative to ℓ^*. This is because the firm's objective is decreasing in z for $\hat{\ell}$ sufficiently small.

Therefore, the firm hires the worker for fewer hours than is optimal in advance of the regulation. Intuitively, the firm takes into account the effect that an increase in labor will have on the overtime multiplier.

2.8.2.2. Manipulation after regulation. Suppose that the firm is regulated but wants to prevent future regulation. When it chooses the contract, it takes into account that the regulator will obtain information from the prevailing contract.

In the ultimatum model, paying overtime to a worker reveals that the marginal productivity of the worker is $w_0 \frac{\ell_0}{\ell_0 - \hat{\ell}}$. As a result, the regulator knows that a minimum wage of $w_0 \frac{\ell_0}{\ell_0 - \hat{\ell}}$ will not affect total surplus, but will increase the surplus of the worker.

As a concrete example, suppose a firm in the U.S. hires a worker for ten dollars per hour for more than sixty hours per week. As a result, the firm must pay the worker time and a half for the last twenty hours that she works. If the regulator sees this, he knows that the firm would be willing to pay time and a half for the first forty hours as well. The regulator can use this information to impose a minimum wage of fifteen dollars per hour. This regulation costs the firm the equivalent of twenty hours of work each week.

This suggests that the regulator has a limited ability to improve on a labor cap if he cannot commit to the mechanism. The firm may be unwilling to offer any overtime to workers if doing so invites regulation.

2.9. Conclusion

We have explored regulation under very general assumptions in a setting where workers are overworked. We show that a minimum wage is the best tool that a fully informed regulator can use to alleviate this issue. Through a comparative statics exercise, we demonstrate that the minimum wage can hurt both workers and firms when it is set either too low or too high. We show that this is particularly important if regulators assume the flexible-hours model and try to interpret the effects of regulation on hours.

This issue of interpretation relates to the empirical literature on measuring the effects of minimum wage policies on hours (e.g., Jardim et al., 2022) and the effects of other policies such as overtime (Hamermesh and Trejo, 2000; Quach, 2020; Trejo, 1991). This paper proposes a framework which can be tested by and used in a welfare analysis of these policies using these empirical estimates.

Our study also cautions that the intensive margin (i.e., hours) and extensive margin must be treated differently with regards to regulation. For example, Jardim et al. (2022) shows that the 2014 increase of the minimum wage in Seattle did not significantly reduce employment, but did significantly reduce hours. Most would say that this is a bad sign. However, a reduction in hours may be good if these workers wanted their hours to be reduced. The objective of minimum wage regulation is not to maximize hours (or even take-home pay). The goal, broadly, is to improve the lives of workers. We demonstrate that this is at odds with hours maximization.

We find the overall optimal minimum wages when the regulator has complete information. Even if the regulator is not willing to sacrifice any total surplus to increase the worker's welfare, there exists a minimum wage that achieves this goal. This policy ensures an efficient market equilibrium where both the firm and worker receive their preferred number of hours.

This analysis joins two theoretical strands in labor economics. First, this paper is connected to the large literature on optimal regulation under imperfect competition with hours set by workers (e.g., Berger et al., 2022) or with fixed hours (e.g., Flinn, 2006; Loertscher and Muir, 2021). Secondly, our work is also related to the smaller literature on labor hours and overwork outside of a regulatory context (Feather and Shaw, 2000; Manning, 2005). The most novel innovation of our approach lies in combining these two strains by studying regulation in a setting with labor hours and overwork.

In addition to the complete information setting, we consider a regulator who has no prior over production and disutility from labor. This regulator instead observes the current market state and knows that a specific reduction in labor hours at the existing wage will benefit the worker.

This analysis contributes to the literature on robust implementation. Following Carroll (2015), this literature focuses on finding policies that can be implemented without any prior on the space of parameters. Guo and Shmaya (2019) study the problem of regulating an inefficient monopolist seller without any prior over supply and demand. Unlike our study, Guo and Shmaya (2019) consider a regulator who knows bounds on supply and demand.

In any robust analysis, the regulator must be able to somehow bound the unknown objects. In Carroll (2015), the principal is able to create an endogenous lower bound on the agent's technology from partial knowledge of the agent's set of available actions. Typically, these bounds are exogenous. This is troubling because the objects may be difficult for the regulator to bound in a reasonable way, and extremely permissive bounds generally produce unreasonable outcomes (e.g., arbitrarily large or small minimum wages).

We demonstrate a method for creating endogenous bounds on supply and demand when the regulator is able to observe the price and quantity that prevail in the market.

To use this bound, we develop a new robust objective, the never Pareto dominated satisficing criterion. This objective is natural in any delegation problem. In general, the principal chooses the largest delegation set for which the principal does not regret the decision to delegate (i.e., the payoff is at least as high as the principal's preferred singleton delegation set). It is common for the principal and agent to have some inherent alignment of incentives such that this is beneficial. In our case, this alignment stems from the fact that our contract bargaining is Pareto efficient.

CHAPTER 3

Robust Delegation

3.1. Introduction

The idea for this chapter comes from the theory of robust regulation in Chapter 2. We demonstrate that the never Pareto dominated satisficing regulation developed in Section 2.6.3.2 is an objective that can be applied to any delegation problem.

We develop a general theory of never Pareto dominated satisficing delegation sets. We show that such a delegation set always exists and that a refinement can be used to obtain uniqueness. In particular, the least restrictive never Pareto dominated satisficing delegation set is unique.

This is particularly useful for a delegation problem where bounds on the preferences of the agent are not known. Intuitively, the condition yields a rule with guaranteed *gains from delegation*. A satisficing rule ensures that the principal will not be worse off by delegating to the agent than if he chose the alternative himself. This is a desirable property when the principal is uncertain about the agent's preferences, but still wants to benefit from the agent's information and some presumed alignment of incentives. By offering the agent a choice from the largest possible satisficing delegation set, the principal maximizes the agent's freedom subject to the satisficing condition. It is common in delegation problems for the principal and agent to have some inherent alignment of incentives such that this is beneficial.

In Section 2.6.3.2, the regulator delegates a bargaining space to two bargaining parties. The regulator is uncertain about the preferences of the parties, but knows that the bargaining is efficient. This efficiency takes the place of alignment of incentives in this chapter. The regulator wants to maximize the gain from the efficient bargaining, but does not want to risk a loss over a command and control economy. The least restrictive satisficing delegation set is thus a natural choice for the regulator.

3.2. Model

Let $\mathcal{A}$ be a set of alternatives that a principal (e.g., regulator) and agent (e.g., aggregated labor market) may choose from. There is some state, $\theta \in \Theta$, that is known to the agent but unknown to the principal. This state may affect the payoff of both the principal and agent. The principal has some choice $\hat{x}$ which he would choose from the set of alternatives if he were not able to delegate. We call this his *outside option*.

In order to elicit this information, the principal may choose a subset, $D \in \mathbb{D} \subset 2^{\mathcal{A}}$, to provide to the agent such that the agent makes a choice from D. This choice is defined by choice function, $\mathcal{C}_\theta : \mathbb{D} \to 2^{\mathcal{A}}$. This choice function is nonempty and satisfies the weak axiom of revealed preference, which we report in the form of Sen's α and β conditions, for each $\theta \in \Theta$.

Assumption 3.1 (Sen's α, IIA)**.** *If $x \in A \subseteq B$ and $x \in \mathcal{C}_\theta(B)$, then $x \in \mathcal{C}_\theta(A)$.*

Assumption 3.2 (Sen's β)**.** *If $x, y \in \mathcal{C}_\theta(A)$, $x \in \mathcal{C}_\theta(B)$, and $A \subseteq B$, then $y \in \mathcal{C}_\theta(B)$.*

These assumptions are typically used to obtain a revealed preference binary relation. In this case, we only impose the conditions on $\mathbb{D} \subset 2^{\mathcal{A}}$. In Chapter 2, Sen's β is vacuously true because all allowed delegation sets yield a unique choice.

The principal has a complete, but not necessarily transitive, weak relation: $\succeq_\theta$. He wants to choose a delegation set with two properties.

Definition 3.1 (Satisficing)**.** A delegation set, $D \in \mathbb{D}$ is satisficing with respect to $\hat{x} \in \mathcal{A}$ if for each $\theta \in \Theta$, there exists a $z \in \mathcal{C}_\theta(D)$ such that $z \succeq_\theta \hat{x}$.

Definition 3.2 (Never dominated)**.** A satisficing delegation set, $D \in \mathbb{D}$ is never Pareto dominated if for all satisficing delegation sets, $S \in \mathbb{D}$, and all $\theta \in \Theta$, $x \in \mathcal{C}_\theta(D) \implies x \in \mathcal{C}_\theta(D \cup S)$.

The satisficing criterion ensures that the outcome is at least as good for the principal and the never dominated condition imposes a refinement that we give the agent as much surplus as possible. In the setting of Chapter 2, weak Pareto of the bargaining protocol implies that any never dominated delegation set is never Pareto dominated.

3.3. Results

We want to show that there exists a delegation set that satisfies Definitions 3.1 and 3.2. We will do this by showing that there is a *least restrictive* satisficing set which contains all other satisficing delegation sets. This set is never dominated because the agent prefers larger delegation sets to smaller ones.

Lemma 3.1. *If the set of available delegation rules, $\mathbb{D}$, is closed under unions, then the set of satisficing delegation sets, $\mathbb{S} \subseteq \mathbb{D}$, is an upper semi-lattice under unions.*

Proof. We need to show that if S_1 and S_2 are satisficing, then $S_1 \cup S_2$ is satisficing. For each θ, consider each $x \in \mathcal{C}_\theta(S_1 \cup S_2)$. Because $x \in S_1 \cup S_2$, it is either in S_1, S_2, or both. Without loss, say it is contained in S_1. By Sen's α, $x \in \mathcal{C}_\theta(S_1)$. By Sen's β, $\mathcal{C}_\theta(S_1) \succeq \mathcal{C}_\theta(S_1 \cup S_2)$.

Because S_1 is satisficing, there exists $z \in \mathcal{C}_\theta(S_1)$ such that $z \succeq \hat{x}$. Because $z \in \mathcal{C}_\theta(S_1 \cup S_2)$, $S_1 \cup S_2$ is also satisficing. □

Intuitively, Lemma 3.1 says that if we have two satisficing delegation sets, we can take the union of the two and still have a satisficing delegation set. This is because every alternative the agent chooses from the larger delegation set chosen in at least one of the smaller delegation sets. Because these two sets are satisficing, these choices must be at least as good as the principal's outside option.

Because of this, we know that the union of all satisficing delegation sets is a satisficing delegation set. Clearly, this union is the least restrictive satisficing delegation set.

Proposition 3.1. *Suppose the set of available delegation rules, $\mathbb{D}$, is closed under unions and set of satisficing delegation sets, $\mathbb{S} \subseteq \mathbb{D}$ is nonempty. There exists a never dominated satisficing delegation set. Moreover, there exists a unique least restrictive never dominated satisficing delegation set.*

Proof. If there is only one satisficing delegation set, then we are done. It is never dominated.

If there is more than one, by Lemma 3.1, there exists an $S^* \in \mathbb{S}$ such that for all $S \in \mathbb{S}$, $S \subseteq S^*$. This delegation set is uniquely least restrictive and is never dominated because $S^* \cup S = S^*$. So, $x \in \mathcal{C}(S^*) \implies x \in \mathcal{C}(S^* \cup S)$. □

We assume in Proposition 3.1 that there is at least one satisficing delegation set. In many delegation settings, this will be $\{\hat{x}\}$. In the setting of Chapter 2, it was the hours cap.

