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Analytical Logistics: Modeling the Geographical and Temporal Decisions of Firms

RESEARCH REPORT

NORTHWESTERN UNIVERSITY EVANSTON • ILLINOIS

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Analytical Logistics: Modeling the Geographical and Temporal Decisions of Firms[a]

by

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Introduction

For many years the emphasis in private sector research and development expenditure in the United States, as elsewhere, was on: the development of ways of improving existing products; the discovery of new products; the development of improved techniques for marketing products; and the discovery of techniques of reducing the costs of producing existing products. Attitudes have changed in recent years. The belief has taken hold that cost savings in the field of physical distribution can be very great and that logistical or physical distribution management comprises "...today's frontier in business. It is the one area where managerial results of great magnitude can be achieved. And it is still largely unexplored territory."[1] To appreciate why this is the case, we should begin by understanding what the field of physical distribution or logistical management is usually considered to encompass. Its intellectual thrust is seen in some of the more popular definitions of the field:

> "...physical distribution management has emerged as the term most generally used to describe...the total cost concept of material flow ...the total movement and storage function...(of) inbound as well as outbound (freight) movements."[2]

Elsewhere the field is described as:

> "Physical distribution management is the term describing the integration of two or more activities for the purpose of planning, implementing and controlling the efficient flow of raw materials, in-process inventory, and finished goods from point of origin to point of destination. These activities may include but are not limited to customer service, demand forecasting, distribution communications, inventory control, material handling, order processing, parts and service support, plant and warehouse site selection... ."[3]

Efficient physical distribution can reduce a firm's costs of operation. It also increases the time and place utility or value of the goods produced by a firm.

> "The value or utility of moving materials available in a completed state is termed form utility. However, the product must also...be available in the right place, at the right time, and be available for purchase...Place and time utility are generally thought to be provided by the distribution activity, while marketing provides possession utility."[4]

There are several reasons for the growing awareness of the importance of physical distribution management in the efficient management of enterprise, public as well as private. Surely one of these reasons is the very high interest rate that prevailed during the recent years of high inflation. The real cost of holding inventories of finished products and of raw materials rose rapidly form the late seventies through 1981. It became extremely important for firms to find ways of buying and shipping and handling materials and finished products on as close to a continuous flow basis as possible in order minimize the interest costs of holding inventory.

Interest rates are a good deal lower today than they were in 1980 and 1981. However, two things concerning interest rates are a virtual certainty. First, as the present recovery proceeds, they are certain to rise in nominal terms, and there is a strong likelihood that they they will also rise in real terms. Second, we are unlikely to see a return in secular terms to interest rates in the five and six percent range. Hence, from both short and long run perspectives, efficiency in the management of inventories of material inputs and finished products will be much more important in the overall efficiency of firms' operations than it was in the decades preceeding the seventies.

The changes that have been occurring in transportation also make physical distribution and the handling of input and output flows much more important in the overall costs of firms. One of the changes is the rise in the price of fuels relative to other inputs and the resulting increases in the real costs of freight transport. These rates rose rapidly in real terms in the latter half of the seventies for a number of reasons, one of them being the short run success of the OPEC cartel in restricting the supply of petroleum and raising its price. The power of the cartel is now waning because of some success in conservation, a relative shift to other fuels, and a significant expansion in the supply of non-OPEC petroleum, coal, and other energy sources. However, the increases in the prices of fuels that have taken place since 1974 are by no means due solely to OPEC's direct impact on petroleum and its resulting indirect impact on substitute fuels. They are also due to a continued, strong secular increase in the demand for petroleum and coal in many production industries -- chemicals, fertilizers, synthetics, etc. -- which compete with the transportation industries for hydrocarbons. Worldwide usage of them in non-transport sectors is very likely to continue to grow strongly in the next several decades. Fuel prices are likely to increase in real terms in that period, and with them freight rates. It follows that the costs of handling and shipping inputs and outputs will continue to increase as a share of overall costs in firms' operations. The need to develop techniques and new approaches to logistics that will slow the increase in costs of materials handling, inventory, product distribution, etc. will continue to grow.

Above it was stated that there are several reasons for the growing importance of transportation in effective physical distribution management. One of those reasons -- the likelihood of continued secular increases in the real costs of freight transport due to rising energy prices -- has now been explained. Changes in the competitive environment in which transportation firms function is a second reason.

In the United States and many other parts of the world there is an increasing tendency to reduce the economic control and regulation of governments and their agents over transportation firms. Competition between carriers is increasingly being relied upon to determine rates, quality of service, market share, etc. The social benefits from increased competition are lower rates and service that is more diverse and offered at rates that more closely reflect the true economic costs of providing alternative qualities of service. However, the greater freedom that carriers have as to the rates they charge, the quality of service they provide, and the routes they choose to serve can entail costs and difficulties for shippers. These costs and difficulties can be of different types, but many of them involve an increase in the risk of doing business for some kinds of shippers.

Consider as an example of such a shipper, the firm that invested large sums to develop production of a low value product in an area that is served by only one railroad, at least for a significant part of the distance from point of production to various markets where the product is sold. Under deregulation and changes in government attitudes, the railroad that serves the area may raise rates very significantly in an effort to achieve revenue adequacy, i.e. to earn a normal rate of return on investment measured in current replacement cost terms. Moreover, the Interstate Commerce Commission (I.C.C.) has accepted the agrument that it is both rational economics and sound public policy for railroads to set rates in accordance with elasticities of demand as well as marginal costs.[5] This approach to railroad rate making, known as Ramsey Pricing, calls for rates to depart from marginal costs of transporting individual products in a way that is inversely proportional to the elasticity of demand for transporting them. That is, other things constant, the more inelastic the demand for transporting a product, the higher the rate that will be charged to transport it. Not only has the I.C.C. now accepted this principle, it has also gone on to add that it will rely on the railroads and shippers rather than its own staff to determine whether the principles of Ramsey Pricing have been adequately implemented by the carriers. Thus, it is now a matter of public policy to accept price discrimination in railroad rate making, at least until such time as the railroads are earning a normal return on investment. We can now return to our main theme concerning transportation deregulation, risk, and logistical management.

In the past, shippers could choose locations for production or other facilities that offered special advantages such as low labor or power costs, amenities, etc. The sites selected were frequently in areas of low density of population and economic activity. Oftentimes they were also far from the main corridors of transportation. Such locations could be chosen with a high degree of certainty that transport conditions would not change radically. The Commission would not permit sudden, sharp increases in rates, radical changes in frequency of rail service, and outright abandonment by railroads without extensive hearings. All of these things are now possible because of the recognition that transport must be free to function with (almost) the same degree of entrepreneurial freedom as other industries if the transport sector is to be socially efficient and healthy. But such freedom for carriers implies significant risks for shippers who have low elasticities of transport demand, whether due to the nature of the products they ship, the locations of their sites of production in relation to markets, sources of material inputs, etc. Such firms may face the risk of rapid and significant changes in the cost, quality, and availability of the transport they require for material inputs, outputs, and personnel.

There are ways of reducing the above risks, but they entail costs. So far as site selection is concerned, shippers can reduce the chances of radical changes in the tranport situations they face by choosing to locate facilities with good, easy access to the main corridors of transportation, the corridors within which there is considerable inter and intra-modal competition between carriers. The greater the choice of transport faced by a firm, the more elastic will tend to be its demand for the form of transport it is using, because of the possibility of shifting partially or wholly from one mode or carrier to another. Such a shipper is less likely to face serious economic price discrimination. However, it is also then likely to have to pay more for the land it requires for facilities. Land values tend to be higher in areas of great transport competition than in remote areas. Generally speaking the same is true of wage rates. Here we make the point that the risks associated with a less regulated transport environment can be reduced, but only at a cost.

Above we dealt with selection of new sites. Firms can also reduce the risks of rapid, significant, adverse changes in the transport situations they face at existing sites. They can do so by signing contracts with carriers that involve guaranteed tonnages and rates. However, this way of reducing the risks of changes in transport rates also entails costs. For example, in a recession a firm may find that the quantity it wishes to ship falls short of the contracted quantity. Furthermore, the firm that has signed a contract may find that it cannot avail itself of the lower price transport alternatives that become available in a recession.

This completes the introductory portion of this report. In it the argument has been made that the functions associated with physical distribution have become much more important in the overall efficiency of modern enterprise, public as well as private, than they were in the past. It has been argued that this growth in importance of physical distribution is due in part to the fact that the costs of performing its various functions have been increasing in real terms and are very likely to continue to increase in both the short and long runs. It has also been argued that the changes in transport regulation make the physical distribution function more risky for certain categories of shippers. The remainder of this report is divided into five parts. The titles and brief descriptions of the parts appear below.

Part 1: Physical Distribution: Progress and Shortcomings

The field's approach to efficiency is summarized, and some of its shortcomings are explained.

Part 2: Reasoning about the Firm in Time and Space

The time-space analog is developed in qualitative terms and the nature of the direct and cross relationships that exist between temporal and geographic variables is discussed. By way of illustration it is shown how a change in the cost of transporting a product through space can influence a number of temporal variables, such as the lenth of production runs at various plants and the amount of inventory held, as well as geographic variables, such as prices charged at markets that are more distant from production sites than other markets.

Part 3: A Model of the Firm in Time and Space

A formal model is presented that shows the interrelationships between the spatial distribution of plants, the temporal distribution of production, pricing and sales over time and space, and the geographic and temporal holdings of inventories.

Part 4: Models for Use

The theoretical understandings achieved in Part 2 and 3 are cast into solution procedures for solving some of the complex logistical problems that arise in the real world. Results are presented for three kinds of problems.

Part 5: Future Research

The emphasis in the academic year 1983-84 will be on empirical work and further development of solution procedures.

Footnotes

[1]P.F. Drucker, "Physical Distribution: The Frontier of Modern Management" in D.J. Bowersox, B.J. LaLonde, and E.W. Smykay, editors, Readings in Physical Distribution Management, (New York: The Macmillan Co., 1969), p. 8.

[2]G.A. Gecowets, "Physical Distribution Management," Defense Transportation Journal 35, no. 4, August 1979, p. 11.

[3]D.M. Lambert, J.R. Stock, Strategic Physical Distribution Management, (Homewood, Illinois: Richard D. Irwin Inc., 1982), pp. 10-11.

[4]National Council of Physical Distribution Management, NCPDM Comment 9, no. 6, November-December 1976, pp. 4,5.

[5]On this see the submissions in I.C.C. Ex Parte 347 by Kenneth Arrow; William Baumol and Robert Willig; Ronald J. Braeutigam and Leon N. Moses.

Part 1: Physical Distribution: Progress and Shortcomings

Physical distribution or logistics specialists are in quite general agreement about the functions that fall in the domain of the field, on the overall objective that should be pursued by them in performance of these functions, and on the major change that has taken place in the way that this objective is pursued. This part of the report begins with an explanation of these three areas of agreement. The statement of functions and the broad agreement by specialists on the objective to be pursued in the performance of those functions is followed by a brief critique of the field. The basic point made is that the accomplishments to date have largely been philosophical and qualitative and what is required in the next phase of development is a growing use of quantitative tools in which trade-offs between changes in such things as inventory policy, locational choices, length of production runs, and choice of transport are actually measured so that the various distribution functions can be truly integrated.

A field and a set of functions whose role in the profitability of business is growing in importance tends to attract intelligent, ambitious and aggressive people. Such people tend to push out the limits of their jobs, expanding the number and importance of the functions they perform. This has been happening in business logistics. Because the field is growing there is some small disagreement as to the functions it encompasses. Thus, there is some disagreement as to whether the forecasting of demand is a function that is a responsibility of physical distribution specialists. Some practitioners feel that demand forecasting is today appropriately considered a part of the field, even though it is probably performed most commonly in sales-marketing departments of firms. Physical distribution specialists will concede to production people responsibility for choosing the techniques of production. However, there is disagreement as to whether the timing of production runs for the different products of a company should be in the hands of the distribution or production specialist. There are such areas of disagreement as to the functions of the physical distribution specialist, but overall there is considerable agreement. The field encompasses the management and organization of the short run temporal and geographic flows and storage of material inputs from specified supply sources to existing plants, and of finished products from plants to specified markets. In the long run the selection of sites for plants and warehouses is another function that should normally be carried out by logistics specialists.

As indicated earlier, a second area of general agreement is in the definition of the objective of the physical distribution function. The goal is the minimization of the total costs of coordinating and integrating the various temporal and geographic stocks and flows that are required to produce products and deliver them to customers. Logistics specialists do not normally determine how products are to be produced, where materials are to be bought, where and in what quantities products are to be sold. However, they are seen as facing a large number of interrelated choices in determining the least cost method of organizing the temporal and geographic flows that link materials acquisition, production, and sales. The trade-offs between these various choices must be specified so that a distribution plan is developed that minimizes total system cost, all aspects of distribution being considered.

The third area of agreement among logistics specialists has to do with the adoption of a systems approach to cost minimization. It is felt that the systems approach really marks the change from the old to the new logistics. Specialists assert that in the past, distribution analysis was carried out as a set of diverse and poorly integrated choices. As a result, the total cost of the overall distribution function was rarely minimized. Thus, in two recent texts on distribution we find:

> "...improvements in techniques of analysis and methodologies facilitated the development of logistics. One such improvement was systems analysis or the systems concept ... Cost then is very often the criterion for evaluating system performance in logistics... (and) the effect of a change in the system usually measured by some criterion such as cost and the emphasis is upon the total cost of the system....If the initial change represents an increased cost then we may be trading off against decreased costs in other variables so that overall costs are reduced."[1]

And in another recent volume:

> "Total cost analysis is the key to managing the physical distribution function. Management should strive to minimize the total costs of physical distribution rather than attempt to minimize the cost of individual physical distribution activities (which)...may be suboptimal and lead to increased total costs.[2]"

But, is the emphasis on total costs, on the systems approach, on the careful evaluation of trade-offs between different costs and functions a reality? Some specialists, as well as the present writer, feel that the achievement of a quantitatively and theoretically sound approach to the total cost concept falls far short of what is needed, and in some cases also falls far short of what is claimed by specialists. Thus, the authors of a recent article state that:

> "Successful implementation of integrated distribution management is based on the analysis of cost trade-offs between and among the various components of the logistics system. However, the total cost concept cannot be implemented until the necessary cost data are available to distribution decision-makers."

The authors of the article go on to say that:

> "Many of the costs required for implementation of distribution cost trade-off analysis were not reported available by many respondents."[3]

The authors of the above paper feel that not enough data is collected to allow a total cost approach to integrated physical distribution management. The present writer believes that something much more fundamental than data is required. What is needed is a methodology, a sound theoretical model that deals with major components of the physical distribution system. An examination of the major writings in business logistics reveals a significant lack of attention to formal models that develop the trade-offs between major components of the physical distribution system in ways that are theoretically and quantitatively sound.

Consider, for example, the much referred to volume commissioned by the prestigious National Council of Physical Distribution.[4] Productivity data in a variety of distribution functions are considered, such as transportation, warehousing, purchasing and inventory management. However, several things must be said about the way in which they are treated. First, the treatment does not represent a quantitative analysis of trade-offs between expenditures on different kinds of functions. Second, even the treatment of the separate functions is largely qualitative. Third, in seeking ways to improve the efficiency of physical distribution, the authors of the report emphasize organization rather than an improved understanding of the nature of the complex trade-offs between the various cost elements and functions. Thus, the report concludes:

> "Perhaps the most effective means of impacting management productivity is by changing the organization of the distribution functions... Merely by recognizing the need for an integrated distribution department, many organizations are able to achieve a better coordination of all affected activities."[5]

Better organization, centralized control over physical distribution, and better data can certainly lead to reduced costs of the total distribution function of a firm. However, the claim of the present writer is that more soundly structured approaches to cost trade-off analysis are essential. Much of the work now done in distribution analysis amounts to little more than a listing of cost elements. A careful and systematic layout of the various costs that should be considered in a total cost approach to distribution decision-making can be useful.[6] The example developed below may lend both clarity and concreteness to the point that is being made about the need for theoretically sound models of trade-offs in distribution.