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APPENDIX A

Chapter 1 Appendix

A.1. Proofs

A.1.1. Proof of Lemma A.1

Lemma A.1. *In any Nash equilibrium, players choose strictly increasing, continuous mixed strategies G_i with common support on some interval $[0, \bar{s}]$. At most one participant can have a mass point and it must be at zero.*

Moreover, the equilibrium is defined by the following indifference condition:

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s;s)} - \int_0^s \frac{v'_{-i}(s;y)}{v_{-i}(s;s)} \tilde{g}_i(y) dy. \tag{1.3}$$

Proof. The argument is standard in the all-pay auction literature, and is presented here for completeness. Our proof is in several steps.

(1) By A1.3, each player $i \in I$ would select scores in $S_i = [0, T_i]$, for T_i finite.

(2) By A1.1 and A1.4, both players prefer to win than lose in a neighborhood of the opponent's score. This is used in points 3, 5, and 7.

(3) *The minimum score in the support of both players' strategies is zero.*

Let $\underline{s}_1, \underline{s}_2$ denote the lower bounds of player 1 and player 2's strategies' supports, respectively. Suppose $\underline{s}_i \geq \underline{s}_j$. Then, if i places no atom in $\underline{s}_j$, $\underline{s}_j = 0$ and j will never want to play anything in $(0, \underline{s}_i)$ by A1.2. If i has no atom at $\underline{s}_i$, then j won't want to play anything on the $(0, \underline{s}_i]$.

Now suppose $\underline{s}_i > \underline{s}_j = 0$ – we will show that one of the two players at least has a profitable deviation. If i has an atom at $\underline{s}_i$, she could bring the bottom of her support closer to zero, increase her payoffs, and not change her probability of winning. If i does place an atom at $\underline{s}_i$ and j does not, then she could move that point-mass at $\underline{s}_i$ closer to zero, spend less on bids and not change her probability of winning. Finally, if both i and j place an atom at $\underline{x}_i$, then (by A1.1) either i could do better by spreading that atom to a ε-neighborhood just above it (if $\underline{s}_i < 1$), or j would prefer to place that mass at 0 instead of $\underline{s}_i < 1$ (if $\underline{s}_i = 1$).

Thus, it must be that $\underline{s}_i = \underline{s}_j = 0$.

(4) *Both players will have the same maximum score in their strategies' support ($\bar{s}$).* Otherwise, the player with the highest upper bound to her support could reduce it and increase her payoff (by A1.2) without impacting her probability of winning.

(5) *There are no mass points on the half open interval* $(0, \bar{s}]$. If i places a mass point at $s_i \in (0, \bar{s})$, then j would find it worthwhile to transfer mass from a neighborhood below s_i to one just above s_i. If i places a mass point at $\bar{s}$, then j would find it worthwhile to transfer mass from a neighborhood below s_i to 0. Either way, there would be an ε-neighborhood below s_i in which j would put no mass. But then it can't be an equilibrium strategy for i to place an atom at s_i in the first place.

(6) *There are no gaps in the density.* Suppose that there is an interval $(s', s'') \subset [0, \bar{s}]$ where player i places no probability mass. Pick this interval such that s'' is the

"largest" point without density [1] This is just to make sure there's some probability mass in the interval just above s''.

We have that j can't have any density on (s', s'') either, or she would rather transfer it all to x'. But then, for i, anything too close to s'' from above is worse than picking s'.

(7) *At most one player will place a mass point at zero.* The two players can't both have a mass point at $\underline{s} = 0$: either player would rather move that mass infinitesimally above it.

Therefore, there are mixed strategies on some interval $[0, \bar{s}]$ where the following indifference condition must hold for every point in the support of Player i:

$$\bar{u}_i(G_{-i}) := \int_0^s v_i(s; y)\, dG_{-i}(y) - c_i(s) \qquad \text{for } s \in [0, \bar{s}]. \tag{1.1}$$

We can uniquely decompose G_{-i} into a continuous measure $\tilde{G}_{-i}$ and a mass-point measure using the Lebesgue decomposition. Because there can be only one mass point at zero, we decompose to

$$\bar{u}_i(G_{-i}) = \int_0^s v_i(s; y)\, d\tilde{G}_{-i}(y) + v_i(0, 0)G_{-i}(0) - c_i(s).$$

Plugging in $s = 0$ reveals $\bar{u}_i = v_i(0, 0)G_{-i}(0)$. So, the condition simplifies to

$$c_i(s) = \int_0^s v_i(s; y)\, d\tilde{G}_{-i}(y).$$

[1] Such a s'' exists since there are no mass-points in i's distribution, implying it is continuous.

Therefore, the right hand side is differentiable on the support of Player i. This implies that

$$\begin{aligned} c_i'(s) &= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left[\int_0^{s+\varepsilon} v_i(s+\varepsilon;y)\,d\tilde{G}_{-i}(y) - \int_0^s v_i(s;y)\,d\tilde{G}_{-i}(y)\right] \\ &= \int_0^s v_i'(s;y)\,d\tilde{G}_{-i}(y) + v_i(s;s)\lim_{\varepsilon\to 0}\varepsilon^{-1}\left[\int_s^{s+\varepsilon} d\tilde{G}_{-i}(y)\right] \end{aligned}$$

holds. Then, $v_i(s;s) \neq 0$ implies that the limit exists and is equal to

$$\begin{aligned} \lim_{\varepsilon\to 0}\varepsilon^{-1}\left[\tilde{G}_{-i}(s+\varepsilon) - \tilde{G}_{-i}(s)\right] &= \lim_{\varepsilon\to 0}\varepsilon^{-1}\left[\int_s^{s+\varepsilon} d\tilde{G}_{-i}(y)\right] \\ &= \frac{c_i'(s)}{v_i(s;s)} - \int_0^s \frac{v_i'(s;y)}{v_i(s;s)}\,d\tilde{G}_{-i}(y). \end{aligned}$$

Then, $\tilde{G}_{-i}$ is differentiable on Player $-i$'s support with a continuous derivative.

Consider the case where Player i plays with full support on $[0,\bar{s}]$. Because continuous differentiability implies absolute continuity on a bounded interval, the above can be rewritten as

$$\tilde{g}_{-i}(s) = \frac{c_i'(s)}{v_i(s;s)} - \int_0^s \frac{v_i'(s;y)}{v_i(s;s)}\tilde{g}_{-i}(y)dy. \tag{1.3}$$

Now, suppose that there are some points not on the support such that (1.1) does not hold with equality. If there exists a positive $\tilde{g}_{-i}$ that solves (1.3), then at any such point, t,

$$\int_0^t v_i(t;y)\tilde{g}_{-i}(y)\,dy = c_i(t) \geq \int_0^t v_i(t;y)\,d\tilde{H}_{-i}(y).$$

The Lebesgue decomposition ensures that $\tilde{H}_{-i}$ can be broken down to an absolutely continuous portion (which must agree with $\tilde{G}_{-i}$) and a continuous singular portion. Therefore, we can equivalently write

$$0 \geq \int_0^t v_i(t;y)\, d\hat{H}_{-i}(y).$$

where $\hat{H}_{-i}$ is the singular part of $\tilde{H}_{-i}$.

We now show that $\hat{H}_{-i}(s) = 0$ for all $s \in [0, \bar{s}]$. Suppose, by way of contradiction that there exists an x such that $\hat{H}_{-i}(x) > 0$. By continuity, $\hat{H}_{-i}(0) = 0$.

Because $\hat{H}_{-i}$ is continuous, there must exist a point t such that $\hat{H}_{-i}(t) = 0$ and $\hat{H}_{-i}(t+\varepsilon) > 0$ for all $\epsilon > 0$. Then,

$$\begin{aligned}
0 &\geq \int_t^{t+\varepsilon} v_{-i}(t+\varepsilon; y)\, dQ_i(y) \\
&\geq \left[\min_{y\in[t,t+\varepsilon]} v_{-i}(t;y)\right] \left[\hat{H}_{-i}(t+\varepsilon) - \hat{H}_{-i}(t)\right] \\
&= \left[\min_{y\in[t,t+\varepsilon]} v_{-i}(t;y)\right] Q_i(t+\varepsilon).
\end{aligned}$$

Therefore, for any $\varepsilon > 0$,

$$\min_{y\in[t,t+\varepsilon]} v_{-i}(t;y) \leq 0.$$

However, $v_{-i}(t;y)$ is continuous and $v_{-i}(t;t) > 0$. This is a contradiction. □

A.1.2. Proof of Lemma 1.2

Lemma 1.2. *Assume a two-player all-pay auction where* $(\tilde{g}_i)_{i\in I}$ *satisfies the indifference condition in* (1.3)*. Then, for each* $i \in I$*, there exists* $\bar{s}_i \in S_i$ *such that*

$$\int_0^{\bar{s}_i} \tilde{g}_i(y) dy = \tilde{G}_i(\bar{s}_i) = 1, \tag{1.5}$$

and $\tilde{g}_i(s)$ *is positive for* $s \leq \bar{s}_i$.

Proof. The finite definite integral cannot diverge because the function is continuous. Also note that (1.3) gives us $g_i(0) = c'_{-i}(0)/v_{-i}(0;0) \geq 0$. This inequality is strict if $c'_{-i}(0) > 0$.

If the inequality is not strict, we find a positive value in a neighborhood of zero. Because c'_{-i} is positive everywhere but zero, we know that g_i is strictly increasing near zero. So there exists some $\delta > 0$ such that $g_i(s) > 0$ for $s \in (0, \delta)$.

We still need to confirm that $\tilde{g}_i(s) > 0$ on the relevant interval $\{s : \int_0^s |\tilde{g}_i(y)|dy \leq 1\}$. Suppose, by way of contradiction, that it is not. Then, by continuity, there must be an initial point $s^\star > 0$ such that $\tilde{g}_i(s^\star) = 0$, $\int_0^{s^\star} \tilde{g}_i(y)dy \leq 1$, and $\tilde{g}_i(s) \geq 0$ for all $s \leq s^\star$. However, this is impossible because

$$\begin{aligned}
\tilde{g}_i(s^\star) &= \frac{1}{v_{-i}(s^\star; s^\star)} \left(c'_{-i}(s^\star) - \int_0^{s^\star} v'_{-i}(s^\star, y)|\tilde{g}_i(y)|dy \right) \\
&\geq \frac{1}{v_{-i}(s^\star; s^\star)} \left[c'_{-i}(s^\star) - \left| \max_{y\in[0,s^\star]} v'_{-i}(s^\star; y) \right| \underbrace{\left(\int_0^{s^\star} |\tilde{g}_i(y)|dy \right)}_{\leq 1} \right] \\
&\geq \frac{1}{v_{-i}(s^\star; s^\star)} \underbrace{\left[c'_{-i}(s^\star) - \left| \max_y v'_{-i}(s^\star; y) \right| \right]}_{>0 \text{ (A1.2)}} > 0.
\end{aligned}$$

We must now show that it is not possible for $\int_0^\infty |\tilde{g}_i(y)|dy \leq 1$. We can do this in one step with Holder's inequality.

$$c_{-i}(s) = \int_0^s v_{-i}(s; y) g_i(y) dy \leq \left(\int_0^s |g_i(y)|dy \right) \left(\max_{y\in[0,s]} v_{-i}(s; y) \right)$$

so $\int_0^s |g_i(y)|dy \geq c_{-i}(s)/(\max_y v_{-i}(s;y))$ which is assumed to be greater than one as s approaches infinity (A1.3). By continuity, there exists an $\bar{s}_i$ such that $\int_0^{\bar{s}_i} |g_i(y)|dy = 1$ (A1.1). □

A.1.3. Proof of Corollary 1.1.1

Corollary 1.1.1. *Consider a two-player all-pay auction where $v_i(s;y)$ is continuously differentiable in both arguments for all $i \in I$ (A1.1 guarantees differentiability in the first argument). Then, we can alternatively express the unique equilibrium as*

$$G_i(s) = \left[\tilde{G}_i(\overline{s}_i) - \tilde{G}_i(\overline{s})\right] + \tilde{G}_i(s),$$

where

$$\tilde{G}_i(s) = \frac{c_{-i}(s)}{v_{-i}(s;s)} + \int_0^s \frac{\partial v_{-i}(s;y)}{\partial y} \frac{\tilde{G}_i(y)}{v_{-i}(s;s)} dy. \tag{1.6}$$

The solution admits the following series representation

$$\tilde{G}_i(s) = \frac{c_{-i}(s)}{v_{-i}(s;s)} + \int_0^s \frac{c_{-i}(y)}{v_{-i}(y;y)} \frac{R_{-i}(s;y)}{v_{-i}(s;s)}\, dy$$

where

$$R_{-i}(s;y) := K^0_{-i}(s;y) + K^1_{-i}(s,y) + K^2_{-i}(s;y) + \dots$$

for $K^0_{-i}(s;y) := \partial v_{-i}(s;y)/\partial y$ and $K^n_{-i}(s;y)$, $n = 1, 2, \dots,$ defined recursively by

$$K^n_{-i}(s;y) := \int_y^s \frac{\partial v'_{-i}(s;z)}{\partial z} \frac{K^{n-1}_{-i}(z;y)}{v_{-i}(z;z)}\, dz.$$

Proof. Equation (1.6) is obtained by applying integration by parts to (1.1). This defines a Volterra Integral Equation which has a unique solution by lemma 1.1. This solution coincides with the one in Theorem 1.1 because Equation (1.1) cannot have two solutions. □

A.1.4. Proof of Theorem 1.2

Theorem 1.2. *Consider a two-player all-pay auction where $v_i(s;y)$ is continuously differentiable in s and y for all $i \in I$. Suppose that the following two conditions hold:*

$$\frac{c_i(s)}{v_i(s;s)} < \frac{c_{-i}(s)}{v_{-i}(s;s)} \tag{1.8}$$

$$\frac{1}{v_i(s;s)}\left|\frac{\partial v_i(s;y)}{\partial y}\right| \leq \frac{1}{v_{-i}(s;s)}\frac{\partial v_{-i}(s;y)}{\partial y} \tag{1.9}$$

for all $s \in (0,\bar{s}]$ and $y \in [0,s]$. Then, Player i has a positive payoff.

Proof. Consider equation (1.6). The main result of Beesack, 1969 allows us to compare the solutions of two VIEs. In our setting, this means that conditions (1.8), (1.9) imply

$$\tilde{G}_2(s) \leq \tilde{G}_1(s) + \frac{c_1(s)}{v_1(s;s)} - \frac{c_2(s)}{v_2(s;s)} < \tilde{G}_1(s).$$

From this, it is clear that $\bar{s}_1 \leq \bar{s}_2$ which implies that player 2 has a mass point. The bound comes from

$$u_1 = v_1(0;0)(1-\tilde{G}_2(\bar{s})) \geq v_1(0;0)\left[\frac{c_2(\bar{s})}{v_2(\bar{s};\bar{s})} - \frac{c_1(\bar{s})}{v_1(\bar{s};\bar{s})}\right].$$

□

A.1.5. Proof of Corollary 1.2.1

Corollary 1.2.1. *Consider a two-player all-pay auction with spillovers. Suppose the players have the same value $v(s;y) \equiv v_1(s;y) = v_2(s;y)$, which is continuously differentiable in both arguments and $c_2(s) > c_1(s)$ for all s. Then Player 2 has a positive payoff only if*

$$\frac{\partial v(s;y)}{\partial y} < 0$$

for some s, y.

Proof. We want to apply Theorem 1.2. Because the prizes are identical, (1.8) is satisfied by the ranked cost assumption and (1.9) holds with equality as long as the derivative is non-negative. □

A.1.6. Proof of Proposition 1.1

Proposition 1.1. *Consider a two-player all-pay auction. Suppose the players value for the prize is given by the function $v(s;y) := v_1(s;y) = v_2(s;y)$, and that $c_2'(s) > c_1'(s)$ for all $s \in S_i \cap S_{-i}$. Then Player 2 has a positive payoff only if all of the following apply*

(i) Costs are not scaled: there does not exist a $0 < \lambda < 1$ such that $c_1(s) = \lambda c_2(s)$ for all s.

(ii) There exist some $t, z \in S_i \cap S_{-i}$ such that

$$\frac{\partial v(t;z)}{\partial z} < 0.$$

(iii) There exist some $t, z \in S_i \cap S_{-i}$ such that $v'(t;z) > c_2'(t) - c_1'(t) > 0$.

(iv) There exists some $s \in S_i \cap S_{-i}$ such that

$$\max_{y \leq s} v'(s;y) - \min_{y \leq s} v'(s;y) > c_2'(s) - c_1'(s) > 0.$$

(v) There exist $t, z \in S_i \cap S_{-i}$ such that

$$\frac{\partial v(t;z)}{\partial t \partial z} < 0,$$

i.e. the common value function is not weakly supermodular.

We first prove that it's not possible to have a reversal when costs are scaled (Condition 1).

Proof. Suppose $v(s;y) := v_1(s;y) = v_2(s;y)$ and $c_2(s) = \lambda c_1(s)$ where $\lambda > 1$. Because the two players have the same kernel, they must share the same resolvent (R). Then,

$$\begin{aligned}
\tilde{G}_1(\bar{s}) &= \frac{c_2(\bar{s})}{v(\bar{s};\bar{s})} + \int_0^{\bar{s}} R(\bar{s},y)\frac{c_2(y)}{v(y;y)}dy \\
&= \lambda\left(\frac{c_1(\bar{s})}{v(\bar{s};\bar{s})} + \int_0^{\bar{s}} R(\bar{s},y)\frac{c_1(y)}{v(y;y)}dy\right) \\
&= \lambda\tilde{G}_2(\bar{s}),
\end{aligned}$$

implying $\tilde{G}_1(\bar{s}) > \tilde{G}_2(\bar{s})$ and thus Player 2 must have a mass point at zero. □

Condition 2 is Corollary 1.2.1. We then show that a reversal is impossible when $v'(t;z) \leq c_2'(t) - c_1'(t)$ for all t, z (Condition 3).

Proof. We would like to show that $\tilde{g}_1(s) - \tilde{g}_2(s) > 0$ for all $s \in [0, \bar{s}]$. This would imply that $\tilde{G}_1(\bar{s}) > \tilde{G}_2(\bar{s})$, which implies that Player 2 must have the mass point at zero.

By (1.3):

$$\tilde{g}_1(s) - \tilde{g}_2(s) = \frac{c_2'(s) - c_1'(s)}{v(s;s)} - \int_0^s \frac{v'(s;y)}{v(s;s)} \left(\tilde{g}_1(s) - \tilde{g}_2(s)\right) dy.$$

This is positive by Lemma 1.2. □

Finally, we prove Conditions 4 and 5 by contradiction. The two proofs share the same initial setup.

Proof. Suppose, by way of contradiction that Player 1 does not have a positive payoff. Then, there exists an $\bar{s}$ such that $\tilde{G}_2(\bar{s}) = 1$ and $\tilde{G}_1(\bar{s}) \leq 1$. Note that $\tilde{G}_1(0) - \tilde{G}_2(0) = 0$ and

$$\tilde{g}_1(0) - \tilde{g}_2(0) = \frac{c_2'(0) - c_1'(0)}{v(0;0)} > 0.$$

Therefore, there exists some $r \in (0, \bar{s}]$ that is the first point such that $\tilde{G}_1(r) - \tilde{G}_2(r) = 0$ and $\tilde{g}_1(r) - \tilde{g}_2(r) \leq 0$.

If we assume $c_2'(r) - c_1'(r) > \max_{y \leq r} v'(r;y) - \min_{y \leq r} v'(r;y)$, then

$$\begin{aligned}
\tilde{g}_1(r) - \tilde{g}_2(r) &= \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \int_0^r \frac{v'(r;y)}{v(r;r)} [\tilde{g}_1(y) - \tilde{g}_2(y)] dy \\
&= \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \int_0^r \frac{v'(r;y)}{v(r;r)} \tilde{g}_1(y) dy + \int_0^r \frac{v'(r;y)}{v(r;r)} \tilde{g}_2(y) dy \\
&\geq \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \max_{y \leq r} \frac{v'(r;y)}{v(r;r)} \tilde{G}_1(r) dy + \min_{y \leq r} \frac{v'(r;y)}{v(r;r)} \tilde{G}_2(r) \\
&= \frac{c_2'(r) - c_1'(r) - [\max_{y \leq r} v'(r;y) - \min_{y \leq r} v'(r;y)]\, \tilde{G}_1(r)}{v(r;r)} > 0
\end{aligned}$$

If we assume $\partial v(r;y)/\partial r\partial y \geq 0$ a.e.

$$\begin{aligned}\tilde{g}_1(r) - \tilde{g}_2(r) &= \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \int_0^r \frac{v'(r;y)}{v(r;r)}[\tilde{g}_1(y) - \tilde{g}_2(y)]dy \\ &\geq \frac{c_2'(r) - c_1'(r)}{v(r;r)} + \frac{1}{v(r;r)} \int_0^r \frac{\partial v(r;y)}{\partial s \partial y} \left[\tilde{G}_1(y) - \tilde{G}_2(y)\right] dy > 0\end{aligned}$$

where we apply integration by parts in the second line. □

A.1.7. Proof that WoA with costly preparation approximates WoA

Proof. A direct application of (1.3) yields the following differential equation:

$$\tilde{g}^{\epsilon}_{-i}(s) = \frac{1}{\ell_i(s) - f_i(s)} \left(\ell_i'(s)\left[1 - \tilde{G}^{\epsilon}_{-i}(s)\right] + \epsilon'(s)\right)$$

Because this is a continuous linear mapping, we can take the limit as $\epsilon'(s)$ approaches zero. This simplifies to the same differential equation used to describe the equilibrium of the WoA (e.g. in Hendricks et al., 1988):

$$\frac{\tilde{g}_{-i}(s)}{1 - \tilde{G}_{-i}(s)} = \frac{\ell_i'(s)}{\ell_i(s) - f_i(s)}.$$

□

A.1.8. Proof of Theorem 1.4

Theorem 1.4. *Consider a symmetric n-player, m-prize all-pay auction with runner-up spillovers. Assume v, c satisfy assumptions A1.1 to A1.4. Let $\hat{G}$ be defined as in Corollary 1.1.1. That is, let $\hat{G}$ be the equilibrium cumulative distribution function of a* *two-player all-pay auction with spillovers:*

$$\hat{G}(s) = \frac{c(s)}{v(s,s)} + \int_0^s \frac{c(y)}{v(y,y)} \frac{R(s,y)}{v(s,s)}\, dy,$$

with

$$R(s,y) = K^0(s,y) + K^1(s,y) + K^2(s,y) + \dots$$

for $K^0(s,y) = \partial v(s,y)/\partial y$ *and* $K^t(s,y)$, $t = 1, 2, \dots,$ *defined recursively by* $K^t(s,y) := \int_y^s (\partial v(s,z)/\partial z)(K^t(z,y)/v(z,z))\, dz$.