Consider a firm that has some number of plants, say two, located in different parts of the country. The firm produces and sells two products which it sells in some number of markets that are dispersed geographically. Each plant is capable of producing both products. However, the two products cannot be produced simultaneously in a single plant. Each plant can ship both products to all markets by truck or rail. Assume initially that the quantity of each of the two products that the firm must deliver for sale in each market is known over some period, perhaps each day of a month. Now consider the highly complex and interrelated kinds of production/distribution/transportation/inventory trade-offs that must be quantitatively analyzed if sound decisions are to be made.

One possible arrangement of activities is to have each of the two plants specialize in the production of one of the two products. Such plant-product specialization eliminates the need for switching from production of one product to production of another within plants, and therefore acts to reduce costs of production. If the firm's production cost functions exhibit a stage of increasing returns to applications of the variable factors of production, specialization of plants to products can bring further cost savings in production. Such specialization will also have an impact on expenditures for transportation, but the impact can go either way, as is explained below.

Specialization of plants to a single product means that more of a given product is shipped out of each plant. Increases in tonnage can mean a lower transport rate per unit of product shipped. Such plant specialization might, for example, lead to a switch to carload quantities and the substitution of lower cost rail for truck transport. On the other hand, specialization of plants to single products means that each plant must ship to all markets. In other words, if the firm chooses to have plants specialize in production, it must forego another kind of specialization, that in which plants specialize in markets they serve. If a plant must ship the product it produces to all markets, it will in general be shipping greater distances and this raises expenditures on transportation. Therefor, even within the transport sector, there is a trade-off involved in plant-product allocation choices: that between the savings that follow from larger quantities being shipped per unit time, and the additional costs due to the greater distance products are shipped.Specialization by product and efforts to achieve lower transport charges per unit product shipped given distances also generally mean that larger inventories of final product are kept, which increases the costs of storage and inventory carrying costs.

Above I have recounted the kinds of choices and trade-offs that are involved in a problem that involves multiple plants, multiple products, multiple modes, multiple markets, and multiple periods of analysis. Were we to present the above problem to a physical distribution specialist and ask for an explanation of the basic trade-offs it involves, the individual would undoubtedly list those presented above. However, a listing is not enough. It is a qualitatively important first step that provides significant insights. Two additional steps must follow if management is to be provided the analytical tools needed to improve the quality of the complex decisions that must be made in physical distribution. The qualitative insights of the interrelationships and trade-offs between elements must be embedded in a formal quantitative model. That model must then be empirically estimated so that the effects of alternative decisions can be simulated, or an effort made to apply optimization techniques.

All too frequently, writings of the academic scholars who specialize in logistics stop at the qualitative stage. Practitioners are told the various kinds of trade-offs that should be investigated. When advice is also given on the kinds of formal, quantitative analysis that should be worked out, the advice and the tools recommended are frequently much too simple for the complexity of the decisions that practitioners must make. Such quantitative modeling as is done is frequently within a given portion of the spectrum of physical distribution issues. For example, it would not be unusual to find an analysis of how an increase in carrying costs effects the quantities of a product that are ordered at various points of time and the frequency of ordering. What is much less likely, is to find quantitative tools for measuring trade-offs between quite different elements in the spectrum of physical distribution problems. For example, suppose the interest rate rises. The firm decides that it may be worthwhile to reduce its holdings of finished product inventories. Such a decision might entail at least a partial shift of shipments from rail to truck. Shifts in transport may mean a reallocation of markets to different production facilities, and a reallocation of production between plants if total costs are to be minimized. Such trade-offs are at the very heart of what physical distribution specialists claim is the essence of their craft, i.e. the minimization of the costs of integrated distribution systems. However, it is rare to find formal models for investigating complex chains of trade-offs. When such models do appear in the literature, they are frequently linear. Such models cannot capture the essence of many distribution problems because these problems entail significant non-linearities, i.e. economies of large scale production, of shipment, of storage, of order size, etc. One last point remains to be made in this section of the report. It has to do with the responsiveness of demand to prices charged, the relationship of distribution to marketing, and the validity of the total cost criterion for physical distribution decision-making.

Some contributers to the logistical literature have expressed a concern over what they view as the excessive concentration of physical distribution studies on the goal of cost minimization. These writers claim that other goals also deserve consideration. Surely one such goal is the maximization of profits.

Most logistics specialists take the position that they manage and attempt to minimize the costs of the flows and inventories associated with the following: (1) the delivery of quantities of final product(s) to various markets as specified by marketing specialists; (2) the production of quantities of various products at alternative plants as specified by production specialists; (3) the acquisition of required amounts of various material inputs into production processes from alternatives cited as specified by the firm's production specialists and buying agents. The integration of these decisions into an overall plan is seen by most physical distribution specialists as being the responsibility of those executives in the firm who are responsible for overall corporate decision-making. However, there are physical distribution experts who believe that their analyses should be capable of being integrated with the decisions of those within the firm whose responsibilities encompass objectives that go beyond the fixed quantities-total cost approach of most business logistics studies.

Surely one area where such integration would be extremely valuable is marketing and price setting. Marketing specialists such as Phillip Kotler of Norhtwestern University accept the economist's approach to the determination of the quantities of product sold by a firm. The economists' demand models involve a responsiveness of quantities sold to such variables as prices charged, advertising expenditures, and product quality. Current physical distribution models are not capable of integrating marketing decisions as to variables that influence quantity of product sold in different markets. Such integration would entail an approach in which marketing and physical distribution decisions are made simultaneously and take into account the interactions of both sets of decisions on the firm's profits. This is very different from an approach in which marketing specialists first make decisions on the variables that determine quantities sold in different geographic or temporal markets, and physical distribution specialists then take these quantities as given and attempt to minimize total logistical costs associated with them. A brief summary statement of the state of the art in logistical modelling and decision-making as seen by the present writer is now in order.

Most logistics specialists attempt to reason in terms of a fully integrated distribution systems. In their work they have the goal of minimizing the total costs of such systems, given the quantities of product to be delivered to various markets, the quantities of material inputs bought from various sources, etc. The idea of trade-offs between various cost elements is an essential part of such cost minimization. Such reasoning represents a significant improvement over the logistical studies that were carried out prior to the development of the systems concept. The position of the present writer is that much remains to be done before the goal of minimizing the total cost of integrated physical distribution systems can be achieved. Most logistical studies do not in fact encompass complex trade-off relationships between diverse elements of the distribution system except in purely qualitative terms. When quantitative models are employed they tend to be within a given segment of the physical distribution system. The well known EOQ model is an example. Where complex trade-offs are involved, the logistical models currently in use tend to be linear in nature. Such models cannot really achieve the goal of cost minimization because physical distribution systems involve significant non-linearities. Finally, the goal of cost minimization may not be the appropriate goal. If the goal of the firm is to maximize profits, then it is necessary to work with demand responsive rather than fixed quantity models. In other words, it is necessary to employ approaches in which decisions as to the variables that influence quantities of product sold (or quantities of inputs bought) and variables that influence distribution cost are both considered.

Footnotes

[1]J.J. Coyle and E.J. Berdi, The Management of Business Logistics, West Publishing Company, 1980, St. Paul, Minnesota, pp. 22, 25.

[2]D.M. Lambert and J.R. Stock, "Strategic Physical Distribution Management", Richard D. Irwin, Inc., Homeward, Illinois, 1982, p. 36.

[3]D.M. Lambert and J.J. Mentzer, "Is Integrated Phusical Management A Reality?" Vol. 2, No. 1, 1980.

[4]Measuring Productivity In Physical Distribution, by A.T. Kearney, Inc., prepared under contract to the National Council of Physical Distribution Management, Chicago, Illinois, 1978.

[5]Ibid., p. 222.

[6]An excellent example of such a systematic listing of distribution costs is found in H.W. Davis and Company, Workbook for a Physical Distribution Productivity Audit, (Marketing Publications, Inc., Washington, D.C., 1980).

[7]H.M. Wagner and T.M. Whitin, "Dynamic Problems in the Theory of the Firm," Naval Research Logistics Quarterly, March 1958, pp. 53-74 and T.M. Whitin, "Dynamic Programming Extensions to the Theory of the Firm," Journal of Industrial Economics, April 1968, 16, pp. 81-99.

[8]It can readily encompass cases in which storage as well as production cost exhibit U-shaped average and marginal costs.

[9]The solution procedures used to solve actual problems involve algorithms built around the techniques of non-linear programming.

Part 2: Reasoning about the Firm in Time and Space[1]

The theory of the firm and its decisions as to output, prices charged etc. is one of the most highly developed and powerful parts of the entire body of microeconomic theory. The literature in the theory of the firm is rich and goes back to the very beginning of economics as a field. Two important sub-areas in the economic theory of the behavior of firms are: (1) the theory of dynamics, or the study of behavior over time; (2) the theory of location and spatial competition, or the study of the decisions that firms make in space. Logistics or physical distribution attempts to deal with both of these kinds of decisions. However, there does not exist in the field of logistics an integrated, soundly constructed theory of how firms go about optimizing simultaneously across space and time. Nor is there at present an economic model of the interactions between temporal and spatial variables. In this part of the report, the reader is introduced to a way of looking at space-time trade-offs. The treatment is qualitative. The kind of firm dealt with in the entire report is explained. A general, largely intuitive explanation is offered of how our firm can adapt to changes in a purely spatial parameter, such as transport cost, by changing its temporal as well as its spatial plan; or how it can adjust to a change in a purely temporal parameter, the cost of storage, by changing its spatial as well as its temporal plan. The formal model of space-time interrelationships and decision making is presented in Part 3.

The firm about which we reason in this report is an imperfect competitor. It faces a downward sloping demand curve(s) for its product(s) and must therefore decide on the price(s) it should charge. The firm has a cost function that exhibits the typical shape of much of the reasoning in microeconomics. That is, whether due to scale economies in the long run, or variable returns to variable factors in the short run, the firms marginal and average cost curves exhibit both declining and increasing stages. In other words, the firm's total cost curves, short and long run, are in part concave and in part convex. This form of the cost function causes considerable difficulty in the theory of profit maximization over time and space, and in the solution of actual problems. Indeed, on the latter point, it is most unusual to find optimization procedures in the operations research literature that employ such cost functions, and the present writers know of only one study of profit maximization over time and none over time and space in which such cost functions are combined with downward sloping demand functions.[2]

In order to maximize profits, the firm with which we deal in all of our work must combine the markets it serves from its production facilities. That is, it must aggregate markets across time, across space, or both in order to maximize profits. Even with such aggregation eah of its plants operates to the left of minimum average cost at each point of time when production takes place.

There are a number of reasons why we choose to cast our theoretical and applied work in terms of an imperfect rather than a perfect competitor, and also to develop our models in terms of the traditional cost function of economic theory. One reason for the choice is that we wish to develop a pure theory of the temporal and spatial choices made by the firm about which we are reasoning. In the present context the use of the word 'pure' is meant to convey the idea that the temporal and spatial choices made by the firm originate largely from conditions within it rather than outside it. This notion requires elaboration.

It may be optimal for the firm about which we are reasoning to choose to operate more than one plant, some number of markets or some market area being served exclusively by each plant. A perfectly competitive firm may be found to be operating more than one plant but if so it is with one major exception, explained below, due to changes in its environment. A perfectly competitive firm can sell as much output as it wants in its market without the need to lower its price. Hence in static conditions the perfectly competitive firm chooses that one plant-market combination that maximizes profits. A perfectly competitive firm can be found to be operating more than one plant, but the likely explanation is that something has changed since it made its original one plant-market profit maximizing decision. Costs of production or technology may have changed or prices in different markets changed, or costs and prices may be expected to change, so that a new one plant-market combination is optimal. However, it may not be possible for the firm to instantaneously begin production and sales in the new optimal plant-market combination and discontinue operations in the old one. Hence for some period of time the firm is found to be a multi-plant firm, but the situation is the result of changes in its environment and adjustment time.[3]

We develop our models in terms of cost functions that involve first falling and then rising marginal and average costs because that is the traditional cost function of economic theory. We want our models to square with and represent a natural outgrowth of the traditional economic theory of the firm. However, there are other, equally important reasons for the choice of cost function. Despite the assumption of a downward sloping demand function, our firm would not choose to sell in multiple markets and to operate multiple plants if its cost functions were strictly linear or quadratic, unless costs or demands changed over time. Our goal, as explained earlier, is the development of a pure model of behavior in time and space. Our firm may choose to sell in more than one market and operate more than one plant because that is what it is optimal for it to do, even if demands and costs are everywhere the same, are expected to remain unchanged over time, and are not subject to any randomness. Our firm may choose to behave dynamically rather than statically even in these conditions, because such behavior maximizes its profits. It produces in excess of sales at certain times and holds inventory. It sells off that inventory, reducing it to zero before it begins another production run. The space-time models developed in this report are perfectly capable of analyzing decision making when cost functions and demand functions change over time.[4] However, we wish to show that the adoption of a dynamic production policy, including inventory accummulation, and the adoption of a multiple market - multiple plant strategy can represent the profit maxing solution even if demand functions and cost functions are identical over space and are invariant over time.

There is nothing terribly surprising about a firm's accumulating product inventory if it expects the demand function it faces to shift up over time, or if it expects costs of production to rise over time. Variations over time can cause even the perfectly competitive firm to accumulate inventories of finished products and of material inputs and, at least transitionally, to operate more than one plant. What is much more fundamental is to demonstate the conditions under which a firm will operate in these ways even when costs and demands are temporally and spatially invariant. We do so in this part of the report in a largely qualitative way, the goal being to offer general insights into the space-time analogy, and the conditions under which a firm operates fewer plants than markets served and operates each of those plants under a dynamic regime. We use the basic logic of profit maximization to offer insights into how a change in a parameter, say the cost of transporting the product to certain markets, can have two sets of effects, direct effects and cross effects. By a direct effect we mean an effect on some geographic variable, the most obvious one being that the firm may choose to reduce or reallocate the sales from any given plant to those markets where transport cost has not increased.[5] By a cross effect of a change in a temporal parameter we mean a change in, for example, the time period over which the output of a production run is sold. Similarly, a change in some spatial parameter, such as the cost of storage, can also have direct effects on some temporal variables and cross effects on spatial variables. We begin this general discussion by dealing with the inventory problem and ignoring spatial considerations.

The firm about which we are reasoning is an imperfectly competitive firm. At each point of time it faces a downward sloping demand curve. The general model developed in this report can handle shifts over time in the firm's demand function, but at this stage we assume the demand function is temporally invariant. The introduction of a regular demand function into the analysis means that the firm may have a temporal pricing-sales policy as well as a production-inventory policy as part of its overall profit maximizing strategy. The former departs from what has been a long standing tradition in inventory theory. Almost all models of inventory behavior assume that the firm sells a fixed amount of product in each period, though that quantity can vary deterministically over time, reflect the workings of a random distribution, or be uncertain. In other words, the great majority of inventory models do not attribute to the firm a pricing policy even though they are clearly dealing with firms that are not perfect competitors. In this regard, it is worth noting that a recent, excellent and very comprehensive volume on inventory theory devotes only two of its approximately 800 pages to a review of models in which firms make price decisions.[6]

The absorbtion of inventory theorists with fixed quantity models is somewhat strange since almost a quarter century has passed since a path-breaking article was written in which a downward sloping demand curve was introduced into a model that had storage, interest and other costs usually found in inventory studies.[7] On the cost side, almost all production-inventory models preclude scale economies in the long run, and the influence of changing factor proportions on the productivity of variable factors in the short run. The two elements that we introduce, downward sloping demand functions and U-shaped average and marginal cost curves are clearly part of the traditional, static economic theory of the imperfectly competitive firm. By coupling them to a model in which the firm can store product we can generate dynamic behavior in pricing, sales, production, and inventory even though the firm's demand and cost functions are temporally invariant. The basic argument is developed around a two period case, and is illustrated with some figures.

Our firm faces a demand function which for convenience in the graphic treatment we assume to be linear. The inverse of this demand function is:

$$(1) \qquad P = \bar{P} - aq$$

where P is price, $\bar{P}$ the intercept on the vertical (price) axis, and q is quantity. The firm faces this demand function in each of the two periods. The firm's cost of production is a cubic cost function that yields the traditional U-shaped average and marginal cost curves.

$$(2) \qquad C = bq - gq^2 + jq^3$$

We assume initially that the firm behaves statically, producing in each period exactly what is sold in that period and carrying no inventory of finished products. This means that q stands for both production and sales.

The firm's total revenue function, R, is obtained by multiplying through by q in equation (1).

$$(3) \qquad R = \bar{P}q - aq^2 \,.$$

Profit, π , is the difference between revenue and cost of production:

$$(4) \qquad \pi = \bar{P}q - aq^2 - bq + gq^2 - jq^3$$

The firm's maximum profit position is found by differentiating π in equation (4) and setting it equal to zero:

$$(5) \qquad \frac{d\pi}{dq} = \bar{P} - 2aq - bq + 2gq - 3jq^2 = 0$$

By solving this equation for q we determine the profit maximizing rate of output which, in the static strategy, is exactly equal to sales in each period. The optimal quantity, q, is then entered into equation (1) and the price the firm charges is also determined. From the point of view of the graphics of the figures employed, it is useful to explain the profit maximization in terms of incremental or marginal cost, MC, and incremental or marginal revenue, MR.

The derivative of the total cost function, equation (2), is marginal cost:

(6) $$\frac{dC}{dq} \equiv MC = b - 2gq + 3jq^{2}.$$

Incremental or marginal revenue, MR, is the derivative of the total revenue function, equation (3):

(7) $$\frac{dR}{dq} \equiv MR = \bar{P} - 2aq.$$

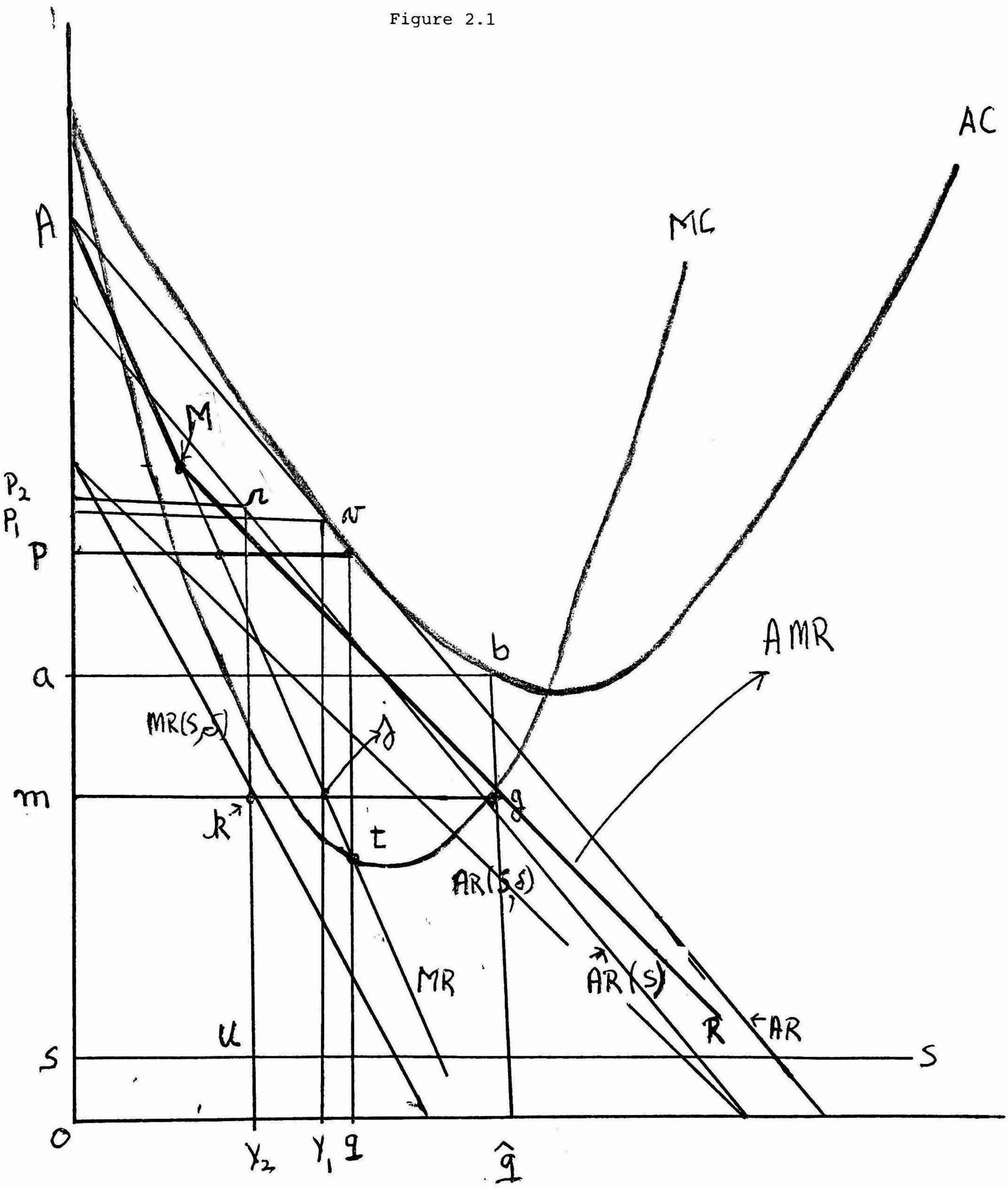
Profits are maximized by operating at that rate of output and sales per period at which marginal cost equals marginal revenue. That is, we set equation (6) equal to (7) and solve for q. The graphics of the static case appear in Figure 1.

In the Figure, AR is the firm's temporally invariant demand curve, equation (1). The firm's marginal revenue curve, equation (7) is MR. The marginal cost curve, equation (6), is MC. The average cost curve is shown as AC. It is obtained by dividing total cost, equation (2) through by q.

(8) $$AC \equiv \frac{C}{q} = b - gq + jq^{2}$$

The classical tangency solution of monopolistic competition theory is depicted in the Figure. That is, at the optimal, static rate of output, q, and price P, the firm earns zero monopoly profit. Since price is equal to average cost at output rate q, the firm only earns a normal rate of return or profit in each period. We turn now to a dynamic strategy. We consider the possibility that the firm would be better off if it produced in period 1 and sold that period's output in periods 1 and 2. In this strategy, the sales in period 2 are made from the stock of inventory "built up" in period 1.

Figure 2.1



AC
MC
A
M
P_2
P_1
P
n
v
a
b
AMR
MR(S,δ)
j
m
g
k
t
AR(S,δ)
MR
$\overrightarrow{AR}(S)$
R
AR
u
s
S
0
Y_2
Y_1
q
$\hat{q}$

In order to carry out the analysis of the dynamic strategy, we require some additional notation. We use s to represent the cost of storing a unit of product over one unit of time. There is no need for our general model to assume linear storage costs.[8] The analysis of the dynamic case requires a second change in notation. Because output of today need not be sold today, we require a different notation for output and sales. The letters y and q will be used to denote sales and output respectively. The letter i will be used to denote the interest rate.

The firm's temporally invariant demand function is exactly the same as before except that it is now written as:

$$(9) \qquad P = \bar{P} - ay \; .$$

Production and sales are assumed to take place instantaneously at the beginning of a period. Hence, sales in period 1 involve neither storage nor discounting and the revenue function of period 1 can be written:

$$(10) \qquad R_1 = \bar{P}y_1 - ay_1^2 \; .$$

Revenue received in period 2 must be discounted by the interest rate. The revenue function of period 2 is then:

$$(11) \qquad R_2 = \frac{\bar{P}y_2 - ay_2^2}{(1 + \delta)} \; .$$

The firm's temporally invariant production cost function is the same as that of equation (2). Its total expenditure on storage is:

(12) $S = sy_2$.

The profit function associated with the dynamic strategy is :

(13) $$\pi = \bar{P}y_1 - ay_1^2 + \frac{\bar{P}y_2 - ay_2^2}{(1 + \delta)} - bq + gq^2 + jq^3 - sy_2 \ .$$

This formulation assumes that storage costs are paid at the same time as costs of production are incurred. The above function is to be maximized. A straightforward way to explain this maximization is to recognize that all of the sales in both periods come from production in the first period. This means that

(14) $q = y_1 + y_2$.

Equation (13) is then rewritten with $y_1 + y_2$ being substituted for q in the cost portion of the equation. Equation (13) is then differentiated partially twice, once with respect to y_1 and once with respect to y_2. Each of the partial derivatives is then set equal to zero. At this point we have a system of two simultaneous equations in two unknowns, y_1 and y_2.[9] With y_1 and y_2 known, the two prices, P_1 and P_2 can be determined from the basic demand equation (9). The total revenue of period 1 and the present value of period 2 total revenue can be calculated since P_1, P_2, y_1, and y_2 are known. Since y_2 is known, the total cost of storage is simply sy_2. Since output, q, is the sum of y_1 and y_2, the total sum spent on production can be calculated by entering the known q into the cost equation. Thus, with all of the elements of equation (13) known, profits can be calculated.

Figure 1, which was used to depict the static strategy, can also be used to illustrate the dynamic strategy. In the figure, AR(s) is the basic demand curve shifted down in a parallel fashion to reflect the cost of holding a unit of product for one unit of time. The per unit storage cost is shown as ss in the figure. AR($s\delta$) is the result of a second adjustment in the basic demand curve, the discounting of revenues so that they are in current value terms. This adjustment involves a change in slope of AR(s) because at a zero price, the discounted and undiscounted revenues are of course identical. MR(si) is the curve that is marginal to AR(si). In the tradition of monopoly theory as applied to multiple geographic markets, MR, the basic marginal revenue curve, and MR($S\delta$), which reflects the revenue function after adjustment for storage and discounting, are summed horizontally. The result is an aggregate temporal marginal revenue curve, AMR. It appears as the heavy line, AMR. The profit maximizing rate of output under the dynamic strategy is determined by the intersection of the marginal cost (of production) curve, MC, with the aggregate marginal revenue curve, AMR. That output rate is shown as $\hat{q}$, and the intersection of MC and AMR is denoted by the letter g.

The total amount of money spent on production is average cost per unit multiplied by the output rate. This is the rectangle oab$\hat{q}$. To this we must add cost of storage which requires that we identify sales in period 2. This is done by running a line from point "g", parallel to the horizontal axis and noting the intersections of that line with the marginal revenue curves of each of the two periods. These intersections are j and k. The sales in the two periods are therefore y_1 and y_2. Expenditure on storage is therefore the rectangle oy_2us. Total expenditure on production and storage is the sum of this rectangle and the former one, $oab\hat{q}$.

Revenues are depicted by running two lines, one from y_1 and one from y_2, up parallel to the vertical axis and noting the intersections of these lines with the two demand curves, AR and AR(si). These two intersections are shown as r and v. The revenue earned in the first period is then y_1vP_1o and in the second period is y_2rP_2o and total revenue is the sum of the two.

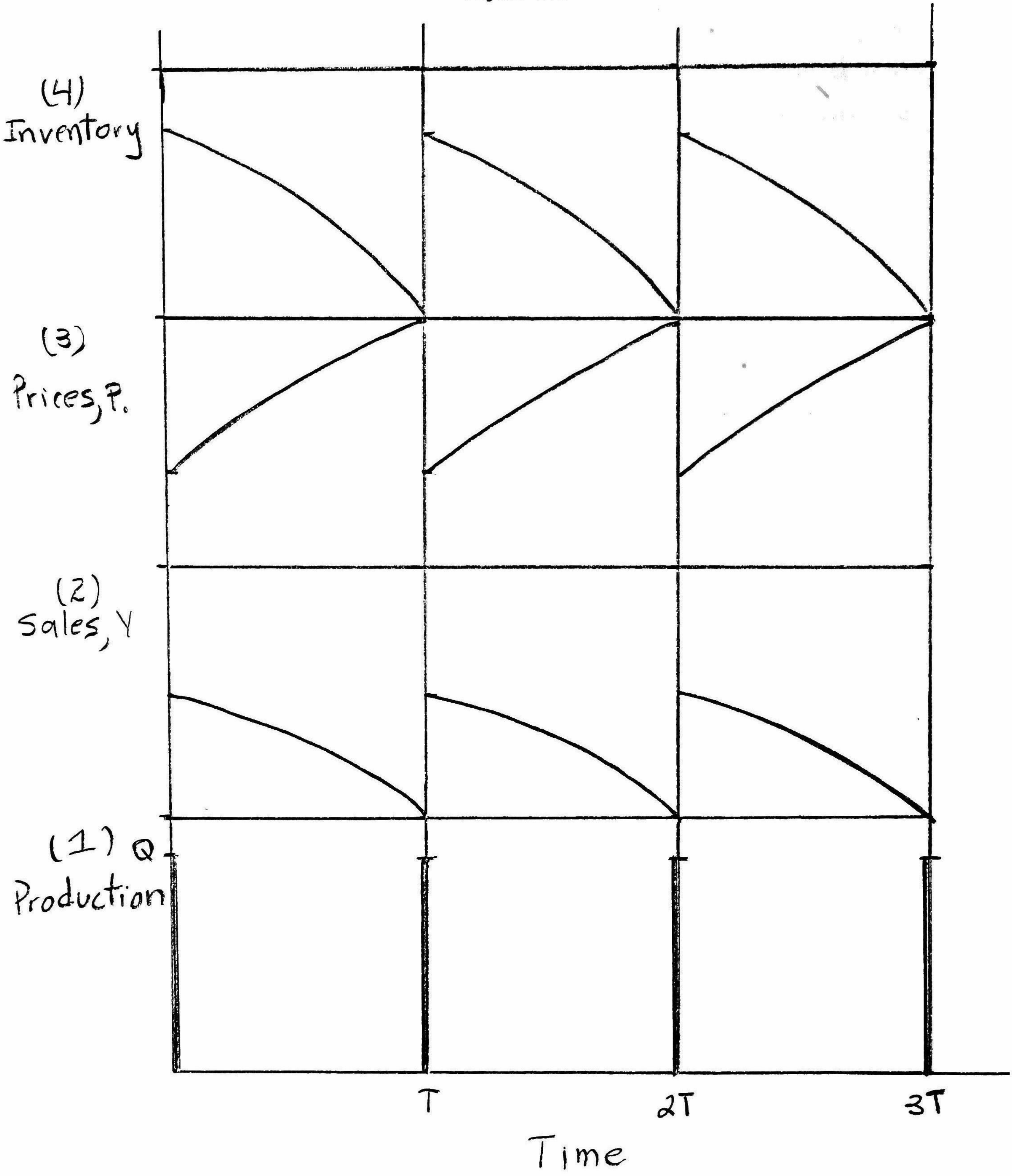
Under certain circumstances it is clear that the total revenues associated with the dynamic, can exceed those of the static strategy. Whether or not they do depends on: (1) the height of the interest rate and storage cost; (2) on the amount of decrease in average cost that can be achieved by concentrating production in time and achieving cost economies. Some additional aspects of the dynamic solution as it compares to the static solution should be noted.

The dynamic strategy may be the more profitable over the planning horizon but there can be losses over early intervals of time. In our example the total costs of production and storage ($oab\hat{q}$ plus oy^2us) exceeds period 1 revenues, oy_1nP_1. The dynamic solution also entails a change in prices over time. Under the static regime the price charged in every period of time is P. With the adoption of a dynamic strategy and the achieving of additional cost economies in production but the need to spend money on storage, there are different prices charged in different periods. In Figure 1 they appear as P_1 and P_2. Both of these prices are greater than P, the price associated with the static strategy. The explanation is that in the present case the dynamic strategy results in an output rate, $\hat{q}$, at which marginal cost of production is $\hat{q}g$. This exceeds the marginal cost, qt, of the static output rate. Thus, in the situation depicted in Figure 1, consumers are worse off under the dynamic strategy than they would have been under the static. By adopting the dynamic strategy, the firm converts a part of consumer surplus into monopoly profit. The reader will recall that the static strategy yielded zero excess profit for the firm.