Then, the symmetric equilibrium of the n*-player,* m*-prize all-pay auction with runner-up spillovers is given by the unique* G *that solves:*

$$\hat{G}(s) = \sum_{j=n-m}^{n-1} \binom{n-1}{j} [G(s)]^j [1 - G(s)]^{n-j-1}. \tag{1.12}$$

Proof. Fix a symmetric auction with n identical players and $m < n$ identical prizes. The expected payoff of a Player who bids s is

$$\int_0^s v(s;y) d\hat{G}(y) - c(s),$$

$\hat{G}$ is the $n-m$ order statistic of a sample of $n-1$ draws from the equilibrium distribution, G. By standard arguments similar to the ones used in Lemma A.1, any symmetric equilibrium will be in mixed strategies, with all players randomizing continuously on an interval $[0, \overline{s}]$ for some $\overline{s} > 0$ and expected payoffs will be zero for all participants. Therefore, the following condition holds for all $s \in [0, \overline{s}]$:

$$\int_0^s v(s;y) d\hat{G}(y) = c(s).$$

As in Lemmas A.1 and 1.1, the equilibrium condition can be rewritten as

$$\hat{g}_{-i}(s) = \frac{c_i'(s)}{v_i(s;s)} - \int_0^s \frac{v_i'(s;y)}{v_i(s;s)} \hat{g}_{-i}(y)dy. \tag{1.3}$$

which has a unique solution. Because this condition is the same as in the two-player case, we know that $\hat{G}$ is the two-player equilibrium. Because it is the $n-m$ order statistic of a sample of $n-1$ draws from G, we can write it as

$$\hat{G}(s) = \sum_{j=n-m}^{n-1} \binom{n-1}{j} [G(s)]^j [1-G(s)]^{n-j-1}. \tag{1.12}$$

All that remains is to show G can be recovered from $\hat{G}(s)$. To see that this is the case, we rewrite (1.12) as $\hat{G}(s) = E[G(s)]$ where

$$E[p] = \sum_{j=n-m}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-j-1}.$$

That is, $E[p]$ is equal to the survival function of the binomial distribution evaluated at $n-m$. This function is known to be strictly increasing in p for $p \in [0,1]$. Therefore, E is invertible. □

A.2. Optimal Contest Design

In this section, we consider how a designer should bias a contest to increase the scores. Several papers have analyzed this problem of assigning prizes to maximize total scores, or the average score of the winner. For example, Mealem and Nitzan, 2014 consider prize redistribution in a two-player all-pay auction with fixed values and symmetric costs. They show equalizing the prize values maximizes the total scores and that the this contest yields weakly more total score than any similar Tullock-type lottery contest. Che and Gale, 2003 investigate the optimal design of contests for innovation procurement, and find that the procurer might want to limit the maximum prize available to the most efficient firms – effectively eliminating any positive rents – in order to increase their own expected maximum surplus. The problem of optimal contest design in all-pay auctions with spillovers has not been previously analyzed.

This is relevant because principals are constrained in the prizes that they can offer. Many of the tools that principals use to make prizes have spillovers. For example, if an employer chooses to construct a compensation package using a cash bonus and stock options, then the inclusion of the stock options will generate spillovers. This section analyzes the optimal prize choice when prizes can be constructed from multiple instruments.

Let $\Lambda_i \subset \mathbb{R}^{\tilde{S}_i}$ denote the set of prize functions available to the designer for player i, and let $V : \prod_{i \in I} \tilde{S}_i \times \prod_{i \in I} \Lambda_i \to \mathbb{R}$ denote the designer's payoff function, i.e., given the pair of scores $\mathbf{s} := (s_1, s_2)$ and the pair of value functions $\mathbf{v} = (v_1(\cdot;\cdot), v_2(\cdot;\cdot))$, $V(\mathbf{s}, \mathbf{v})$ denotes the designer's derived net benefit from the contest.

We make the following (mild) assumptions:

Assumption A.1 (Completeness, DA.1)**.** *For each $i \in I$, set of prizes Λ_i, is convex and its closure contains an element with $v_i(\cdot;\cdot) \equiv 0$.*

Assumption A.2 (Productive scores, DA.2)**.** *For each $i \in I$ and $\mathbf{v} \in \prod_{i \in I} \Lambda_i$, the designer's objective function $V(\mathbf{s}, \mathbf{v})$ is strictly increasing in s_i.*

Assumption A.3 (Costly prizes, DA.3)**.** *For each* $i \in I$, $\mathbf{s} \in \prod_{i\in I} \tilde{S}_i$ *and* $v_{-i} \in \Lambda_{-i}$, $V(\mathbf{s}, \mathbf{v})$ *is decreasing in* v_i[2].

The primary complication with the construction in this paper is the mass point is difficult to compute. Fortunately, if the mechanism designer can discriminate between the two players, an optimal mechanism will have no atoms in many specifications. This is formalized in the following proposition.

Proposition A.1. *Assume a two-player contest where a fully informed principal with payoff function* V *chooses the prize* $v_i \in \Lambda_i$ *for each* $i \in I$. *Assume that* Λ_i *and* V *satisfy assumptions DA.1 to DA.3, and that for all* i *and all* $v_i \in \Lambda_i$, *assumptions A1.1 to A1.4 hold. Then, no contestant in equilibrium can have a positive payoff. Equivalently, no player will have a point-mass as part of their strategy.*

Proposition A.1 implies that there will be no strictly dominant player in any discriminating contest design problem where the principal benefits from the efforts of participants and pays for prizes. This proposition comes from the fact that the equilibrium strategy of the dominant player is locally invariant to changes in her prize value. Intuitively, for any contest with a strictly dominant player, there exists a more competitive contest where their prize is reduced and scores are larger.

Proof. Take an optimal choice of $\mathbf{v} := (v_i)_{i\in I} \in \prod_{i\in I} \Lambda_i$. Suppose, by contradiction, that player i has a strictly positive payoff. Her strategy is defined by

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s;s)} - \int_0^s \frac{v'_{-i}(s;y)}{v_{-i}(s;s)} \tilde{g}_i(y)dy,$$

[2]That is, if $v_i, \hat{v}_i \in \Lambda_i$ are such that $v_i(s;y) \leq \hat{v}_i(s;y)$ for all $(s,y) \in \tilde{S}_i \times \tilde{S}_{-i}$, then $V(\mathbf{s},(v_i,v_{-i})) \geq V(\mathbf{s},(\hat{v}_i,v_{-i}))$.

which does not depend on v_i. Because player $-i$ has an atom, we know that $\tilde{G}_i(\bar{s}) - \tilde{G}_{-i}(\bar{s}) > 0$. Therefore, there exists a $\gamma \in (0,1)$ such that $\tilde{G}_i(\bar{s}) = \tilde{G}_{-i}(\bar{s})/\gamma$.

Then, the principal could offer $(\gamma v_i, v_{-i})$ without changing the equilibrium strategy of player i. By the costly prizes Assumption DA.3, this is weakly preferable given a fixed distribution of s_{-i}.

By construction, player 2's new equilibrium strategy is $\tilde{G}_{-i}(\bar{s})/\gamma$. This first-order stochastically dominates player $-i$'s original strategy. In fact, it is the same distribution, but with the mass point removed. The productive scores assumption implies that this mechanism is strictly preferred. □

Proposition A.1 demonstrates that the expected welfare of all agents is zero in a large class of contest design problems[3]. It also suggests the optimality, from a design perspective, of handicapping the most efficient players (as in, the players with lower costs and lower marginal costs) The idea is very much analogous to the conclusion in Che and Gale, 2003, for example: handicapping the player that has the technological upper hand causes the less efficient player to become more aggressive, and to choose higher scores than they would otherwise.

A.3. Removing assumptions

An auction which satisfies all assumptions A1.1–1.4 but one may have equilibria which fail to meet our characterization. An auction which fails to satisfy any of A1.2–1.4 can have multiple equilibria.

[3]Which is not to say that there are no settings where it would not apply to. For example, the designer could wish to maximize the agents' expected welfare. In this case, the principal's objective function would violate costly prizes. It would usually also violate productive scores.

Assumption 1 (Smoothness). We assume continuous differentiability of v_i and c_i. Continuity is not sufficient to ensure that the equilibrium has interval support. For example, consider the case where the prize is fixed at $v_i = 1$ and the costs are given by the density function of some distribution which uniformly assigns probability one to a dense subset of $[0, 1]$ with Lebesgue measure zero.[4] This cost function is continuous because the distribution assigns uniform weight to infinitely many points. It is also strictly increasing because the support is dense. However, it is not absolutely continuous. Then, the aforementioned distribution is an equilibrium, which has support only on a set of measure zero.

Assumption 2 (Monotonicity). The case where $v_i'(s_i; y) > c_i'(s_i)$ for some s_i is considered in Siegel (2014) without spillovers. In this case, the equilibrium distribution has gaps and is thus not an interval. In the presence of spillovers, non-monotonicity may generate pure-strategy equilibria or result in non-uniqueness. For example, consider the symmetric game where

$$v_1(s; y) = v_2(s; y) = v(s; y) = \begin{cases} 1 + s - y & \text{if } s \leq 1 \\ (3 - s)s - y & \text{if } s > 1 \end{cases}$$

and $c_1(s) = c_2(s) = c(s) = s$. Note that this prize value satisfies all assumptions except for monotonicity, which is violated on $[0, 1]$. There are two asymmetric pure strategy equilibria where one player bids 0 and the other player bids 1.[5]

Assumption 3 (Interiority). Consider the symmetric all-pay auction with spillovers where $v_1(s; y) = v_2(s; y) = v(s; y) = 2\sqrt{y}$ and $c_1(s) = c_2(s) = c(s) = s$. Then, there is a pure strategy equilibrium where both players play zero. Moreover, there is also a mixed

[4] For example a uniform distribution over the countable union of cantor sets shifted by each of the rationals modulo one.

[5] This game also has a symmetric mixed-strategy equilibrium.

strategy equilibrium at

$$G_1^\star(x) = G_2^\star(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1, \end{cases}$$

i.e. equilibria are no longer unique.

In the other case where costs are no higher than the prize value in the limit (that is, $\lim_{s_i \to \infty} \sup_{y \in \tilde{S}_{-i}} v_i(s_i; y) \geq \lim_{s_i \to \infty} c_i(s_i)$), players might find it profitable to submit unbounbed bids. This can result in non-existence.

Assumption 4 (Discontinuity at ties). Consider the symmetric all-pay auction with spillovers where $v_1(s; y) = v_2(s; y) = v(s; y) = \mathbf{1}_{y \leq 1} 4\sqrt{1-y}$ and $c(s) = s$. Then, there is a symmetric equilibrium where

$$G_1^\star(x) = G_2^\star(x) = \begin{cases} \frac{1-\sqrt{1-x}}{2} & \text{if } x \in [0,1) \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 1 \end{cases}$$

such that both players have an atom at one, the point where the prize is worth zero in the event of a tie. The usual argument that the two players cannot have atoms at the same point fails here because a small increase in either player's bid does not increase the probability of winning a prize *of positive value*. There are also two asymmetric equilibria in this game of the form

$$G_i^\star(x) = \begin{cases} \frac{1-\sqrt{1-x}}{2} & \text{if } x \in [0,1) \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 1 \end{cases} \qquad G_{-i}^\star(x) = \begin{cases} 1 - \frac{\sqrt{1-x}}{2} & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1. \end{cases}$$

That is, one player has an atom at zero while the other has an atom at 1. Any convex combination of the symmetric equilibrium and the above is also an equilibrium.

A.4. Numerical Approximation

It is possible to approximate the solution by iterating numerically on this sequence:

$$\tilde{g}_{n+1}(s) = \frac{1}{v(s;s)} \left(c'(s) - \int_0^s v'(s;y)\tilde{g}_n(y)dy \right)$$

starting from $\tilde{g}_0 = 0$ to find the true $\tilde{g}$. There is a much simpler and faster way.