The adoption of a dynamic strategy need not necessarily result in a reduction of consumer welfare. If the static equilibrium output occurs on the falling portion of the firm's marginal cost function, the adding up of temporal markets can result in a decrease in the marginal cost of production. Then, the consumers served "today" from "today's" production will benefit by paying a lower price under the dynamic regime than they would if the firm behaved statically. The customers served "tommorow" from today's production can be better or worse off than they would be with a static strategy. They will pay a lower price and be better off than they would be with the static solution, if the reduction in marginal cost of production achieved under the dynamic strategy exceeds the storage cost involved in serving them. Additionally, we may note that the price paid by "tomorrow's" customers. The price paid by tomorrow's customers may differ from today's price by more or less than the cost of storage, depending on the form of the demand function. In Figure 1, P_2 differs from P_1 by an amount that is less than the cost of storage, but only because we have asumed a linear demand function.

We have been dealing with a simple two period model. Its purpose was to introduce the reader to the time-space trade-offs, and the direct and cross effects of changes in spatial and temporal paramaters. In part 4 we extend the discrete approach of this simple model. Programming techniques are presented that can help solve discrete problems in which there are many spatial, temporal markets. The formal model of Part 3 is cast in terms of continuous time. The advantage of this approach is that it allows us to solve for the optimal length of T, the production-sales cycle, with the use of calculus techniques. At this point it is worth noting what happens to prices, sales, and inventories in a model with continuous time when, as a result of the interest rate, certain complications due to a fixed planning horizon do not arise. Figure 2 shows the time patterns of these variables.

Figure 2.2



The first of the four panels pertains to production, which takes place instantaneously at the beginning of the plan, and then occurs at times 2T, 3T, etc. The on-off nature of the production pattern is the result of the assumptions that are being made in the present stage of the explanation of the model to simplify the exposition. In Part 4 of this report we show, for example, how a 30-day planning model that involves a firm with two products and costs of production that are higher on week-ends, leads to a switching back and forth over several days in the production of the two products, the only down time coming on week-ends.

The second panel in Figure 2 is for sales, which fall over time. In a profit maximizing model, prices rise over time because the cost of serving future customers from past production involves more storage cost. There are of course situations in which it is not possible, or in which it is too costly, to change prices over time. In those situations, the models with which we are working may be constrained to have a single price over time. The cycling of production and inventory holdings can still occur, but they are then due solely to cost economies.

The third panel shows the increases in prices over each time period T. As explained above, they are the result of the increased storage cost associated with serving customers whose purchases are more and more distant in time from when the output used to serve them was produced. The final panel, 4, is for inventories. They are at a maximum at the beginning of each cycle when production occurs, decline over time, and go to zero at T, at which point production again takes place and a second cycle is begun. This completes the preliminary exposition of the purely temporal portion of our model. We turn now to a qualitative explanation of the purely geographic component. The exposition can be brief because the reasoning is very similar to that of the pure temporal case.

Suppose a firm is contemplating selling a product in two markets, and that the cost of shipping the product is quite high relative to the cost of shipping the raw materials employed in making the product. In addition, the prices of other factors of production employed in making the product exhibit relatively little geographic variation. As a result, market orientation is the optimal choice in location. However, there is still the unresolved issue of how many plants the firm should establish. If the two markets are A and B, there are three choices: (1) establish a plant in A; (2) establish a plant in B; (3) become a multiple plant firm, with a plant in each market. If transport costs on the product are quite high, or if the demand and U-shaped production cost functions in each market are such that a plant serving each market would operate on the rising portion of its average cost curves, the optimal choice is to have a plant in each market.

This last is the spatial equivalent of the static strategy of our temporal reasoning. There we saw that if storage costs are high, or each period's equilibrium output occurs on the rising portion of the firm's average cost function, the monopolist will find it most profitable to behave statically, serving each period's customers form that period's production. If transport costs are high, the monopolist may find it most profitable to adopt the analogous spatial strategy, that of geographic isolation. Each market is served from production in that market, and there is no transportation of the finished product. On the other hand, if the economies of massing production in one place are very great relative to transport costs, it may be profitable to establish a plant in one market and serve all customers in both markets from that plant. This is the spatial analog of the dynamic strategy in temporal reasoning. It leads to variations in prices and sales over space because transport cost increases with distance. Hence, if it is possible and not too costly for the firm to vary prices geographically, profit maximization calls for a policy in which prices go up and sales go down as distance from plant to customers increases. As was the case with storage costs in the temporal case, prices in the geographic case increase by more than or less than the increase in transport cost. There are of course limits to the space-time analogy, the most obvious one being that in transport it is normally possible to ship in all directions, though not necessarily at the same cost. That is much less common in the temporal case. Consumption today cannot be satisfied from future production. That is one of the reasons why most production-inventory scheduling models do not permit back-ordering.[10]

If a production process can be carried out in ways that involve high capital intensity and economies of scale in production, there may be great social benefits associated with massing production in time and space. Indeed, if storage costs, the interest rate, and transport costs are such that production and sales can only take place in temporal and spatial isolation, it may not be economically possible to produce the product at all. We turn now to a discussion of the effects of changes in the costs of carrying goods through time and space on the firm's behavior.

The firm can adapt to such changes in a variety of ways. Some of them involve short run and some long run strategies. We distinguish between the two types on the basis of investment in plant and equipment. The short run is defined as a period of time that is not long enough to permit the firm to add new plants or change the scale of old plants through investment. We deal first with short run adaptations to a change in the cost of transport.

An increase in any cost, whether of production, of storage, or of transport, must have the effect of reducing output, the sales, and the profits of the firm. An across the board increase in, for example, the cost of shipping the firm's product will reduce the output and total sales of each of the firm's plants, but that does not mean that sales to every space-time market pair must decline. The example developed below shows that sales in some space-time market pairs can increase as a result of an increase in one of the geographic or temporal parameters. Such reallocations and other effects of changes in parameters are most easily explained at this stage with the aid of a simple example.

Suppose the firm operates two plants, one located in place A and one in B. Each of these places is also a market, and there are two other markets, one in C and one in D. To facilitate exposition, we assume that the cost of production and the cost of storage at A are identical to these costs at B. There is a single discount rate. There is one demand function for the firm's product across all geographic-time market pairs. The cost of transporting a unit of product within A is identical to that within B. The cost of transporting a unit of product from A to C is the same as that from B to D, and higher than the A to A or B to B. Product can be shipped more cheaply from A to C than from B to C. The same assumption is made about shipments from A to D. The firm has a profit maximizing mode of time-space behavior.

Because of the assumptions made above, the plant at A serves demand at A and C exclusively, and the plant at B serves B and D exclusively. The firm finds that in serving the purely geographic markets and carrying no inventory, each plant operates on the rising portion of its marginal and average total cost curves. Profits are maximized by operating as a static-spatial monopolist. Each period's demands are satisfied from that period's production. The price of the product is equal at A and B. Prices are equal at C and D, and these prices are higher than those at A or B because of transport cost. Sales are of course lower in the two outlying markets than at A and B.

We now change the situation by assuming an increase in the cost of transporting product to the two outlying markets. The static demand functions associated with markets C and D shift down and the new static aggregate marginal revenue functions lie below the old ones. We suppose that the increase in transport cost is sufficiently great that each of the aggregate marginal revenue curves intersects a plant marginal cost function on its falling portion. Now, if storage costs are not too high, it may become profitable for the firm to add a temporal strategy to its geographic strategy of serving the demands at A and C from production at A, and serving the demands at B and D from production at B. A production-sales cycle of duration T, as in Figure 2, is adopted. The plant at A serves demands at A and C exclusively at each moment of time. The plant at B is in a comparable position with regard to demands at B and D. The price of the product at A equals the price at B at each moment of time. However, these prices now rise over time during the interval from production at time zero, (0), to T because of storage cost. Sales at A and B fall over the interval, and inventory, which was highest at the moment of production, (0), declines over time. The inventory stored at each plant falls to zero at T. A new multi-plant production cycle is begun at 2T, just as in the non-spatial dynamic monopoly case discussed earlier. Above we made some comments on patterns of prices. There are some additional effects on prices and sales that should be considered if we are to offer meaningful comments on the direct and cross effects of the change in transport cost.

Let us first consider the static price at A, or at B, in comparison with the pattern of dynamic prices that are part of the new strategy that is adopted after the increase in transport cost. It is useful to begin by comparing the price situation at A (or at B) under the new dynamic situation with that under the original static equilibrium. The key comparison is between the static price, P, and the price at time (0), P_0, the instant of production in the dynamic case. We distinguish two cases.

In Case 1, the static aggregate marginal revenue curve, AMR, intersects marginal cost of production at a point where output is q and where marginal cost of production is equal to qt, as shown in Figure 3. After imposition of an increased transport cost from A to C, as well as from B to D, the firm goes over to a strategy in which there are dynamic as well as spatial elements. The dynamic aggregate marginal revenue function, AMR, intersects the marginal cost function at a point where output is $\hat{q}$ and marginal cost of production is $\hat{q}g$.[24] In Case 1, $\hat{q}g$ is less than qt. Total system output, $\hat{q}$, (and of course sales) after the imposition of the higher cost of transport from A to C, are lower than the static output, q. When this occurs, the price paid by consumers at A at time (0) will be less than the price, P, paid by consumers in each period under the static regime. Of course sales will also be greater at time (0) under the dynamic than under the static strategy. The lower price and higher sales at A associated with the dynamic strategy at the time (0) can persist for some periods. Eventually, however, costs of storage may cause the prices charged at A under the dynamic strategy to exceed those of the static strategy. Total sales to consumers at A and the average price paid by them over one production-sales cycle can be less than, equal to, or greater than total sales in the comparable number of periods under the static regime. Let us now consider the changes brought about in market C as a result of the higher transport cost.

If the increase in the cost of shipping from A to C is high enough, there may be no sales in market C after the change. However, even if there are positive sales in C, it is clear that the amount sold will be less than under the static regime. Price at C may be lower under the dynamic regime at time (0), but they must rise above the static price over the time interval of production and sales must be lower because of the increase in transport cost and positive storage cost. The average price at C under the dynamic strategy must exceed the static price. The comparison of sales under the two regimes must be for the appropriate time interval. If the time run of production-sales under the dynamic strategy is from (0) to T, then the sales made over this time must be compared to sales in the same time interval under the static regime.

Above it was stated that a change in a spatial parameter, such as the cost of transport, will tend to have direct effects on spatial variables that are under control of the firm, and can also have cross effects on temporal variables. Let us now summarize the effects associated with Case 1.

Direct (Spatial) and Cross (Temporal) Effects of the Increase in Transport Cost: Case 1

1 The firm goes over to a strategy that combines temporal with spatial elements. The total output, sales, and profits of the firm fall.

2. There is a geographic reallocation of sales. Sales to consumers decline absolutely in the markets, C and D in our example, where transport cost has risen. Average price paid by consumers in C and D over the production-sales interval exceed the static price. However, prices in these markets may be lower than the static price over some sub-interval in the time from (0) to T.

3. The average price paid over the production-sales cycle by consumers in the markets reached by unchanged transport costs, A and B in our example, may be greater than, equal to, or less than the static price. This means that sales in A and B under the dynamic strategy can be greater than, equal to or less than under the static regime.

4. As a result of the increases in transport cost, the firm's demand for storage capacity increases, and its demand for transport from A to C and B to D declines.

5. The firm's demand for liquid funds increases because, under the dynamic regime, time is required to recover from sales, the money spent on production, storage, and transport.

6. In the long run, the firm might react to the increases in transport cost by building additional plants at C and D. In a sense the firm then goes over to a pure dynamic strategy, i.e. a strategy of concentrating production in time and not in space.

Case 2. An Increase in Storage Cost

The basic conditions of our second case differ from those of the first. We now assume that our firm has been maximizing profits by operating across time and space. In addition, the equilibrium rate of output from each production run at each plant is on the falling portion of the marginal cost curve. In this situation, an increase in the per unit price or cost of storage, s, causes the output of all production runs at all plants to fall. This causes the marginal cost of production to rise. The result of the increase in marginal production cost is that the price of the product in every geographic market at time (0) and for some interval of time thereafter is higher, and sales lower than before the increase in storage cost. However, there can be intervals of time in which the opposite is true.

The latter can occur because the increase in storage cost causes the length of the production-sales run to be shortened. If production took place originally at times T, 2T, 3T etc., it may take place at H, 2H, 3H afterwards, where $H < T$. With the shortening of the cycle, there is zero expenditure on storage at time H. There may also be less expenditure on storage for some instants of time after H. It is possible for the reduced expenditure on storage to more than compensate for the increase in the marginal cost of production. Hence, for some instants of time, the price of the product can be lower and sales higher than they were before the increase in per unit cost of storage.

The increase in the per unit cost of storage causes the firm to reduce the amount that it produces, ships, and sells over its planning horizon. Storage-inventory activities and transport services are in this case complementary factors of production. The increase in the cost of storage causes the firm to also reduce its use of the services that carry goods through space.

Case 3: <u>Transport Cost Increase with Ambigious Results</u>

This case resembles the preceding one in that the firm has been behaving as a spatial and temporal monopolist, and the equilibrium output of each plant is on the falling portion of its marginal cost function. In these circumstances an increase in transport cost, say from A to C, the latter being the market served exclusively by A, and from B to D will have certain impacts that are clearcut. However, there can be a second kind of impact that can go in either of two ways. We consider the clearcut effects first.

An increase in transport cost must cause the output of each of the firm's plants to fall. The declines in plant output cause the marginal cost of production to rise because we have assumed that each plant is operating on the falling portion of its marginal cost function. The increase in marginal cost of production means that the price of the product must increase in each market at time (0) and for some interval of time thereafter. Moreover, prices increase and sales fall off in A and in B at time (0) even though these markets experienced no increase in transport cost. Of course, the decline in sales is relatively greater in the markets where the increase in transport cost occurred, that is C and D in our example.

The above are the straightforward effects of the increase in transport cost. The ambiguity involves the cross effects on the timing of production of an increase in transport cose. If the length of run was originally from (0) to T, it may be shortened to H, at which time production takes place again, or it may be lengthened to V. Which of these two alternatives occurs depends on two conditions. The first is the steepness of the marginal cost function in the vicinity of the original plant equilibrium output. The second condition is the rapidity with which sales fall off with time because of storage costs. We may think of the latter condition as the time elasticity of demand. Now, if the marginal cost function is very steep in the vicinity of the original equilibrium and if sales fall off rapidly with time, then an increase in the per unit cost of transport will tend to shorten the length of the production-sales run, say from T to H. In that event, prices can be lower at H, and for some interval of time thereafter, and sales higher than they were before the increase of transport cost. The explanation is that because production resumes earlier after the increase in transport cost, it is possible that the savings in storage expenditure associated with serving some temporal markets more than compensates for the increase in marginal cost of production and increased transport cost. The increase in transport cost causes the quantity of goods shipped to all markets over the entire planning horizon to fall. It causes a decline in the holdings of inventory and expenditures on storage. Thus, in this case, transport and storage are complementary factors of production.

If the marginal cost function in the vicinity of the original equilibrium is gently sloped, and sales fall off slowly with time, an increase in transport cost can cause the length of the production-sales run to increase, say from T to V. In other words, storage and transport are in this case substitutes for one another in the production process. The increase in the cost of transport causes the firm to shift into a mode of operation that involves relatively more holding of goods over time and less shipping of them in space. When this occurs, price at time T is higher and sales lower than they were before the increase in transport cost. On the other hand, price at time V is lower and sales are higher than they were before the increase in transport cost.

In order to introduce the reader to the notion of temporal strategies, spatial strategies, and how they can interact, we have up to this time largely adopted the logic of partial equilibrium analysis in this part of the report. That is, in discussing the pure temporal model we assumed a given number and spatial distribution of plants. Similarly, in the pure spatial model we largely ignored dynamic elements. In Part 3, to which we now turn, we adopt a long run view of the firm, allowing it to choose simultaneously a locational-temporal plan that is profit maximizing.

We consider a multiplant monopoly that sells and produces over space and time. Each market is identified by a space-time ordered pair, and each production - sales run such as the time from (0) to T in Figure 2, is identified by the plant location of the run as well as the time of completion of the run. We view both space and time to each pair to be represented by a single coordinate, and for analytical convenience we now treat space and time coordinates as continuous.

We solve the firm's problem as a nested maximization problem. The first step in this procedure is to consider a plant producing from a single production run and selling its output over a prespecified time horizon. The plant is described by a production cost function which is characterized by both U-shaped marginal and average costs, i.e., there are at least some economies of scale. If there are no such economies, then it is optimal for the firm to produce at each space-time pair, i.e., to operate under both spatial and temporal isolation. Each market is characterized by a downward sloping demand curve which admits a downward sloping marginal revenue curve. These are the only restrictions we put on the shape of the demand curves. To show the pure effect of scale economies, we assume that the market demand curve and the production run cost functions are space and time invariant. These assumptions imply that each plant is located in the middle of the market region it services, and that over time it is not optimal for one plant to service a market some of the time while another nearby plant services the market at other times.

We assume that there are freight delivery costs associated with shipping the product through space from plant to market. We model these costs as if they are directly proportional to distance from the plant so that there are no delivery costs to a market located at the plant and these costs rise linearly as one goes away from the plant. In an analogous fashion we assume that there are inventory storage costs associated with shipping the product through time. We model these costs as if they are proportional to inventory stocks. We also assume that the firm faces a constant interest rate over time, so that both future revenue and cost flows are appropriately discounted. Then we can make storage and freight costs appear even more similar. Since inventory stock at any time subsequent to the completion of the production run is just the output of the run minus sales up to that time, we can associate storage charges with sales at times subsequent to the completion of the production run, rather than with inventory stocks held at that time. The only difference then between freight charges and storage charges is that, due to discounting, current value storage charges rise exponentially rather than linearly with time.

For a specified output from the production run, the firm must deterine how to sell off this output over its market region within the prespecified time horizon. Since freight and storage charges act to separate markets, it is natural to assume that the firm price discriminates both spatially and temporally. We will proceed with the analysis under this assumption. It should be pointed out, however, that a similar analysis can be performed under other assumptions. For example, one can rule out price discrimination and impose a generalized F.O.B. condition that prices in different spatial or temporal markets must differ by the difference in the costs of serving those markets. Alternatively, one can allow partial price discrimination by imposing a policy of price uniformity temporally, spatially, or both. For now we do not concern ourselves with the influence that different economic environments have on the price policy selected by the firm. That is, we ignore questions of the possible influence of potential competition or regulation on the firm's choice of a pricing policy.

The first order condition which determines the optimal sales policy for the firm is that at any market, discounted marginal revenue net of freight and storage charges, also appropriately discounted, must be equal to discounted net marginal revenue at any other market.[13] This common value of discounted net marginal revenue is determined so that aggregate sales over the market region up to the prespecified time horizon just exhausts the production run output. This common value of <u>discounted</u> net marginal revenue equals the aggregate marginal revenue for this output level.

By varying production run output levels, one traces out over the market region and the time horizon an aggregate marginal revenue curve. This aggregate marginal revenue curve must be downward sloping since this property holds for the marginal revenue curve for each market. Profit from this production run is maximized by finding the intersection of this aggregate marginal revenue curve and the marginal cost curve such that marginal revenue cuts marginal cost from above.[14] This completes the first stage of the optimization.

If one performs a comparative static-dynamic analysis of the first stage problem by varying the size of the market region and the time horizon as well, a profit function can be generated. This function depends directly on the market radius, the distance from a plant to its most extreme market, and the time horizon. Indirectly, the profit function depends on the unit freight rate, the unit storage cost, the discount rate as well as the shape of the market demand curve and the production run cost function.

For this given profit function, the firm maximizes over the two variables: the market radius, r, and the time horizon between completion of production runs, T. These variables are chosen to maximize the total runs per unit length. We explain this objective as follows.

Suppose the entire market area of the firm (as opposed to the market region of a given plant), and the total time horizon (as opposed to the time between production runs) is specified. The firm wants to locate its plants and time its production runs so as to maximize the discounted present value of profits. If the number of plants is determined so that each plant serves a market region of identical length, and if the number of production runs is determined so that the time between production runs is the same then: (1) the market radius of each plant is one half of the total market area divided by the number of plants; and (2) the time between production runs is the total time horizon divided by the number of runs.

Alternatively, the number of plants is one half the total market area divided by the market radius, and the number of production runs is the total time horizon divided by the time between consecutive runs. Considering the choice of number of plants, aggregate profit is profit per plant multiplied by the number of plants and that is equal to:

$$\text{profit per plant} \times \frac{\text{market area}}{\text{market radius}} \, .$$

What is done over the spatial dimension is to maximize profit per unit length. This is an assumption that has also been adopted in purely spatial studies of the multiplant firm. The assumption is made in order to avoid an endpoint problem. The problem arises because the number of plants is not a continuous variable, and therefore the assumption that all plants serve an identically sized market region is not correct. To get around this problem one might assume that the entire market area served by the firm is unbounded but then so are aggregate profits. Hence, the true optimum is approximated by assuming that the number of plants is a continuous variable. This is not a bad approximation when the optimally determined radius, and hence the market region of individual plants, is small relative to the entire market area.

The endpoint problem crops up again in determining the number of production runs. However, the temporal aspect of the problem involves discounting, and this permits us to assume an unbounded total time horizon and still have bounded profits. This is what is typically done in analysing profit maximizing decisions over time. We turn now to our formal model of space-time equilibrium.

Footnotes

[1]This and the next part of the report are written in collaboration with L. Arvan, Department of Economics, University of Illinois, Champaign, Illinois.

[2]L. Arvan and L.N. Moses, "Inventory Investment and the Theory of the Firm," American Economic Review, March 1982.

[3]Earlier the point was made that there was one major exception to the general rule that in static conditions, perfectly competitive firms maximize profits by choosing one plant-market combination. That exception has to do with randomness and attitudes toward risk. Suppose that the profits associated with different plant-market combinations are subject to random variations and that a given entrepreneur has a utility function involving risk and the expected rate of profits. Now, if the plant-market choices available to the entrepreneur are discrete, it may be optimal to blend these choices, i.e. to operate as a multiplant firm even though costs and prices do not change systematically over time. The multiple plant-market combinations in which the firm then operates are a form of portfolio.

[4]Part 4 contains examples of models in which costs change over time.

[5]The word 'may' is underlined because the results of changes in temporal and spatial parameters can prove to be counter intuitive.

[6]R. Peterson and E.A. Silver, Decision Systems for Inventory Management and Production Planning, New York, John Wiley and Sons, Inc., 1979.

[7]H.M. Wagner and T. M. Whitin, "Dynamic Problems in the Theory of the Firm," Naval Research Logistics Quarterly, March 1958, pp. 53-74 and T.M. Whitin, "Dynamic Programming Extensions to the Theory of the Firm," Journal of Industrial Economics, April 1968, 16, pp. 81-99.

[8]It can readily encompass cases in which storage as well as production cost exhibit U-shaped average and marginal costs.

[9]The solution procedures used to solve actual problems involve algorithms built around the techniques of non-linear programming.

[10]It is not possible to satisfy consumption today from production tomorrow, but it is certainly possible to make sales today from production or deliveries that will take place in the future. The practice is most common in durable goods, both producer and consumer.

[11]In this footnote we offer a qualitative explanation of the aggregate marginal revenue curve associated with the dynamic-spatial strategy. It is obtained in a number of steps. First, the instantaneous demand function is adjusted for per unit transport cost A to A. Second, the transport cost adjusted demand function at A is adjusted for storage cost and the interest rate for each period, say for a time interval from time (0) to time T. Third, the associated marginal revenue curves for each of these adjusted demand functions at A is obtained and all are then summed to obtain the aggregate temporal marginal revenue curve at A. Exactly the same adjustments are made for the instantaneous demand function at C except that the relevant transport cost in the above first step is the cost of shipping a unit of the product from A to C. The temporal aggregate revenue function at C is then summed horizontally to that of market at A to obtain the spatial-temporal marginal revenue function.

[12]The reader will recall that up to the present time we have been assuming that the firm's storage cost function is linear in quantity and time.

[13]This implies that sales fall with distance from the plant that serves a particular market and with time from completion of production.

[14]Shutdown conditions must also be examined, but this is difficult to do without equations. They are examined in Part 3 below.

Part 3: Economies of Scale and A Model of the Firm in Time and Space

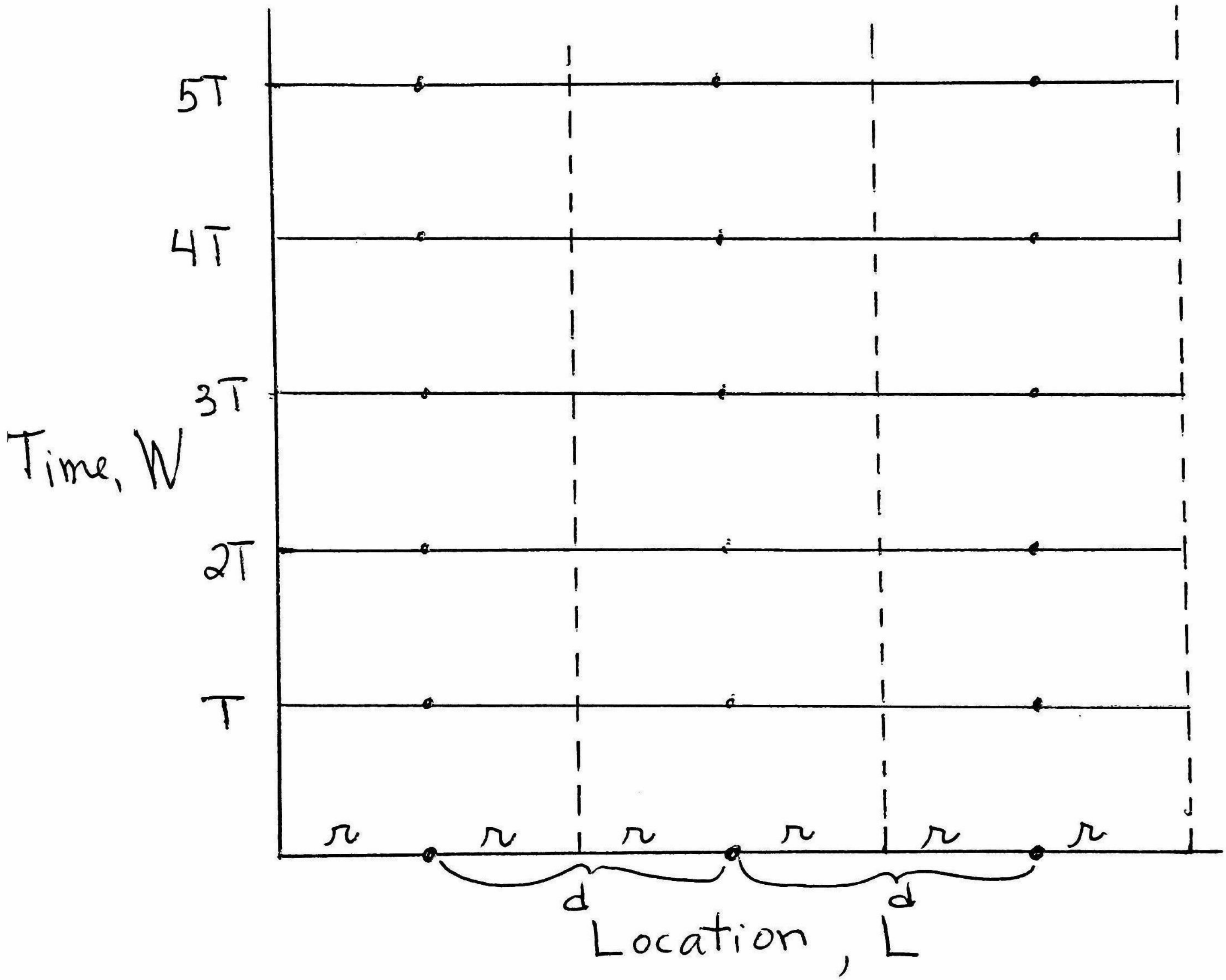
This part of our report is divided into two sections. The formal model of the firm that maximizes profit by concentrating production in time and in space is presented in Section 1. Section 2 is devoted to an investigation of the short and long run comparative statics-dynamics of the model. That is, we examine the effects on the firm's behavior of changes in the various costs of carrying goods through space and time. Five cases are considered in Section 2. The first four deal with issues that are essentially short run in nature. The fifth case deals with long run adjustments. The effects considered in this case are very complex. The results that we are able to achieve are, unfortunately but understandably, ambiguous.

Section 1: The Model

Consider a firm that can sell over a market region of length L, a time horizon of duration W, and which faces cost and demand functions that are time and space invariant. That is, for each space-time pair at which the firm can produce, it would incur identical costs if it produced identical outputs and it would sell identical quantities if it charged identical prices.[1] Given this assumption and an additional simplifying assumption mentioned below, each plant that the firm constructs will service a market region of identical length. In addition, sales from each production-sales run lasts for the identical length of time. We call the constant interplant distance d. $\frac{d}{2}$, the distance from the plant to each of the two furthest markets served by that plant, is called the market radius and is denoted by r. The constant duration of production-sales runs is denoted by T, and since we rule out backordering (i.e., the satisfaction of today's consumption from future outputs) there is no temporal equivalent to radius. Figure 3.1 below depicts the spatial-temporal choices the firm faces. Distance and the spacing of plants are shown on the horizontal axis. Time is measured on the vertical axis. Each column of dots is associated with a plant location. We refer to the time between successive dots in a column as the length of a production-sales run. That is, each dot within a column denotes an instant of time at which production takes place. At that time, the stock of inventory from the preceding production time is exhausted. The dashed vertical lines and the two solid vertical lines denote plant market boundaries. Each plant services the market region contained within its left most and right most boundary. The solid horizontal lines denote temporal production-sales run boundaries. The output of a plant's production run is sold at all locations within the plant's market region until the time of the next production run. Production takes place when output of the previous run has been exhausted by sales.

Given such a grid specifying interplant spacing and the time between production-sales runs, the firm chooses the output of each run and the associated sales policy that maximize profit. Product is costly to ship through space, costly to store through time, and the firm discounts future revenue and cost flows. The output, sales policy, and profit associated with a single run at a given plant depends on freight costs, storage costs, and discount rates as well as demand and production cost function. We denote the maximal profit associated with a single plant production-sales run by $\pi(r,T)$.

Figure 3.1: Time-Distance Grid



If there are n plants each serving a market region of identical length, and each plant is located in the center of the region, then $d = \frac{L}{n}$ and $r = \frac{L}{2n}$. Alternately, $n = \frac{L}{2r}$. Since production-sales runs at each plant are completed at the identical time, to maximize aggregate profits of all plants from a single production-sales run, the firm maximizes $n\pi(r,T) = \frac{L}{2r}\,\pi(r,T)$. Since $\frac{L}{2}$ is fixed, maximization of $n\pi(r,T)$ is equivalent to maximization of $\frac{\pi(r,T)}{r}$. This definition of aggregate profit assumes n is an integer. When r is chosen so that $\frac{L}{2r}$ is not integer valued, it is not strictly correct to assume that all plants will produce identical outputs and consequently also incorrect to assume that they are all equally spaced. We want to avoid this integer problem caused by assuming that the endpoints of the entire market region served by the firm are fixed. Consequently we assume that when the firm optimizes with regard to interplant spacing it maximizes $\frac{\pi(r,T)}{r}$, an assumption that has been employed by others interested in location and spatial competition.[2]

Since the firm discounts future revenue and cost flows at a constant discount rate equal to δ, the discounted present value of all production runs from a given plant equals:

$$\text{(1)} \qquad \sum_{j=0}^{n-1} e^{-\delta jT}\,\pi(r,T) = \frac{1-e^{-\delta nT}}{1-e^{-\delta T}}\,\pi(r,T)$$

where $n = \frac{W}{T}$.[3] The integer problem crops up here as well. However, if W is very large relative to T, this discounted present value is approximately equal to $\frac{1}{1-e^{-\delta T}}\,\pi(r,T)$. We take this to be the objective of the firm with regard to its choice of time between production-sales runs. This is the objective when the time horizon is infinite and consequently no endpoint problem arises.