Consider our original equation

$$\int_0^s v_{-i}(s;y)\tilde{g}_i(y)dy = c(s)$$

and consider this 3×3 discrete approximation of this problem for $s \in [0,1]$

$$\frac{1}{3} \underbrace{\begin{bmatrix} v_{-i}(\frac{1}{3},\frac{1}{3}) & 0 & 0 \\ v_{-i}(\frac{2}{3},\frac{1}{3}) & v_{-i}(\frac{2}{3},\frac{2}{3}) & 0 \\ v_{-i}(1,\frac{1}{3}) & v_{-i}(1,\frac{2}{3}) & v_{-i}(1,1) \end{bmatrix}}_{\mathbf{V}} \cdot \underbrace{\begin{bmatrix} \tilde{g}_i(\frac{1}{3}) \\ \tilde{g}_i(\frac{2}{3}) \\ \tilde{g}_i(1) \end{bmatrix}}_{\mathbf{g}} \approx \underbrace{\begin{bmatrix} c_{-i}(\frac{1}{3}) \\ c_{-i}(\frac{2}{3}) \\ c_{-i}(1) \end{bmatrix}}_{\mathbf{c}}$$

So, we can approximate $\tilde{g}_i(s)$ with

$$\mathbf{g} = 3\mathbf{V}^{-1}\mathbf{c}$$

To get a good estimate, we do the same thing with an $N \times N$ grid for N large on some interval $[0, T]$.[6]

$$\begin{bmatrix} \tilde{g}_i(\frac{1}{N}) \\ \tilde{g}_i(\frac{2}{N}) \\ \vdots \\ \tilde{g}_i(T) \end{bmatrix} \approx N \begin{bmatrix} v_{-i}(\frac{1}{N}, \frac{1}{N}) & 0 & \cdots & 0 \\ v_{-i}(\frac{2}{N}, \frac{1}{N}) & v_{-i}(\frac{2}{N}, \frac{2}{N}) & \cdots & 0 \\ & & \ddots & \\ v_{-i}(T, \frac{1}{N}) & v_{-i}(T, \frac{2}{N}) & \cdots & v_{-i}(T; T) \end{bmatrix}^{-1} \cdot \begin{bmatrix} c_{-i}(\frac{1}{N}) \\ c_{-i}(\frac{2}{N}) \\ \vdots \\ c_{-i}(T) \end{bmatrix}$$

Once you get $(\tilde{g}_1, \tilde{g}_2)$ you just have to:[7]

(1) take the cumulative sum and divide by N to get $(\tilde{G}_1, \tilde{G}_2)$

```
G1, G2 = cumsum(g1)/N, cumsum(g2)/N
```

(2) truncate both distributions at the last value where both are ≤ 1

```
G1, G2 = G1[G1 <= 1 & G2 <= 1], G2[G1 <= 1 & G2 <= 1]
```

(3) add to each CDF vector so that both end with 1 (add the atom)

```
G1, G2 = (G1 - G1[-1] + 1), (G2 - G2[-1] + 1)
```

[6] The Python package `allpy` implements the approximation algorithm in this section and computes mixed-strategy equilibria of all-pay auctions with spillovers

[7] Sample Python code provided below each item.

APPENDIX B

Chapter 2 Appendix

B.1. Proofs

B.1.1. Useful lemmas

In this section, we prove some lemmas that are used in the proofs of the general bargaining results in Section 2.6. The main purpose of these lemmas is to ensure that the proofs extend to the setting of Peters and Wakker (1991). In the ultimatum case, $M(x, y) \equiv x$ and these lemmas are trivial.

Lemma B.1. *The functional,*

$$\mathcal{L}[\phi] \equiv \arg\max_{l, t \geq \phi(l)} M(f(l) - t, t - c(l)),$$

is sequentially continuous in the sense that $\phi_n \to \phi$ *(under the Hausdorff metric) implies* $\mathcal{L}[\phi_n] \to \mathcal{L}[\phi]$.

Proof. Suppose that M is continuous and satisfies weak Pareto, then $\mathcal{L}$ is continuous by Berge's theorem of the maximum. Continuity implies sequential continuity because the set of convex functions is a metric space under the Hausdorff metric.

In the setting of Peters and Wakker (1991), sequential continuity of bargaining ensures that the payoffs are sequentially continuous in ϕ. Continuity of hours and payment is immediate from continuity and strict concavity of the payoffs. □

Lemma B.2. *A point on the unconstrained Pareto frontier of M lies above all points on the Pareto interior in some neighborhood of itself if and only if it is the overall optimum.*

Proof. This is immediate from continuity and strict quasiconcavity of M. The purpose of this lemma is to show this result under the assumptions of Peters and Wakker (1991): Pareto optimality, IIA, and sequential continuity.

First, note that the overall Pareto frontier is the curve in (π, u) space where total surplus is maximized. Because of transferable utility, this is a line with slope negative one. Without loss, say that the space of feasible payoffs is the simplex.

Let z be a point on the Pareto frontier that is above all points in the interior contained in an ϵ-ball centered at z for some $\epsilon > 0$. Then, construct an $\epsilon/2$-ball such that the point is at the right with a triangle removed as in the left of Figure B.1. As δ decreases towards zero, the bargaining protocol must still choose z as it is the most preferred point. Continuity of the protocol implies that z must be chosen in the limit as well. Because this limit contains this segment of the Pareto frontier, z must exceed all points above it (preferred by the worker) on the frontier. The same argument can be applied for the points below using the $\epsilon/2$-ball to the right of Figure B.1.

Therefore, z is a local maximum on the frontier. By quasiconcavity of M, it is the global maximum on the frontier. By monotonicity, it is the overall optimum. $\square$

B.1.2. Proof of Proposition 2.1

Proposition 2.1. *Let ϕ be a redistributive regulation that implements ℓ. There exists a minimum wage, $\bar{w}$, that implements ℓ such that $\bar{w}\ell \geq \phi(\ell)$.*

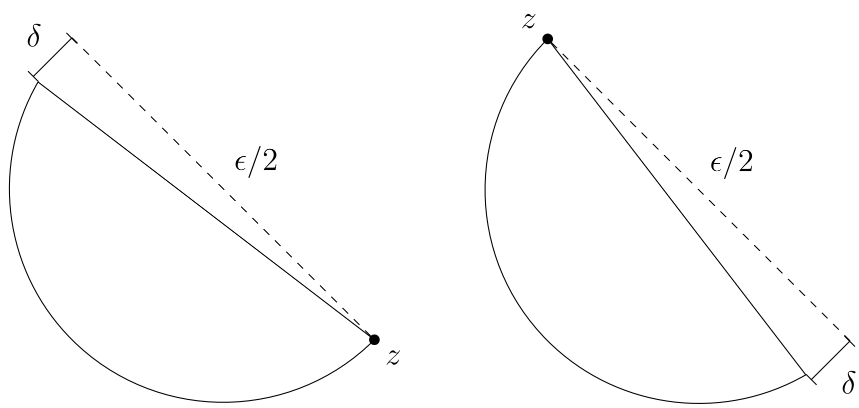

Figure B.1. Figure of $\epsilon/2$-balls for Lemma B.2. As $\delta \to 0$, this sequence approaches the full half circle, which contains a segment of the Pareto frontier.

Proof. We prove this in the general setting of Section 2.6 where the worker and firm contract according to a bargaining process defined by M. The proof for Section 2.4 is the special case where $M(x, y) \equiv x$.

Because ϕ improves the welfare of the worker, the policy is binding. Then, ℓ maximizes $M^{\phi}(x) \equiv M(f(x) - \phi(x), \phi(x) - c(x))$.

By Lemma B.1, the function,

$$L(w) \equiv \arg\max_{l} M(f(l) - wl, wl - c(l)),$$

is continuous. Because $L(f'(0)) = 0$, if a minimum wage implements labor hours, z, we can find a minimum wage to implement any labor hours less than z.

There are two cases.

In the first case, $\ell = \ell^*$. We established that $\bar{w} \equiv c'(\ell^*)$ implements ℓ^* and that $\phi'_{-}(\ell^*) \leq c'(\ell^*)$. The conclusion follows from convexity.

In the second case, $\ell \neq \ell^*$. Consider the minimum wage $\phi(\ell)/\ell$. Clearly, we are done if this also implements ℓ. If it implements labor greater than ℓ, continuity of L guarantees that an even larger minimum wage implements ℓ. Then, the only case left to check is that this minimum wage implements labor less than ℓ. This is impossible. Suppose by way of contradiction that the minimum wage does reduce labor. Then,

$$\arg\max_x M\left(f(x) - \frac{\phi(\ell)}{\ell}x, \frac{\phi(\ell)}{\ell}x - c(x)\right) < \ell.$$

However, this level of labor and transfer were available under ϕ (by convexity) and $(\ell, \phi(\ell))$ is available in the above. That both are the unique optima of their respective problems is a violation of IIA. □

B.1.3. Proof of Proposition 2.2

Proposition 2.2. *A flexible-hours model with cost, c_b, generates the same labor response curve as an ultimatum model with the same production function and cost, c_u, where*

$$c_u(x) \equiv c_b'(x)x.$$

Proof. In the flexible-hours model, the problem of the firm is to select a wage, w, and hours-cap, $\bar{\ell}$ knowing that the worker will best respond with labor hours, ℓ, according to $\ell = \min\{(c')^{-1}(w), \bar{\ell}\}$. Equivalently, by convexity of c, we can write that the firm chooses w, ℓ subject to the constraint $w \geq c'(\ell)$. Using the substitution $\tau = w\ell$, we can write the firm's problem under any policy, ϕ, as

$$\max_{\tau,\ell} f(\ell) - \tau \text{ s.t. } \tau \geq c'(\ell)\ell \text{ and } \tau \geq \phi(\ell).$$

This is equivalent to an ultimatum model where the worker's labor cost is $c'(\ell)\ell$. We use the assumption that this object is convex in order to ensure that this equivalent ultimatum model is valid. □

B.1.4. Proof of Proposition 2.3

Proposition 2.3. *The efficient minimum wage for the flexible-hours model with cost c_b (i.e., $\bar{w}_b^*$) locally minimizes welfare in the ultimatum model with cost c_u.*

Proof. By Proposition 2.2, we know that the two models generate the same contracts, (τ, ℓ). The efficient minimum wage in the flexible-hours model maximizes labor hours. Therefore, this same policy also maximizes labor hours in the ultimatum model. We want to show that the labor-maximizing minimum wage implements labor hours that exceed the efficient quantity. Because the total surplus is completely determined by the labor hours, this would ensure that the labor-maximizing minimum wage is furthest from efficiency of all policies in its locus. That is, it would be a local minimum.

This fact is proven in Corollary 2.2.3, but we can also find it directly by differentiating the firm's problem in the ultimatum model to find $\frac{\partial \ell}{\partial \bar{w}}\big|_{\bar{w} \to^+ c(\ell^*)/\ell^*} = \frac{\ell^*}{c'(\ell^*) - c(\ell^*)/\ell^*} > 0$ (by convexity of c). □

B.1.5. Proof of Proposition 2.4

Proposition 2.4. *For any labor response curve, L:*

- $c_b(x) < c_u(x)\ \forall x > 0$;
- $\bar{w}_b^* < \bar{w}_u^*$;
- $\ell_b^* > \ell_u^*$; *and*

- *for* $i \in \{b, u\}$ *and all* $\bar{w} \geq 0$,

$$\left.\frac{d\,[f(L(w)) - c_i(L(w))]}{dw}\right|_{w=\bar{w}} > 0 \implies \left.\frac{d\,[f(L(w)) - c_{-i}(L(w))]}{dw}\right|_{w=\bar{w}} < 0.$$

Proof. We go point by point.

- Immediate from convexity of c.
- Immediate from the shape of L and Proposition 2.2. The efficient minimum wage must be larger in the ultimatum model to bring labor back down to the efficient level.
- Immediate from the shape of L and Proposition 2.2. Alternatively, it is ensured by the fact that the marginal labor cost is greater in the flexible-hours model.
- Immediate from the shape of L and Proposition 2.2. Note that total surplus is increasing in the minimum wage in the flexible-hours model iff hours are increasing. This always decreases total surplus in the ultimatum model. The only segment where total surplus is increasing in the ultimatum model is the interval that begins at the labor maximizing minimum wage and ends at the efficient minimum wage. Total surplus is decreasing on this interval in the flexible-hours model because labor hours are decreasing.