It should be noted that the firm maximizes average profit per unit distance when making its interplant spacing choice but maximizes aggregate discounted profits in its choice of time between production runs. While there is much similarity between the two choices we will not obtain symmetric conditions for the optimal r and T. Note however, that when W is finite and $\delta = 0$, then the two problems are essentially identical.

Consequently we model the firm's problem as follows:

(2) $$\underset{r,T\geq 0}{\text{maximize}} \quad \frac{\pi(r,T)}{(1-e^{-\delta T})r} \, .$$

The first order conditions for an interior optimum of this problem are:

(3) $$\pi_r - \frac{\pi(r,T)}{r} = 0$$

and $$\pi_T - \frac{\delta e^{-\delta T}\pi(r,T)}{(1-e^{-\delta T})} = 0$$

The second order sufficient conditions require that:

(4) $$\begin{bmatrix} \pi_{rr} & \pi_{rT} - \dfrac{\pi_T}{r} \\ \pi_{rT} - \dfrac{\pi_T}{r} & \pi_{TT} + \dfrac{\delta^2}{e^{\delta T}-1}\pi \end{bmatrix} \quad \text{is negative definite.}$$

To study this problem more closely we examine the determinants of $\pi(r,T)$. Assume for now that r and T are fixed. We first look at the variational problem of how to sell over time and space when the output level from a production run, Q, is held fixed. The firm solves the following problem:

$$(5) \qquad \underset{y(\;)\geq 0}{\text{maximize}}\ 2\int_0^T \left\{\int_0^r e^{-\delta t}[p(y(z,t))-fz]y(z,t)dz(-sI(t))\right\}dt$$

subject to: $\quad I(t) = Q - 2\int_0^t \int_0^r y(z,u)dzdu$

$\qquad I(t) \geq 0 \quad \text{for} \quad t \in [0,T).$

(6) where $\quad$ z is an index of distance from the plant,

t is an index of time from production,

y(z,t) is sales at (z,t),

f is the unit freight rate,

s is the unit storage cost,

I(t) is inventory at t,

δ is the discount rate, and

p(y) is the sales price when sales are y.

We explain this problem as follows. All revenue flows at time t are discounted to time zero by the discount factor $e^{-\delta t}$. Gross revenue in current value from sales at (z,t) is p(y(z,t))y(z,t). Each unit of product shipped from the plant to a market z units from the plant requires payment of a freight charge with unit price in current value equal to fz. Thus the total freight charge associated with sales at (z,t) in current value is fzy(z,t). When we aggregate over all market locations within the plant's market region, the discounted revenue net of freight charges at time t is $2e^{-\delta t}\int_0^r [p(y(z,t))-fz]y(z,t)dz$.

For each unit of product held in inventory at time t the firm pays a storage charge with unit price s in current dollars. Given inventory level of I(t), at time t the discounted value of storage charges is $e^{-\delta t} sI(t)$. Inventory at t equals the original quantity available for sale, Q, minus the amount sold up to t, $2 \int_0^t \int_0^r y(z,u)dzdu$. The requirement that inventory is nonnegative for all t in [0,T) amounts to requiring that over this interval not more is sold in total than was available for sale, i.e., $2 \int_0^T \int_0^r y(z,t)dzdt \leq Q$.

It is convenient to assume that the last inequality holds as an equality so that all product available for sale is actually sold. Doing so allows us to treat inventory charges in terms of sales flow rather than in terms of inventory stock, and makes them more directly analagous to freight charges. Since I(u) can be rewritten as:

$$(7) \qquad 2 \int_u^T \int_0^r y(z,t)dzdt = I(u)$$

the total discounted present value of storage charges is obtained in terms of sales via a change in the order of integration. Thus:

$$(8) \qquad \int_0^T e^{-\delta u} sI(u)du = \int_0^T e^{-\delta u} s[2 \int_u^T \int_0^r y(z,t)dzdt]du$$

$$= 2 \int_0^T \int_0^r sy(z,t) \int_0^t e^{-\delta u} dudzdt$$

$$= 2 \int_0^T \int_0^r \left(\frac{1-e^{-\delta t}}{\delta}\right) sy(z,t)dzdt$$

$$= 2 \int_0^T \int_0^r e^{-\delta t} \left(\frac{e^{\delta t}-1}{\delta}\right) sy(z,t)dzdt.$$

In current value, the unit storage cost associated with sales at time t is $(\frac{e^{\delta t}-1}{\delta})s$. Given this way of writing inventory costs our problem can be rewritten as:

$$(9) \qquad \max_{y(\ )\geq 0} \ 2\int_0^T\int_0^r e^{-\delta t}[p(y(z,t))-fz-\frac{s(e^{\delta t}-1)}{\delta}]y(z,t)dzdt$$

$$\text{subject to } 2\int_0^T\int_0^r y(z,t)dz,dt = Q.$$

We call the value of the objective when evaluated along the optimal sales trajectory, TR(r,T,Q).

Note that in formulating this problem the only restriction placed on sales at (z,t) is that they are nonnegative. Hence we allow for both interspatial and intertemporal price discrimination. Besides being theoretically appealing, price discrimination has the additional advantage of allowing us to describe the optimal sales policy with intuitive first order conditions. Note however that if such discrimination is not possible, due to regulation or to fear of potential entry, the model can be reformulated to require uniform delivered prices.[4]

The first order conditions governing the optimal sales policy are:

$$(10) \qquad e^{-\delta t}[p(y(z,t)) + p'(y(z,t))y(z,t)-fz-\frac{s(e^{\delta t}-1)}{\delta}] \leq k$$

for all (z,t),

$$y(z,t) \geq 0 \quad \text{for all } (z,t), \text{ and}$$

$$y(z,t)\{k-e^{-\delta t}[p(y(z,t))+p'(y(z,t))y(z,t)-fz-s(e^{\delta t}-1)]\} = 0$$

for all (z,t).

These first order conditions along with the condition $2\int_0^T \int_0^r y(z,t)dz,dt = Q$ determine the optimal sales policy. We have assumed that the spot demand curve and its corresponding marginal revenue curve are downward sloping. Hence the first order conditions imply that at each time t, (gross) marginal revenue rises linearly in distance since unit freight charges have been assumed to rise linearly in distance. Therefore, sales fall with distance from the plant. Likewise at each location, z, current value (gross) marginal revenue rises over time as the current value of storage charges rise and future net revenue flows are discounted more heavily than current net revenue flows. Hence sales fall from the instant of production to the instant at which sales exhaust the output of a production run.

Since spot marginal revenue curves are downward sloping we can conclude $\frac{\partial^2 TR}{\partial Q^2} < 0$. To determine the optimal output, the firm solves the following maximization problem.

$$\max_{Q \geq 0} \text{imize } TR(r,T,Q) - C(Q). \tag{11}$$

The first order conditions for this problem are:

$$\frac{\partial TR}{\partial Q} - C'(Q) = 0. \tag{12}$$

The second order conditions are:

$$\frac{\partial^2 TR}{\partial Q^2} - C''(Q) < 0. \tag{13}$$

We assume that $\frac{C(Q)}{Q}$ is U-shaped and there are no fixed costs. This requires $C'(Q)$ to be U-shaped as well. Hence there will be two, one, or no solutions to the first order conditions. When there are none, the firm shuts down. When there is one solution the second order conditions will be satisfied, i.e., $\frac{\partial TR}{\partial Q}$ cuts $C'(Q)$ from above. When there are two solutions only one will satisfy the second order conditions. The other will actually be a local minimum of the objective function. At the smaller of the two output levels which satisfy the first order conditions, $\frac{\partial TR}{\partial Q}$ cuts $C'(Q)$ from below. At the larger it cuts from above. Hence the larger of the two output levels is the only candidate for an interior optimum.

We can now return to the choice of r and T. Let $y(z,t,r,T)$ denote optimal sales at location-time pair (z,t), when the market radius is r and the time between production-sales runs is T. We drop the explicit functional dependence on r and T and write these sales as $y(z,t)$. Let $Q(r,T) = 2 \int_0^T \int_0^r y(z,t)dzdt$ be the optimal production, given r and T. Again we drop the explicit functional dependence on r and T and write this production as Q. We can now write:

$$\pi(r,T) = 2 \int_0^T \int_0^r e^{-\delta t}[p(y(z,t))-fz-\frac{s(e^{\delta t}-1)}{\delta}]y(z,t)dzdt - C(Q). \tag{14}$$

$$\pi_r = 2 \int_0^T e^{-\delta t}[p(y(r,t))-fr-\frac{s(e^{\delta t}-1)}{\delta}]y(r,t)dt \tag{15}$$

$$- C'(Q)\cdot 2 \int_0^T y(r,t)dt.$$

$\frac{\partial Q}{\partial r} = 2 \int_0^T y(r,t)dt$ and the terms involving $\frac{\partial y(z,t)}{\partial r}$ are not included above because, by the envelope theorem, this change has no effect on overall profits. Likewise

$$\pi_T = 2 \int_0^r e^{-\delta t}[p(y(z,T))-fz-\frac{s(e^{\delta T}-1)}{\delta}]y(z,T)dz \tag{16}$$

$$- C'(Q)\cdot 2 \int_0^r y(z,T)dz$$

where $\frac{\partial Q}{\partial T} = 2 \int_0^r y(z,t)dt.$

We can also obtain the second own and cross partials as follows:

$$(17) \qquad \pi_{rr} = 2 \int_0^T e^{-\delta t}[-fy(r,T)]dt - C''(Q) \frac{\partial Q}{\partial r}\frac{dQ}{dr}$$

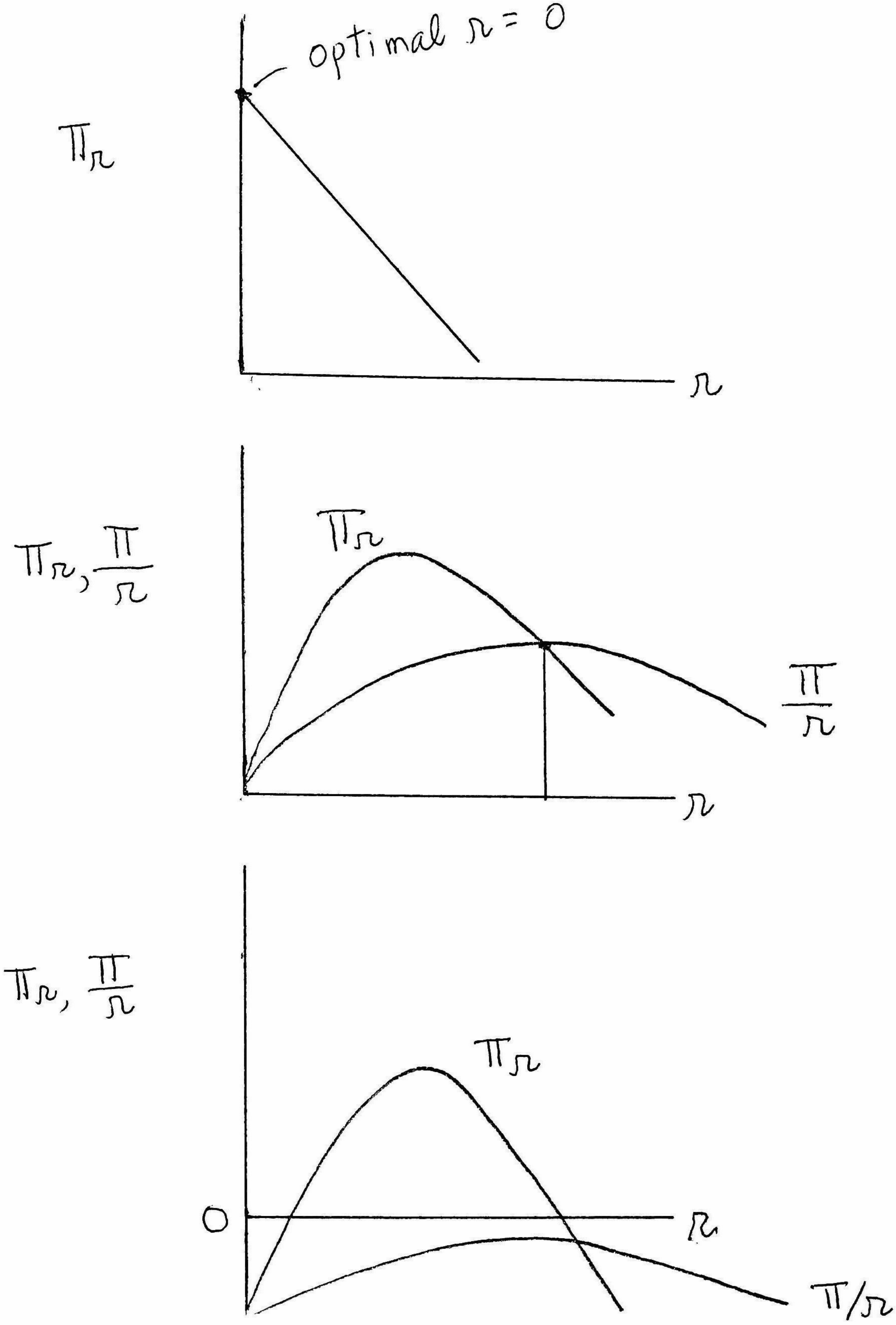
where $\frac{dQ}{dr} = 2 \int_0^T \int_0^r \frac{\partial y(z,t)}{\partial r} dzdt + \int_0^T y(r,t)dt.$

Note that $\frac{dQ}{dr} \neq \frac{\partial Q}{\partial r}$. $\frac{dQ}{dr}$ includes two effects. The first occurs because sales at each (z,t) pair change when r changes. The second occurs because the market area changes when r changes. $\frac{\partial Q}{\partial r}$ only includes this second effect.

We presume that $\frac{dQ}{dr} > 0$ in the relevant range. Intuitively, raising r shifts the aggregate marginal revenue curve for the entire market region to the right. Since the marginal cost curve is unaffected optimal output must rise.

From the above we can conclude that $\pi_{rr} > 0$ only if $C'' < 0$. Since we are assuming that the marginal cost curve itself is U-shaped, one and only one of the situations depicted in Figure 3.2 can occur. The graphs in the Figure are constructed from the policy which solves the first-order marginal conditions. Obviously, in this case these conditions do not ensure an optimum. We focus exclusively on an interior optimum (optimal r > 0). A necessary condition for an interior optimum is $\pi_{rr}|_{r=0} > 0$, while at the optimal point $\pi_{rr}|_{r=\text{optimum}} < 0$. Thus, second order conditions will not be satisfied universally and they must be checked before a comparative static-dynamic analysis is performed.

Figure 3.2


optimal $r = 0$
Π_r
r
$\Pi_r, \frac{\Pi}{r}$
Π_r
$\frac{\Pi}{r}$
r
$\Pi_r, \frac{\Pi}{r}$
Π_r
0
r
Π/r

$$(18)\qquad \pi_{rT} = 2e^{-\delta T}[p(y(r,T)-fr-\frac{s(e^{\delta T}-1)}{\delta}]y(r,T)-C''(Q)\frac{\partial Q}{\partial r}\frac{dQ}{dT}$$

$$- C'(Q)\frac{\partial^2 Q}{\partial r \partial T}$$

where $\frac{\partial^2 Q}{\partial r \partial T} = 2y(r,T)$.