□

B.1.6. Proof of Proposition 2.5

Proposition 2.5. *For* $i \in \{b, u\}$, $\bar{w}_i^*$ *is the largest minimum wage such that the worker is not underworked in model* i.

Proof. In the flexible-hours model, it is well known that minimum wages above the efficient minimum wage result in the worker obtaining fewer work hours than desired (i.e., labor is demand constrained).

In the ultimatum model, a minimum wage larger than $f'(\ell^*)$ will obviously result in hours fewer than ℓ^* because f is concave. Because $f'(\ell^*) = c'(\ell^*)$, the worker is also underworked with any such contract by convexity of c. □

B.1.7. Proof of Theorem 2.1

Theorem 2.1. *There is a unique TS-maximizing satisficing policy. It is*

$$\phi^*(x) \equiv \begin{cases} w_0 x & \text{if } x \leq \hat{\ell} \\ w_0\hat{\ell} + \frac{w_0\ell_0}{\ell_0 - \hat{\ell}}(x - \hat{\ell}) & \text{if } \hat{\ell} < x \leq \ell_0 \\ \infty & \text{if } x > \ell_0. \end{cases}$$

Proof. Refer to the Proof of Theorem 2.4 in Section B.1.11. This proof demonstrates that ϕ^* is both satisficing and minimal. By convexity of ϕ, the minimal *phi* also has the minimal derivative pointwise of any policy. Because the policy is satisficing – and therefore redistributive – the firm's problem can be written as

$$\max_{\ell} f(\ell) - \phi^*(\ell).$$

Any increase to the derivative of *phi* will weakly decrease the implemented ℓ. Because this ℓ is weakly less than ℓ^*, this reduction in labor hours reduces total surplus. Therefore, ϕ^* is a TS-maximizing, satisficing policy.

We now just need to ensure that the policy is unique. Suppose, by way of contradiction, that it is not. Then, there exists another satisficing policy, ψ, which is greater than ϕ at all points after some point, l, but implements the same total surplus for every possible production function and labor disutility.

Because ψ is satisficing, $l \in [\hat{\ell}, \ell^*)$. Then, select an f such that $f'(l+\epsilon) = \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}}$ for some $\epsilon > 0$. This implements labor $l + \epsilon$ under ϕ^* but less under ψ. It is clear that such an f is possible. □

B.1.8. Proof of Theorem 2.2

Theorem 2.2. *Let ϕ be an efficient regulation that implements τ. The worker is overworked under ϕ if and only if there exists another efficient regulation ψ such that $\psi(\ell^*) > \tau$.*

Proof. The converse is trivial. If $\tau_0/\ell^* < c'(\ell^*) = f'(\ell^*)$, $\phi(x) = f'(\ell^*)x$ binds and implements ℓ^*.

For the forward statement, we know that the policy ϕ implements ℓ^* and improves the welfare of the worker. Therefore, $z \equiv (f(\ell^*) - \phi(\ell^*), \phi(\ell^*) - c(\ell^*))$ is the constrained optimum but is not the overall optimum. By Lemma B.2, For any $\epsilon > 0$, there is an ϵ-ball around z such that z is not maximal. We want to show that that $f'(\ell^*) \geq \phi'_-(\ell^*)$.

Suppose, by way of contradiction, that $f'(\ell^*) < \phi'_-(\ell^*)$. Intuitively, decreasing ℓ in a neighborhood of ℓ^* is the same as relaxing the policy constraint because the marginal effect on profit and worker surplus are equal. As a result, there is an ϵ-ball around z available under ϕ, which is a contradiction. Formally, we can access the upper part of the ϵ-ball by transferring ϵ to the worker (because the constraint is in only one direction).

We can access the lower part of the epsilon ball by decreasing ℓ by $\frac{\epsilon}{\phi'_-(\ell^*)-f'(\ell^*)}$:

$$f\left(\ell^* - \frac{\epsilon}{\phi'_-(\ell^*) - f'(\ell^*)}\right) - \phi\left(\ell^* - \frac{\epsilon}{\phi'_-(\ell^*) - f'(\ell^*)}\right) = f(\ell^*) + \phi(\ell^*) + \epsilon - \mathcal{O}(\epsilon^2)$$
$$\phi\left(\ell^* - \frac{\epsilon}{\phi'_-(\ell^*) - f'(\ell^*)}\right) - c\left(\ell^* - \frac{\epsilon}{\phi'_-(\ell^*) - f'(\ell^*)}\right) = \phi(\ell^*) - c(\ell^*) - \epsilon - \mathcal{O}(\epsilon^2).$$

This comes from taking a Taylor approximation near ℓ^* and using the fact that $f'(\ell^*) = c'(\ell^*)$. For ϵ sufficiently small, we can access any angle in the ϵ-ball on the Pareto interior. This is a contradiction.

Therefore, $c'(\ell^*) \geq \phi'_-(\ell^*)$. Therefore, for all $x \leq \ell^*$, $\phi(x) \leq c'(\ell^*)x$ by convexity. Because $\tau_o < \phi(\ell^*)$, we conclude $\tau_0/\ell^* < c'(\ell^*)$. Therefore, the worker is overemployed.

□

B.1.9. Proof of Corollary 2.2.3

Corollary 2.2.3. *Let $L : \mathbb{R}_+ \to \mathbb{R}_+$ define the level of labor at each minimum wage and $w_0 \in [c(\ell^*)/\ell^*, c'(\ell^*))$ be the wage in the absence of regulation. Then, the labor response function, L, is continuous and*

$$L(x)\begin{cases} = \ell^* & \text{if } x \in [0, w_0] \\ > \ell^* & \text{if } x \in (w_0, f'(\ell^*)) \\ = \ell^* & \text{if } x = f'(\ell^*) \\ < \ell^* & \text{if } x \in (f'(\ell^*), f'(0)) \\ = 0 & \text{if } x \geq f'(0). \end{cases}$$

Proof. Continuity comes from Lemma B.1 and constant before the constraint binds is immediate from IIA.

We show the third point. The proof is similar to Theorem 2.2. At w_0, consider a small increase in the minimum wage. If the worker's incentive compatibility constraint binds, then the effect of the wage on the equilibrium is the same as under monopsony. Therefore, $\psi'_+(w_0) = \frac{\ell^*}{c'(\ell^*)-w_0} > 0$.

Then, suppose the constraint does not bind, we know that labor cannot decrease because this is Pareto dominated. Therefore, it weakly increases at this point.

As in the proof of Theorem 2.2, increasing the labor perfectly counteracts an increase in the minimum wage in a neighborhood around the point where it first binds. We want to show that $w \in (w_0, f'(\ell^*))$ leads to $\ell > \ell^*$.

Suppose, by way of contradiction, a $w \in (w_0, f'(\ell^*))$ supports a contract $z \equiv (\ell^*, w\ell^*)$. By Lemma B.2, for every $\epsilon > 0$, there is an ϵ-ball around z such that it is not maximal over all Pareto interior points. Therefore, there must be some angle within this ϵ-ball that cannot be accessed under minimum wage, w. However, this is not true. We can access the upper part of the ϵ-ball by transferring ϵ to the worker and can access the lower part of the ϵ-ball by increasing ℓ by $\frac{\epsilon}{f'(\ell^*)-w}$:

$$f\left(\ell^* + \frac{\epsilon}{f'(\ell^*) - w}\right) - \left(\ell^* + \frac{\epsilon}{f'(\ell^*) - w}\right) w = f(\ell^*) + w\ell^* + \epsilon - \mathcal{O}(\epsilon^2)$$
$$\left(\ell^* + \frac{\epsilon}{f'(\ell^*) - w}\right) w - c\left(\ell^* + \frac{\epsilon}{f'(\ell^*) - w}\right) = w\ell^* - c(\ell^*) - \epsilon - \mathcal{O}(\epsilon^2).$$

For ϵ sufficiently small, we can access any angle in the ϵ-ball on the Pareto interior. This is a contradiction.

The other points are justified in the statement of the theorem. □

B.1.10. Proof of Theorem 2.3

Theorem 2.3. *Any minimum wage in* $w^*(\alpha)$ *where*

$$w^*(\alpha) \equiv \arg\max_{w \geq f'(\ell^*)} \quad \alpha(w\ell - c(\ell)) + (1-\alpha)(f(\ell) - w\ell)$$

$$s.t. \quad \ell = \arg\max_{l} M(f(l) - wl, wl - c(l))$$

is a redistributive optimal regulation for $\alpha \in (0.5, 1]$.

The proof consists of two steps. We first find the optimal minimum wages (rather than the optimal policies). We then use Proposition 2.1 to show that any optimal minimum wage is also an optimal policy.

First, find the optimal minimum wages. The expression for an optimal minimum wage is, by definition,

$$w^*(\alpha) \equiv \arg\max_{w} \quad \alpha(\tau - c(\ell)) + (1-\alpha)(f(\ell) - \tau)$$

$$\text{s.t.} \quad \ell, \tau = \arg\max_{l,t} \{M(f(l) - t, t - c(l)) \text{ s.t. } t \geq wl\}$$

Second, simplify the expression. We would like to get rid of the minimum wage constraint and substitute $\tau = w\ell$. However, the minimum wage is not always binding. However, in optimizing, we can restrict attention to minimum wages greater than or equal to $f'(l^*)$ because: (1) this minimum wage maximizes total surplus and (2) a lower minimum wage redistributes less surplus to the worker because it involves less pay and more hours than desired by the worker.

Theorem 2.2 establishes that $f'(l^*)$ is a binding minimum wage. Therefore, the minimum wage constraint is binding on this restricted domain. Therefore, we can rewrite our expression to the one in the Theorem.

Finally, show that any optimal minimum wage is an optimal policy. By way of contradiction, suppose that an optimal minimum wage is not an optimal policy. Then, there exists a "superior" policy that strictly dominates the optimal minimum wage (in that it makes the objective larger). By Proposition 2.1, this "superior" policy is, itself, weakly dominated by a (possibly different) minimum wage. This is a contradiction because it implies that there is a minimum wage which strictly dominates an optimal minimum wage.

B.1.11. Proof of Theorem 2.4

Theorem 2.4. *There is a unique policy which is satisficing and NPD by any satisficing regulation. It is*

$$\phi^*(x) \equiv \begin{cases} w_0 x & \text{if } x \le \hat{\ell} \\ w_0 \hat{\ell} + \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}}(x - \hat{\ell}) & \text{if } \hat{\ell} < x \le \ell_0 \\ \infty & \text{if } x > \ell_0. \end{cases}$$

Proof. We showed in the body of the paper that ϕ^* is satisficing. To repeat the logic, note that for all $x \in (\hat{\ell}, \ell_0]$,

$$c(x) - c(\hat{\ell}) < \frac{x - \hat{\ell}}{\ell_0 - \hat{\ell}} \left[c(\ell_0) - c(\hat{\ell}) \right] < \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}}(x - \hat{\ell}) = \phi(x) - \phi(\hat{\ell}).$$

The increased transfers make up for the extra work. So, ϕ^* is satisficing.

We first show that any satisficing policy is greater or equal to ϕ^*. For this part, we can use the ultimatum model because it must be robust to any bargaining framework. The functions $f_\varepsilon(x) \equiv (w_0 + \varepsilon)x$ and

$$c_\varepsilon(x) \equiv \begin{cases} w_0 x - \frac{\varepsilon \ell_0}{2\hat{\ell}} x & \text{if } x \leq \hat{\ell} \\ w_0 x + \frac{\varepsilon}{2}(x - \ell_0) & \text{if } \hat{\ell} < x \leq \ell_0 \\ w_0 x + 2\varepsilon(x - \ell_0) & \text{if } x > \ell_0 \end{cases}$$

are feasible for all $\epsilon > 0$. Satisficing implies $\phi(x) \geq w_0 x$ for all $x < \hat{\ell}$ and $\phi(x) > w_0 x$ for all $x > \hat{\ell}$. However, if $\phi(\ell) > w_0 \ell$, for all ℓ there exists an ε such that the firm shuts down. So there must be some t such that $\phi(t) \leq w_0 t$ to be chosen in this case. For satisficing to hold, this implies $\phi(\hat{\ell}) = w_0 \hat{\ell}$. Now, consider $f_N(x) \equiv Nx$ and

$$c_N(x) \equiv \begin{cases} 0 & \text{if } x \leq \hat{\ell} \\ \frac{w_0 \ell_0}{\ell_0 - \hat{\ell}}(x - \hat{\ell}) & \text{if } \hat{\ell} < x \leq \ell_0 \\ w_0 \ell_0 + (N+1)(x - \ell^\star) & \text{if } x > \ell_0 \end{cases}$$

which are feasible. If there exists an $x > \hat{\ell}$ such that $\phi(x) < w_0 x + c_N(x)$, then there exists an N such that it will be chosen by the firm. Thus satisficing requires $\phi(x) \geq w_0 x + c_N(x)$ for all N. Therefore, satisficing implies that there must be a cap at ℓ_0. $\square$

B.1.12. Proof of Proposition 2.8

Proposition 2.8. *Let ϕ_i^* be the never Pareto dominated satisficing regulation for worker i. Suppose that the request, $\hat{\ell}$, is below the labor of each worker and that all workers are overworked. The unique never Pareto dominated regulation which is satisficing for all workers is*

$$\phi^*(x) = \max_i \phi_i^*(x).$$

Proof. The function, $\phi^*(x)$, is convex (because it is the maximum of convex functions) and satisfies $\phi^*(0) = 0$. Therefore, it is a regulation. We now show that it is satisficing.

For $x \leq \hat{\ell}$, there is a minimum wage equal to the maximum wage paid to any worker. Because the worker is overworked, this wage is less than $f'(\ell^*) < f'(\hat{\ell})$. Therefore, the firm prefers $\hat{\ell}$ to any point below x. If the worker works less than $\hat{\ell}$ as a result of the regulation, it is because she prefers this (by weak Pareto). Therefore, any contract with $\ell \leq \hat{\ell}$ is satisficing under this policy.

For $x > \hat{\ell}$, there is a region that is the maximum of all of the policies for each worker. These policies were designed such that all allowed contracts in this region were weakly preferred to the requested contract for all workers. By taking the maximum, we ensure that every allowed contract is weakly preferred by all workers.

This policy is never Pareto dominated because it is minimal. Suppose, by way of contradiction, that a smaller policy existed which was satisficing for all workers, then it would be smaller at some points than the minimal satisficing policy of at least one worker. This is a contradiction.

The proof that the minimal policy is uniquely never Pareto dominated follows the same argument as Theorem 2.4. □

B.2. Robust maxmin regulation

Suppose that a regulator with no knowledge of f, c regulates an ultimatum monopsony. The regulator knows that the assumptions on f, c hold. The objective of the regulator is

$$\begin{aligned} \max_{\phi} \inf_{f,c} \quad & \tau - c(\ell) \\ \text{s.t.} \quad & \ell = \arg\max_{x} f(x) - \max\{c(x), \phi(x)\} \\ & \tau = \max\{c(x), \phi(x)\}. \end{aligned}$$

The regulator has a lower bound on marginal productivity, $f'(x) \geq \bar{f}'(x)$, and an upper bound on marginal cost, $c'(x) \leq \bar{c}'(x)$ for all x. The regulator can make, $\bar{f}'$ weakly decreasing and $\bar{c}'$ weakly increasing using his knowledge that the underlying functions are concave. Assume that both bounds are left continuous.

Proposition B.1. *A minimum wage with a labor cap is a maxmin policy. One such policy is defined by*

$$\phi(x) = \begin{cases} \bar{f}'(t)x & \text{if } x \leq t \\ \infty & \text{if } x > t \end{cases}$$

where $t \in \arg\max_z \bar{f}'(z)z - \bar{c}'(z)$.

Proof. If $\bar{f}'(0) \leq \bar{c}'(0)$, the infimum in the regulator's objective is zero. The worker's payoff is at least zero by the individual rationality constraint. So, any policy is maxmin.

Suppose that $\bar{f}'(0) > \bar{c}'(0)$. By left continuity, there exists at least one t that solves $t \in \arg\max_z \bar{f}'(z)z - \bar{c}'(z)$. A minimum wage of $\bar{f}'(t)$ maximizes total surplus when the bound holds exactly by granting the worker the contract $(t, \bar{f}'(t)t)$. In this state, this gives the worker a surplus of $\bar{f}'(z)z - \bar{c}'(z)$. It is not possible to achieve more than this in maxmin because the minimal state is at least as bad as this one.

To guarantee this return, we need to ensure that the firm does not choose a larger level of labor, $\ell > t$, such that $\bar{f}'(t) < \bar{c}'(\ell)$. We did not place an upper bound on marginal productivity. So, this needs to be guaranteed with some sort of convex policy. We could, for example, place a labor cap at t or at the largest q which satisfies $\bar{c}'(q) \leq \bar{f}'(t)$. It's also possible to integrate $\bar{c}'$ and add this to the wage after t. □

B.3. Complete information regulation under heterogeneity

Suppose that there are $N \geq 2$ types of workers employed by a firm. Let c_i denote the cost of the i-th worker type. The costs and the production function, f, satisfy A1-3. For convenience, order the types in terms of efficient labor hours. That is, for $j > k$, $\ell_j^* \geq \ell_k^*$ where $\ell_j^* \equiv \arg\max_z f(z) - c_j(z)$ and $\ell_k^* \equiv \arg\max_z f(z) - c_k(z)$. The workers contract in accordance with the ultimatum model.

Assume that the regulator treats all workers equally. That is, the objective of the regulator is

$$\alpha \left(\sum_{i=1}^{N} \tau_i - c_i(\ell_i) \right) + (1-\alpha) \left(\sum_{i=1}^{N} f(\ell_i) - \tau_i \right) = \sum_{i=1}^{N} (2\alpha - 1)\tau_i + (1-\alpha) f(\ell_i) - \alpha c_i(\ell_i)$$

where $\alpha \in (0.5, 1]$.

Lemma B.3. *All workers who receive a positive surplus under regulation, ϕ, have the same contract: $(\tilde{\ell}, \tilde{\tau})$. This contract is defined by $f'(\tilde{\ell}) = \phi'(\tilde{\ell})$ and $\tilde{\tau} = \phi(\tilde{\ell})$. If that contract is a minimum wage, any worker, i, with zero surplus has $\ell_i \leq \tilde{\ell}$.*

Proof. If a worker, i, has a positive surplus, $\phi(\ell_i) > c_i(\ell_i)$. Therefore, the IR constraint does not bind and c_i has no effect on the firm's problem. Therefore, all workers with positive surplus must have the same contract defined by first order conditions $f'(\tilde{\ell}) = \phi'(\tilde{\ell})$ and $\tilde{\tau} = \phi(\tilde{\ell})$.

If a worker, k, has zero surplus under minimum wage, $\bar{w}$, $\tau_k = c_k(\ell_k) \geq \bar{w}\ell_k$. Suppose, by way of contradiction that $\ell_k > \tilde{\ell}$. By concavity of f,

$$\bar{w} = f'(\tilde{\ell}) > f'(\ell_k).$$

Because the regulation exceeds the worker's marginal productivity, the firm would prefer to hire the worker for fewer hours if this regulation were to bind. Therefore, the regulation does not bind and is therefore weakly below the cost of worker k. Then, $f'(\ell_k)$ is efficient and this minimum wage is above the efficient minimum wage of k. This is a contradiction.

□

Lemma B.3 shows that the homogeneous and heterogeneous worker problems are fundamentally similar. All workers who benefit from a regulation receive the same contract. So, there is no way to design a regulation that provides different redistributive contracts to different workers.

If only the utility of the worker matters to the regulator, heterogeneity essentially has no effect on the problem.

Proposition 2.6. *For $\alpha = 1$, there is at least one optimal regulation that is a minimum wage. Any optimal minimum wage is the same as the optimal minimum wage in a single worker problem. This single worker has the average cost of all workers with positive utility under the regulation.*

Proof. When $\alpha = 1$, the regulator maximizes the average utility of all workers. By Lemma B.3, all workers with positive payoffs have the same contract. For every subset of workers $S \subseteq 2^{1,\ldots,N}$, the regulator can solve

$$\max_{\phi} \sum_{i \in S} \phi(\tilde{\ell}) - c_i(\tilde{\ell}) \text{ s.t. } f'(\tilde{\ell}) = \phi'(\tilde{\ell}) \tag{B.1}$$

which is equivalent to

$$\max_{\phi} \phi(\tilde{\ell}) - \frac{1}{|S|} \sum_{i \in S} c_i(\tilde{\ell}) \text{ s.t. } f'(\tilde{\ell}) = \phi'(\tilde{\ell}).$$

This is the same as the single worker problem where c is replaced by the average of c_i for all $i \in S$. Therefore, the optimum is always a minimum wage.

Some of these problems may be *invalid* in the sense that the regulation does not actually benefit all workers in S. Lemma B.3 ensures $\ell_i \leq \tilde{\ell}$ for any worker, i, with zero utility. As a result, there is no way to reach these workers with a regulation. Therefore, the problem for S is invalid only if there is no optimal regulation that benefits all workers in S. Therefore, the optimal minimum wages are valid solutions to (B.1). The regulator chooses the set of S that maximize the objective. □

Proposition 2.6 suggests that multiple optimal minimum wages may exist. This is because the set of workers affected by the regulation may differ across optimal policies.

Example B.1 (Two optimal minimum wages)**.** Suppose a firm with $f(x) \equiv x - \frac{x^2}{2}$ contracts the services of two workers. Worker 1 has cost $c_1(x) \equiv \frac{7x^2}{2}$ and worker 2 has cost $c_2(x) \equiv \frac{x^2}{2}$.

The three candidate optimal regulation problems are

$$\max_{x,\bar{w}} \bar{w}x - \frac{1}{|S|}\sum_{i \in S} c_i(x) \text{ s.t. } \bar{w} = f'(x).$$

The solutions to the three candidate problems are $\ell_1 = \frac{1}{9}, \ell_2 = \frac{1}{3}, \ell_{1,2} = \frac{1}{6}$ with wages $\bar{w}_1 = \frac{8}{9}, \bar{w}_2 = \frac{2}{3}, \bar{w}_{1,2} = \frac{5}{6}$. The utility of worker 2 under $\bar{w}_1$ is

$$\bar{w}_1\ell_1 - c_2(\ell_1) = \frac{8}{9}\frac{1}{9} - \frac{(1/9)^2}{2} = \frac{5}{54} > 0.$$

This means the regulation is invalid because it should benefit only worker 1. The second benefits worker 2, but does not benefit worker 1 because

$$\bar{w}_2\ell_2 - c_1(\ell_2) = \frac{2}{3}\frac{1}{3} - 7\frac{(1/3)^2}{2} = -\frac{1}{6} < 0.$$

Therefore, it is valid and the benefit of this regulation is

$$\bar{w}_2\ell_2 - c_2(\ell_2) = \frac{2}{3}\frac{1}{3} - \frac{(1/3)^2}{2} = \frac{1}{6}.$$

The combined benefit of the joint regulation is

$$\begin{aligned}(\bar{w}_{1,2}\ell_{1,2} - c_1(\ell_{1,2})) + (\bar{w}_{1,2}\ell_{1,2} - c_2(\ell_{1,2})) &= \left(\frac{5}{6}\frac{1}{6} - 7\frac{(1/6)^2}{2}\right) + \left(\frac{5}{6}\frac{1}{6} - \frac{(1/6)^2}{2}\right) \\ &= \frac{1}{24} + \frac{1}{8} = \frac{1}{6}.\end{aligned}$$

This is the same as the effect of the optimal minimum wage for worker 2. Therefore, both $\bar{w}_2 = \frac{2}{3}$ and $\bar{w}_{1,2} = \frac{5}{6}$ are optimal minimum wage policies. While both are optimal, their distributive effects are different. △

The heterogenous worker problem has different implications for $\alpha \in (0.5, 1)$. The largest effect comes in the case of total surplus maximization ($\alpha \to 0.5$). An immediate implication from Lemma B.3 is that a regulation can be efficient and benefit more than one worker only if the workers who benefit have the same efficient hours. If efficient labor hours are strictly ranked, at most one worker can benefit from a regulation that maximizes total surplus.