Finally,

$$(19)\qquad \pi_{TT} = 2\int_0^r [-\delta e^{-\delta T}[p(y(z,T))-fz-\frac{s(e^{\delta T}-1)}{\delta}]y(z,T)-sy(z,T)]dz$$

$$- C''(Q)\frac{\partial Q}{\partial T}\frac{dQ}{dT}.$$

In order to perform the comparative statics-dynamics we must be able to sign $\pi_{rT} - \frac{\pi_T}{r}$.

$$(20)\qquad \pi_{rT} - \frac{\pi_T}{r} = 2\{e^{-\delta T}[p(y(r,T))-fr-\frac{s(e^{\delta T}-1)}{\delta}]-C'(Q)\}y(r,T)$$

$$- C''(Q)\frac{\partial Q}{\partial r}\frac{dQ}{dT} - 2\int_0^r \frac{\{e^{-\delta T}[p(y(z,T)-fz-\frac{s(e^{\delta T}-1)}{\delta}]-C'(Q)\}y(z,T)dt}{r}$$

Discounted revenue net of freight, storage, and marginal production cost (undiscounted) must be falling with distance from the plant, z. This follows since: (a) by the first order conditions, sales fall with z; and (b) if discounted net revenue were to rise with z over some interval, $[z_1,z_2]$, then by setting sales over this entire interval equal to sales at z_2 the firm would increase profits. Profits would rise because unit freight charges over $[z_1,z_2]$ are greatest at z_2 so that net revenue at z would be at least as great as net revenue at z_2, while production costs would fall since a cutback in sales would imply a cutback in production as well.

We conclude that if $C''(Q) \geq 0$ certainly $\pi_{rT} - \frac{\pi_T}{r} < 0$. When $C''(Q) < 0$, $\pi_{rT} - \frac{\pi_T}{r}$ may change signs. This completes the presentation of the model.

Section 2: Comparative Statics-Dynamics

As stated earlier, five cases are considered in this section. The first four adopt what is essentially a short run approach in that either T, the time of a production-sales run, or r, the market radius of each plant, is held fixed.

Case 1

This case focuses on pure spatial adjustments under the assumptions that T is fixed and $\delta = s = 0$. Our first order conditions then imply $y(z,t) = y(z,0)$ for all t. Hence

$$(21) \qquad \pi(r,T) = 2 \int_0^T \int_0^r [p(y(z,t))-fz]y(z,t)dzdt-C(Q)$$

$$= 2T \int_0^r [p(y(z,0)-fz]y(z,0)dz - C(Q)$$

where $Q = 2T \int_0^r y(z,0)dz$.

The firm maximizes $\frac{\pi(r,T)}{r}$ with respect to r. The relevant first order condition is:

$$(22) \qquad 2T[p(y(r,0))-fr-C'(Q)]y(r,0) - \frac{2T \int_0^r [p(y(z,0))-fz]y(z,0)dz+C(Q)}{r} = 0.$$

The relevant second order condition is:

(23) $$-2Tfy(r,0) - 2TC''(Q)y(r,0)\frac{dQ}{dr} < 0.$$

Thus $f > C''(Q)\frac{dQ}{dr}$ is required. At an interior optimum we obtain the following comparative static result.

(24) $$\text{sign}\frac{dr}{df} = \text{sign}\left[T\left[-r-C''(Q)\frac{dQ}{df}\right]y(r,0) - \frac{T\int_0^r -zy(z,0)dz}{r}\right]$$

$$= \text{sign}\left[\left[-r-C''(Q)\frac{dQ}{df}\right]y(r,0) + \frac{\int_0^r zy(z,0)dz}{r}\right].$$

Intuitively, $\frac{dr}{df}$ should be negative. Raising freight rates raises unit delivery costs more the further one gets from the plant since unit delivery costs at z are fz. An increase in the freight rate, f, makes markets further from the plant <u>relatively</u> less attractive than markets closer to the plant. Hence it pays for the firm to space plants closer together to avoid these high delivery costs, though this strategy cuts down on the firm's ability to exploit plant scale economies and consequently leads to higher average production costs. A sufficient condition for this intuition to hold is that $ry(r,0) > \frac{\int_0^r zy(z,0)dz}{r}$ and that $C''(Q) \leq 0$. We will actually assume a stronger condition on the sales path, namely, that $zy(z,0)$ is increasing in z. This condition says that total outlays on freight increase with distance from the plant, which holds for typical demand curves.

Perverse results concerning the sign of $\frac{dr}{df}$, i.e., $\frac{dr}{df} > 0$ can occur if $ry(r,0) < \frac{\int_0^r zy(z,0)dz}{r}$ or if $C''(Q) > 0$. If the former holds then an increase in the freight rate hurts the firm relatively less at the boundary of the market region than it does on average over the market region. If the latter holds, raising f for a fixed market region lowers marginal production cost since output falls. Though gross marginal revenue at the boundary of the market region rises when the freight rate rises, net marginal revenue actually falls. This is the case because by the first order conditions, marginal production cost must be equal to net marginal revenue. In these circumstances, it is possible for total revenue earned on sales at the boundary, net of delivery and marginal production costs, to fall off less with an increase in the freight rate than the net revenue lost on average over the entire market region. If this is the case, it actually pays for the firm to increase the spacing between plants, i.e., the market region increases with an increase in the freight rate.

Case 2

This case is the temporal analog of Case 1. It considers pure temporal adjustments to changes in the cost of storage when the freight rate is set equal to zero and r is fixed. In this case we will have $y(z,t) = y(0,t)$ for all z. Hence

$$(25) \qquad \pi(r,T) = 2\int_0^T\int_0^r e^{-\delta t}[p(y(z,t)) - \frac{s(e^{\delta t}-1)}{\delta}]y(z,t)dzdt - C(Q)$$

$$= 2r\int_0^T e^{-\delta t}[p(y(0,t)) - \frac{s(e^{\delta t}-1)}{\delta}]y(0,t)dt - C(Q)$$

where $Q = 2r\int_0^T y(0,t)dt$.

The firm maximizes $\frac{1}{1-e^{-\delta T}}\pi(r,T)$ with respect to T. The relevant first order condition is:

$$(26)\qquad 2r[e^{-\delta T}[p(y(0,T))-\frac{s(e^{\delta T}-1)}{\delta}]-C'(Q)]y(0,T)$$

$$-\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}[2r\int_0^T e^{-\delta t}[p(y(0,t))-\frac{s(e^{\delta t}-1)}{\delta}]y(0,t)dt-C(Q)] = 0.$$

The relevant second order condition is:

$$(27)\qquad 2r[-\delta e^{-\delta T}[p(y(0,T))-\frac{s(e^{\delta T}-1)}{\delta}]-s-C''(Q)\frac{dQ}{dT}]y(0,T)$$

$$+\frac{\delta^2 e^{-\delta T}}{1-e^{-\delta T}}[2r\int_0^T [p(y(0,t))-\frac{s(e^{\delta t}-1)}{\delta}]y(0,t)dt-C(Q)] < 0.$$

This condition is a bit more complicated than the equivalent condition of the previous case since now we are maximizing <u>aggregate</u> discounted present value profits rather than profits per unit length. At an interior optimum we obtain the following result:

$$(28)\qquad \text{sign}\,\frac{dT}{ds} = \text{sign}[2r[e^{-\delta T}(\frac{e^{\delta T}-1}{\delta})-C''(Q)\frac{dQ}{ds}]y(0,T)$$

$$+\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\cdot 2r\int_0^T e^{-\delta t}(\frac{e^{\delta t}-1}{\delta})y(0,t)dt].$$

As in the spatial case one would expect intuitively that $\frac{dT}{ds} < 0$. A sufficient condition for this result to hold is that $C''(Q) \leq 0$ and $\frac{(e^{\delta t}-1)}{\delta}y(0,t)$ is increasing with t. That is, current value storage charges associated with <u>sales</u> at t when the interest rate is δ are increasing with time from production. Then $\frac{(e^{\delta t}-1)}{\delta}y(0,t) < \frac{(e^{\delta T}-1)}{\delta}y(0,T)$ and consequently

$$\frac{\delta e^{-\delta T}}{(1-e^{-\delta T})}\left[2r\int_0^T e^{-\delta t}\frac{(e^{\delta t}-1)}{\delta}y(0,t)dt\right] <$$

$$\frac{\delta e^{-\delta T}}{(1-e^{-\delta T})}\left[2r\int_0^T e^{-\delta t}\frac{(e^{\delta T}-1)}{\delta}y(0,T)dt\right]$$

$$= e^{-\delta T}\left[2r\frac{(e^{\delta T}-1)}{\delta}y(0,T)\right].$$

Perhaps a more revealing way to compare the temporal and spatial analyses of Cases 1 and 2 is to rewrite the first order conditions as follows:

For the spatial analysis,

$$\frac{\partial\pi}{\partial r} - \frac{\pi}{r} = 0. \tag{29}$$

For the temporal analysis,

$$\frac{\partial\pi}{\partial T} - \frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\pi = \frac{\partial\pi}{\partial T} - \frac{\delta T}{e^{\delta T}-1}\frac{\pi}{T} = 0. \tag{30}$$

In the temporal case, marginal profit per length of time between production-sales runs equals a fraction, $\frac{\delta T}{e^{\delta T}-1}$, of average profit. Since $\lim_{\delta T\to 0}\frac{\delta T}{e^{\delta T}-1} = 1$ when $\delta = 0$ we get $\frac{\partial\pi}{\partial T} - \frac{\pi}{T} = 0$. In this case the spatial and temporal models coincide. Since $\lim_{\delta T\to\infty}\frac{\delta T}{e^{\delta T}-1} = 0$ when δ is very large, we get approximately $\frac{\partial\pi}{\partial T} = 0$. That is, the firm maximizes profit by acting as if there is only one production run when δ is large. We now consider the comparative dynamics of changes in δ.

If the marginal revenue curve has a finite valued y intercept (e.g., linear demand) then $\frac{\partial\pi}{\partial T} = 0$ for some finite T. Furthermore, the T where this occurs is decreasing in δ. Hence intuitively one has the following: if $\frac{\partial\pi}{\partial T} \gg 0$ then $\frac{\partial T}{\partial\delta} > 0$. That is, raising δ increases the relative importance of profits from the first production-sales run. When $\frac{\partial\pi}{\partial T} \gg 0$ these profits can be increased significantly by increasing T. When $\frac{\partial\pi}{\partial T} \simeq 0$, changing T has no significant effect on profits from the first production run. In this case $\frac{\partial T}{\partial\delta} < 0$.

Alternatively, one can view these effects in terms of Figure 3.3. Raising δ shifts down both $\frac{\partial\pi}{\partial T}$ and $\frac{\pi}{T}$. It also lowers the fraction $\frac{\delta T}{e^{\partial T}-1}$. It is reasonable that $\frac{\partial\pi}{\partial T}$ shifts down more than $\frac{\pi}{T}$. Thus, isolating this shifting effect, T has a tendency to fall. On the other hand reducing $\frac{\delta T}{e^{\delta T}-1}$, has a tendency to increase T.

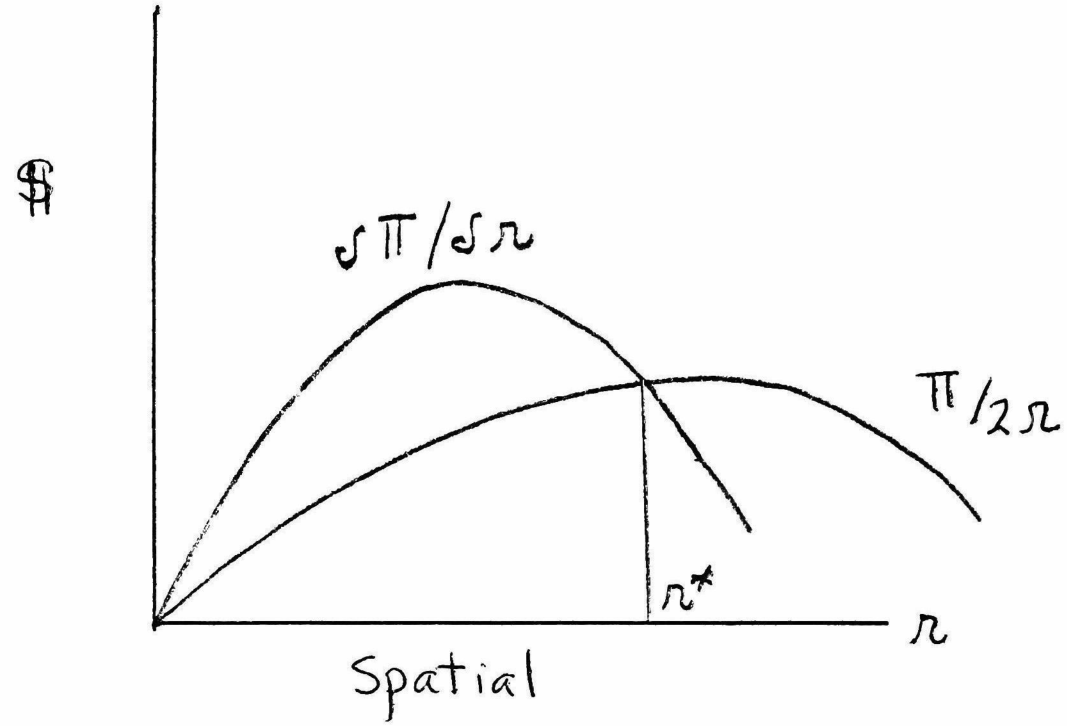
Case 3

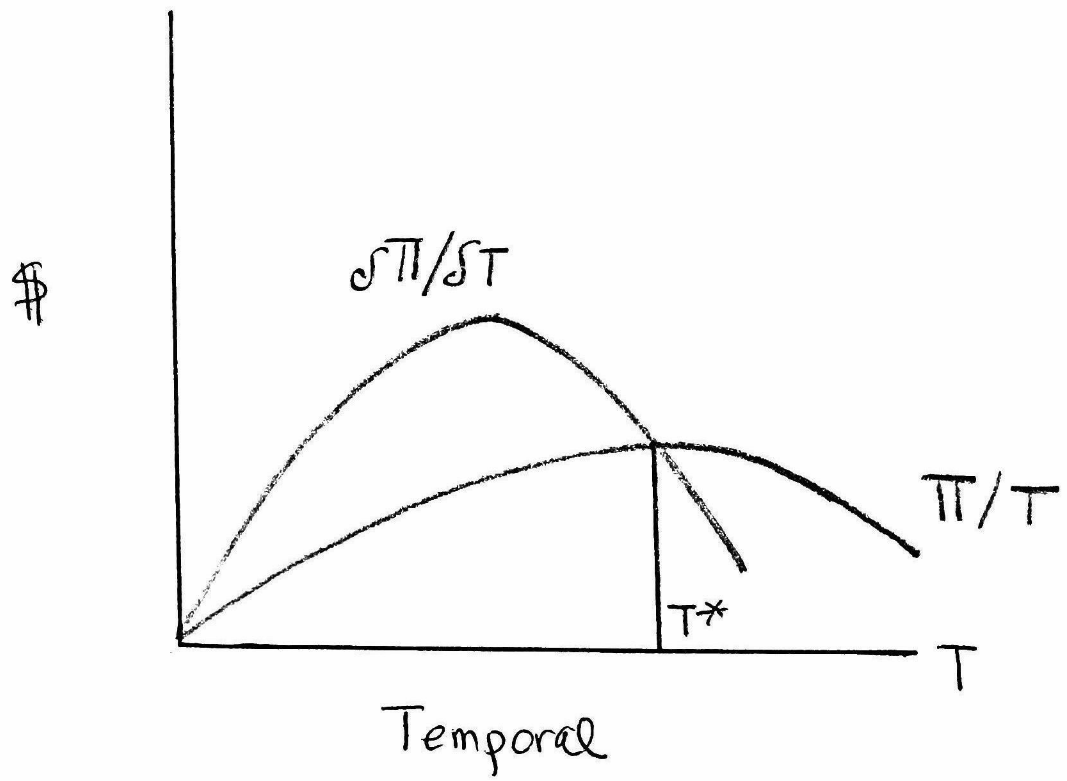
Here we return to an analysis of the effects of changes in f, the freight rate. Again T, the length of time of a production-sales run, is held constant. However, this case differs from Case 1 in that s and δ are positive.