However, if heterogeneity is small, the problems for $\alpha \in (0.5, 1)$ are always similar to the homogeneous case.

Proposition B.2. *For each $\alpha \in (0.5, 1]$, there exists an $\epsilon > 0$ such that $\max_x |c_i(x) - c_k(x)| < \epsilon$ for all $i, k \leq N$ implies there is at least one optimal regulation that is a minimum wage. Any optimal minimum wage is the same as the optimal minimum wage in a single worker problem. This single worker has the average cost of all workers.*

Proof. We only need to show that for any $\alpha \in (0.5, 1]$, there exists an ϵ such that the regulator gives every worker a positive payoff. Giving worker i a positive payoff requires $\phi(\ell_i) - c_i(\ell_i) > 0$. Therefore, any regulation that increases the payoff of one worker increases the payoff of all workers for ϵ sufficiently small. Therefore, we just need to show that there exists a binding optimal regulation.

The fact that there is such a regulation is obvious. Consider a minimum wage of $f'(\ell_N^*)$. This regulation gives strictly positive benefit to all players. Loss in total surplus can be made arbitrarily small. Therefore, this dominates not regulating for any $\alpha \in (0.5, 1]$. $\square$

From this proof, we can see that the required bounds on heterogeneity become more strict as α gets closer to 0.5. From Proposition 2.6, we know that no conditions are needed when $\alpha = 1$.

Lemma B.2 does not hold for $\alpha \to 0.5$. In this case, there is no exactly efficient regulation that increases the utility of workers when heterogeneity in costs is arbitrarily small. However, an exactly efficient regulation that increases worker utility may exist when heterogeneity is not small.

Proposition 2.7. *Let $\alpha \to 0.5$. Suppose $\ell_N^* \neq \ell_{N-1}^*$. If $\min_{i=1}^{N-1} \{c_i(\ell_i^*)/\ell_i^*\} > f'(\ell_N^*)$, the optimal regulation is a minimum wage of $f'(\ell_N^*)$. Otherwise, consider*

$$\phi(x) \equiv \begin{cases} Conv(x) & \text{if } x \leq \ell_{N-1}^* \\ Conv(\ell_{N-1}^*) + f'(\ell_N^*)(x - \ell_{N-1}^*) & \text{if } x > \ell_{N-1}^*, \end{cases}$$

and Conv is the largest convex function to fit under $\{(\ell_i^*, c_i(\ell_i^*))\}_{i=1}^{N-1}$ with the restriction that the slope is capped at $f'(\ell_N^*)$. If $\phi(\ell_N^*) > c_N(\ell_N^*)$, then ϕ is an optimal regulation.

Proof. Lemma B.3 and efficiency ensure minimum wage redistribution can only be used to benefit worker N. Affecting any other worker at all will reduce efficiency. Therefore, the regulation must be nonbinding for all other workers. If if the first condition holds, the efficient minimum wage is optimal because it is the largest efficient regulation for N.

If the second condition holds, any regulation for any other worker, i with slope $f'(\ell_i^*)$ at ℓ_i^* will also affect player N. Regulation, ϕ, is as large as possible while lying below all other points and having slope $f'(\ell_N^*)$ at ℓ_N^*. □

Proposition 2.7 demonstrates two important properties of these problems. First, the optimal regulation may not be a minimum wage. The reason that a piecewise linear regulation may be optimal in the heterogeneous worker setting is that it reduces the effects of regulation the contracts of workers with zero payoff. Such regulation only affects firms. So, the issue becomes less relevant as α increases. The second property is that minimum wage regulation is also optimal in settings where heterogeneity is sufficiently large. In this case, the problem is separable and the lowest cost worker can be regulated alone.

B.4. Oligopsony

We now add an entrant firm with production function g such that $f'(x) \geq g'(x)$ for all x. The incumbent must provide the worker enough surplus so that the entrant cannot make any profitable offer to the worker. Therefore, the profit-maximization problem of the incumbent under oligopsony is

$$\max_{\ell,\tau} f(\ell) - \tau \text{ s.t. } \tau \geq \max\{c(\ell) + u[\phi; g], \phi(\ell)\} \tag{B.2}$$

where u is maximum surplus that the entrant can offer. That is,

$$u[\phi; g] = \max_{\ell} g(\ell) - c(\ell) \text{ s.t. } g(\ell) \geq \phi(\ell). \tag{B.3}$$

Note that the regulation, ϕ, enters twice into the incumbent's problem. As in the monopsony case, it pushes the worker's salary up. However, it also constrains the maximum in (B.3). This means that the regulation reduces competitive pressure. This tension between regulation and competition is the main difference between oligopsony and monopsony.

B.4.1. Pre-regulation benchmark

In the absence of regulation, the incumbent offers the efficient level of labor and matches the best offer of the entrant. So, the incumbent offers contract (ℓ^*, τ_g^*) with $\ell^* \equiv \arg\max_x f(x) - c(x)$ and $\tau_g^* \equiv c(\ell^*) + u[0; g]$ where u is defined by (B.3).

Unlike under monopsony, it is now possible that the worker is underemployed. That is, she might prefer to work more hours at the average wage offered by the incumbent.

Lemma B.4. *Suppose an incumbent with production function, f offers labor quantity, ℓ and receives profits, Π. Then, the worker is underemployed if and only if the incumbent earns profit, $\Pi < f(\ell) - c'_+(\ell)\ell$ and is overemployed if and only if $\Pi > f(\ell) - c'_-(\ell)\ell$.*

For the incumbent, the right hand sides of the above inequalities do not depend on the production function of the entrant, g. However, the equilibrium profit is weakly decreasing in g. Therefore, markets with more competitive entrants (larger g) have underemployment and markets with less competitive entrants (lower g) have overemployment. Because the entrant receives zero profit, its best offer would underemploy the worker.

Because the entrant has lower marginal productivity than the incumbent, the entrant's best offer involves weakly less labor than the incumbent's. This fact combined with Figure

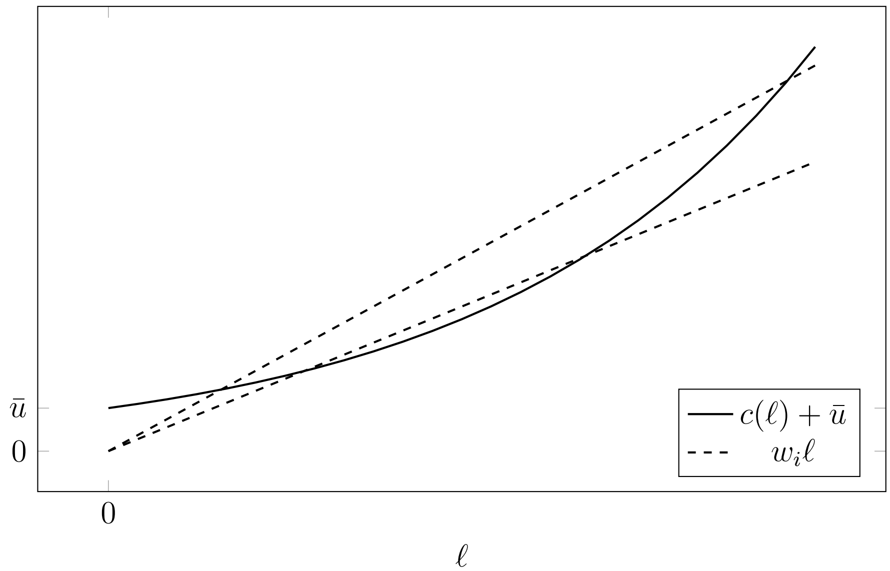

Figure B.2. A plot of labor holding the worker's utility constant. The lower intersections are contracts that underemploy the worker while the intersections at the upper part of the curve overemploy the worker at the same wages.

B.2 demonstrate the incumbent pays a lower wage than the entrant if the employee is underemployed. This is because the incumbent is compensating the worker with more labor, and therefore can pay less. If the worker is overemployed, the incumbent's wage may be greater than the entrant's.

B.4.2. Minimum wage regulation

The competitive constraint ensures that $\tau - c(\ell) \geq u[\phi; g]$. If this condition is binding for some policy, ϕ, it is impossible for the policy to increase the welfare of the worker over the pre-regulation benchmark because the worker's surplus is the left hand side of the constraint and u is weakly decreasing in phi. When the competitive constraint does not bind, the entrant is irrelevant.

Therefore, for any policy that increases the welfare of the worker, the oligopsony outcome and monopsony outcome are the same. Because of this, the justification for restricting attention to minimum wage policies under Monopsony, Proposition 2.1, also holds under Oligopsony.

On the other hand, the effects of minimum wages that do not improve the welfare of workers are very different under Monopsony and Oligopsony. The most apparent difference between the two is that a minimum wage can strictly reduce worker welfare because workers have strictly positive welfare in the pre-regulation benchmark.

Because of this, it is not obvious that the market can be efficiently regulated.

Proposition B.3. *Let (ℓ, τ) be the contract offered by the incumbent under minimum wage, $\bar{w} \geq 0$, and let ℓ^* be the efficient level of labor. Assume f is differentiable at ℓ^*. Then, there exists a larger minimum wage $\bar{w}' > \bar{w}$ that implements ℓ^* if and only if (ℓ, τ) overemploys the worker.*

Section 2.2 shows that, as in the neoclassical model, there is a binding minimum wage that achieves the efficient level of labor under a monopsony. It is well known that this is impossible under perfect competition when labor demand is a function.[1]

For $\bar{w} = 0$, Proposition B.3 demonstrates that if a worker is overemployed in the absence of regulation, the market is uncompetitive enough for a minimum wage regulation to be efficient. If the worker is underemployed in the absence of regulation, the market is competitive enough that efficient minimum wage regulation is impossible.

[1] When f is not differentiable, demand is a correspondence. That is, there may be an interval of wages assigned to any level of labor. In this case, imagine that supply intersects this demand at the bottom of this interval. Then, it's clear that you can impose a minimum wage to the top of the interval.

This means that a regulator need only know if workers desired hours exceed their actual hours in the pre-regulation benchmark to determine if efficient minimum wage regulation is possible. The proposition goes further to say that if there is already a minimum wage in place, a regulator can tell whether this existing regulation is above or below the efficient minimum wage just by observing whether employees are underemployed or overemployed.

Proposition B.4. *There f, g, c and ϕ such that, relative to the pre-regulation benchmark,*

- *worker hours are greater;*
- *the wage offered by the incumbent is strictly lower;*
- *the incumbent's profits are strictly larger.*

However, if the worker is underemployed in the pre-regulation benchmark, then none of the above are possible.