Now, the relevant first order condition is:

$$(31) \qquad 2\int_0^T e^{-\delta t}\left[p(y(r,t)-fr-s\frac{(e^{\delta t}-1)}{\delta}\right]y(r,t)dt-C'(Q)\cdot 2\int_0^T y(r,t)dt$$
$$-\frac{2\int_0^T\int_0^r e^{-\delta t}\left[p(y(z,t)-fz-s\,\frac{(e^{\delta t}-1)}{\delta}\right]y(z,t)dzdt-C(Q)}{r} = 0.$$

Figure 3.3


$
$\delta\Pi/\delta r$
$\Pi/2r$
r^*
r
Spatial


$
$\delta\Pi/\delta T$
Π/T
T^*
T
Temporal

The relevant second order condition is:

$$(32) \qquad 2\int_0^T e^{-\delta t}[-fy(r,t)]dt - C''(Q)\frac{dQ}{dr}\cdot 2\int_0^T y(r,t)dt < 0.$$

At an interior optimum the following results are obtained:

$$(33) \qquad \text{sign}\,\frac{dr}{df} = \text{sign}[2\int_0^T [-e^{-\delta t}r - C''(Q)\frac{dQ}{df}]y(r,t)dt + \frac{2\int_0^T\int_0^r e^{-\delta t}zy(z,t)dzdt}{r}].$$

This is quite similar to the results reported above for the analysis of the pure spatial effects of changes in f. If $zy(z,t)$ is increasing in z for each t and $C''(Q) \leq 0$ then $\frac{dr}{df} < 0$.

$$(34) \qquad \text{sign}\,\frac{dr}{ds} = \text{sign}[2\int_0^T [-\frac{(1-e^{-\delta t})}{\delta} - C''(Q)\frac{dQ}{ds}]y(r,t)dt + \frac{2\int_0^T\int_0^r \frac{(1-e^{-\delta t})}{\delta}y(z,t)dz,dt}{r}].$$

Intuitively $\frac{dr}{ds} > 0$. As long as $C''(Q) \geq 0$ this is necessarily the case because

$$\frac{\int_0^r (\frac{1-e^{-\delta t}}{\delta})y(z,t)dz}{r} > (\frac{1-e^{-\delta t}}{\delta})y(r,t)$$

since sales fall with distance from the plant.

We also can sign $\frac{dr}{d\delta}$!

$$\text{(35)}\qquad \operatorname{sign}\frac{dr}{d\delta} = \operatorname{sign}\Big[2\int_0^T \Big\{-te^{-\delta t}\Big[p(y(r,t)-fr-s\Big(\frac{e^{\delta t}-\delta t-1}{t\delta^2}\Big)\Big]$$

$$- C''(Q)\frac{dQ}{d\delta}\Big\}y(r,t)dt$$

$$+\frac{2\int_0^T\int_0^r te^{-\delta t}\Big[p(y(z,t))-fz-s\Big(\frac{e^{\delta t}-\delta t-1}{t\delta^2}\Big)\Big]y(z,t)dzdt}{r}\Big]$$

Note that $[p(y(z,t))-fz-s(\frac{s^{\delta t}-\delta t-1}{t\delta^2})]y(z,t) \geq [p(y(z,t))-fz-s(\frac{e^{\delta t}-1}{\delta})]y(z,t)$.

(The second expression is net delivered total revenue and is declining in z.) Since $y(z,t)$ is declining in z the first expression is also declining in z. Hence $\frac{dr}{d\delta} > 0$ as long as $C''(Q) \geq 0$.

Case 4

This is the case that is the temporal analog of Case 3. Now, r is fixed and T is variable. Again, f and δ are positive.

The relevant, first order condition for ths case is:

$$\text{(36)}\qquad 2\int_0^r \Big\{e^{-\delta T}\Big[p(y(z,T))-fz-s\Big(\frac{e^{\delta T}-1}{\delta}\Big)\Big]-C'(Q)\Big\}y(z,T)dz$$

$$-\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\Big\{2\int_0^T\int_0^r e^{-\delta t}\Big[p(y(z,t))-fz-s\Big(\frac{e^{\delta t}-1}{\delta}\Big)\Big]dzdt-C(Q)\Big\} = 0.$$

The relevant second order condition is:

$$\text{(37)}\qquad 2\int_0^r \Big\{-\delta e^{-\delta T}\Big[p(y(z,T))-fz-s\Big(\frac{e^{\delta T}-1}{\delta}\Big)\Big]-s-C''(Q)\frac{dQ}{dT}\Big\}y(z,T)dz$$

$$+\frac{\delta^2 e^{-\delta T}}{1-e^{-\delta T}}\Big\{2\int_0^T\int_0^r e^{-\delta t}\Big[p(y(z,t))-fz-s\Big(\frac{e^{\delta t}-1}{\delta}\Big)\Big]dzdt-C(Q)\Big\} < 0.$$

At an interior optimum the following results are obtained.

$$\text{(38)} \qquad \text{sign}\,\frac{dT}{ds} = \text{sign}\Big[2\int_0^r \Big\{-\Big(\frac{1-e^{-\delta T}}{\delta}\Big) - C''(Q)\frac{dQ}{ds}\Big\} y(z,T)\,dz - \frac{\delta e^{-\delta T}}{1-e^{-\delta T}} \cdot 2\int_0^T\int_0^r \Big(\frac{1-e^{-\delta t}}{\delta}\Big) y(z,t)\,dz\,dt\Big].$$

This is also similar to Case 2. If $\frac{e^{\delta t}-1}{\delta}\, y(z,t)$ is increasing in t for each z then $\frac{dT}{ds} < 0$ as long as $C''(Q) \le 0$.

$$\text{(39)} \qquad \text{sign}\,\frac{dT}{df} = \text{sign}\,\Big[2\int_0^r \Big\{-ze^{-\delta T} - C''(Q)\frac{dQ}{df}\Big\} y(z,T)\,dz - \frac{\delta e^{-\delta T}}{1-e^{-\delta T}} \cdot 2\int_0^T\int_0^r - e^{-\delta t} z y(z,t)\,dz\,dt\Big].$$

$\frac{dT}{df} > 0$ as long as $C''(Q) \ge 0$.

This completes the short run analysis.

Case 5: The Long Run

In the long we have to take account that both r and T may vary. Recall our second order conditions are:

$$\text{(40)} \qquad \begin{bmatrix} \pi_{rr} & \pi_{rT} - \frac{\pi_T}{r} \\ \pi_{rT} - \frac{\pi_T}{r} & \pi_{TT} + \frac{\delta^2 e^{-\delta T}}{1-e^{-\delta T}}\,\pi \end{bmatrix} \text{ is negative definite.}$$

This requires π_{rr}, $\pi_{TT} + \frac{\delta^2 e^{-\delta T}}{1-e^{-\delta T}}\pi < 0$ and $\pi_{rr}[\pi_{TT} + \frac{\delta^2 e^{-\delta T}}{1-e^{-\delta T}}\pi] - [\pi_{rT} - \frac{\pi}{r}]^2 > 0$ at an interior optimum.

Algebraically all the comparative static-dynamics results are given below.

$$\text{(41)} \qquad \operatorname{sign}\frac{dr}{df} = \operatorname{sign}\Big[\overset{+}{\left(\frac{\pi_f}{r} - \pi_{rf}\right)}\overset{-}{\left(\pi_{TT} + \frac{\delta^2 e^{-\delta T}}{1-e^{-\delta T}}\pi\right)} - \overset{-}{\left(\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\pi_f - \pi_{Tf}\right)}\overset{-}{\left(\pi_{rT} - \frac{\pi_T}{r}\right)}\Big].$$

$$\text{(42)} \qquad \operatorname{sign}\frac{dT}{df} = \operatorname{sign}\Big[\overset{-}{\left(\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\pi_f - \pi_{Tf}\right)}\overset{-}{\pi_{rr}} - \overset{+}{\left(\frac{\pi_f}{r} - \pi_{rf}\right)}\overset{-}{\left(\pi_{rT} - \frac{\pi_T}{r}\right)}\Big].$$

$$\text{(43)} \qquad \operatorname{sign}\frac{dr}{ds} = \operatorname{sign}\Big[\overset{-}{\left(\frac{\pi_s}{r} - \pi_{rs}\right)}\overset{-}{\left(\pi_{TT} + \frac{\delta^2 e^{-\delta T}}{1-e^{-\delta T}}\pi\right)} - \overset{+}{\left(\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\pi_s - \pi_{Ts}\right)}\overset{-}{\left(\pi_{rT} - \frac{\pi_T}{r}\right)}\Big].$$

$$\text{(44)} \qquad \operatorname{sign}\frac{dT}{ds} = \operatorname{sign}\Big[\overset{+}{\left(\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\pi_s - \pi_{Ts}\right)}\overset{-}{\pi_{rr}} - \overset{-}{\left(\frac{\pi_s}{r} - \pi_{rs}\right)}\overset{-}{\left(\pi_{rT} - \frac{\pi_T}{r}\right)}\Big].$$

The interest rate effects are too complicated to include here. We have taken the liberty of putting in the "usual" signs of all terms above the corresponding term. When this is the case $\frac{dr}{df}, \frac{dT}{ds} < 0$ and $\frac{dr}{ds}, \frac{dT}{df} > 0$. We continue to assume the condition on sales that total freight out-

lays at distance z rise with z, and total current value storage charges at time t rise with t.

The "usual" signs may alter when C" >> 0 (to the right of min MC) or C" << 0 (to the left of min MC).

When C" >> 0 the only terms which may switch signs are $\frac{\pi_f}{r} - \pi_{rf}$ which may become negative and $\frac{\delta e^{-\delta T}}{1-e^{-\delta T}} \pi_s - \pi_{Ts}$ which may also become negative. When this happens $\frac{dr}{df}, \frac{dT}{ds} > 0$ is possible as is $\frac{dr}{ds}, \frac{dT}{df} < 0$.

When C" << 0, $\pi_{rT} - \frac{\pi_T}{r} > 0$ is possible as is $\frac{\delta e^{-\delta T}}{1-e^{-\delta T}} \pi_f - \pi_{Tf}$, $\frac{\pi_s}{r} - \pi_{rs} > 0$. When this is the case it is possible that $\frac{dr}{df}, \frac{dT}{df}, \frac{dr}{ds}, \frac{dT}{ds} < 0$. As should be expected, given the much greater complexity of the problem, the results achieved in the long run analysis are ambiguous when compared to those of the short run.

Footnotes

[1]As stated in Part 2 of this report, the essential logic of the model is unchanged if cost and demand functions differ over time and space. In Part 4, a solution procedure is developed and the solutions to some sample problems are presented that show this to be the case.

[2]M. L. Katz, "Multiplant Monopoly in a Spatial Market," The Bell Journal of Economics, vol. no. 2, Autumn 1980, pp. 519-535.

[3]When W is small the assumption that each production-sales run is of identical duration is inappropriate

[4]L. Arvan and L. N. Moses, "Inventory Investment and the Theory of the Firm," American Economic Review, May 1982.

Part 4: Some Uses of the Space-Time Modelling Effort[1]

The Transportation Center research on physical distribution has two major goals. First, we wish to formulate models of firms' decision making that are capable of taking spatial and temporal factors into account. In effect, we wish to build models that bring together the richness and understanding of two bodies of economic literature about the firm, the theory of location and the theory of dynamics. Our second goal is to formulate solution procedures that incorporate the essential elements of our theoretical work and can be used to solve complex, real world problems. This last point requires further comment.

In Part 1 of this report, the point was made that the systems or total cost approach to physical distribution is viewed by most specialists in the field as representing the intellectual threshold to modern logistical reasoning and practice. We accepted this point but stated our reservation, that in fact the systems approach is rarely used in complex problems. It is rare to see examples of logistical modelling in which quantitative estimates of the tradeoffs between diverse logistical elements are based on a formal logical structure. The point was also made that when tradeoffs are estimated they are frequently based on the assumption of linearity in the various cost elements, whereas it is widely known that some of the most important cost elements in a logistical plan involve significant non-linearities. The sample problems discussed in this part of our report use cost functions that involve scale economies in the long run or variable returns to variable factors in the short run.

It was also noted in Part 1 above that some writers have objected to what they view as an excessive concern on the part of specialists with the minimization of the total costs of distribution, with all other goals being ignored. In this regard we observed that logistical models are fixed quantity models. We accepted the commonly held view that it is appropriate for the specialist in physical distribution to treat as given the prices set by marketing people in various markets and therefore the quantities sold. However, we added that the models with which logistics specialists work should be capable of being put into a profit maximizing framework. Among other things this means that they should be capable of incorporating demand responsive elements. Hence in the work done in the present part of the report, we go beyond fixed quantity models. The economist's and modern marketing specialist's downward sloping price-quantity demand function is introduced into logistical models.

In earlier parts of the report, the point was made that the theory of the model with which we are working was developed for the case of demand and cost functions that are time invariant. This assumption was made because we were interested in showing how non-linearities in costs could of themselves produce a pattern of dynamic behavior for the imperfectly competitive firm. We referred to our model as a pure model of dynamics because the conditions for dynamic behavior largely originated from within the firm. In most models of firm dynamics, the firm's behavior is the result of exogenous changes in costs, demands, etc. faced by it. While our model allows for dynamic behavior to originate largely from within the firm, it and the solution procedures we adopt are fully capable of handling changes in outside conditions. Two of the examples of temporal problems discussed below introduce changes in costs. Demand shifts could just as easily be introduced.

Our theoretical model has production taking place at a moment of time, output then being sold off over a period of time. This on-off character of the model's operations allows for the analysis of a very realistic problem, the production of multiple products in a single plant, and the switching back and forth from the production of one to the production of another. One of the problems solved below involves periodic increases in costs as well as product switching in production.

In all of the problems discussed below, we seek optimal solutions, solutions that maximize profits. Because we have cost functions that involve both convex and concave segments, and because we allow the firm to vary price and hence amounts sold as well as production, the profit surfaces involved in all of our problems are very complex. They can have numerous local optima. It is easy for the computer to get stuck at a local maximum and not converge to a higher one. The solutions obtained are highly conditioned by the initial conditions imposed, i.e. the starting place for a problem. Different starting places yield different solutions. We believe that we have developed a solution procedure that in many, perhaps most, instances overcomes this difficulty. This procedure is described in general terms below.

In all of our applied work we have used a non-linear software package called GRG II. It finds solutions that satisfy the first order conditions for a maximum or a minimum. The package itself does not get around the difficulty of local optima, and the fact that the solution obtained in any given run is so heavily conditioned by the initial conditions of the problem. To cope with this problem, we have had to develop our own solution procedures and link them to GRG II.

In almost all of our work we employ a multiple two stage maximization procedure. In the first of these stages, a cost minimization is carried out to determine, in the case of a spatial problem, which plants would serve which markets if a given output, $\bar{Q}_i$, had to be sold in each market.[2] By limiting the amount sold in each market to some prespecified amount, we place linear constraints on our non-linear objective function, making it easier to find a solution. After finding a least cost solution for a prespecified set of sales and hence an overall limit on the total or system output, we then carry out a second stage. Here, profits are maximized and output is unconstrained. The minimum cost solution provides the initial conditions for the second stage unconstrained profit maximization.

Above we referred to our solution procedure as a multiple two stage procedure. The word multiple was employed because it is run more than once, using different $\bar{Q}_i$'s or output-sales constraints in stage one, the cost minimization stage. In this way a profit "function" is generated with profit measured on the vertical axis and system output or the sum of the $\bar{Q}_i$'s on the horizontal. The shape of profit "curve" is examined for a maximum. The output flows corresponding to "the" maximum profit point on a profit hill is taken to be the optimum solution. We complete this part of the discussion with some additional comments on the linear constraints employed in the cost minimization stage of our solution procedure.

It is a simple rule of profit maximizing behavior that an imperfectly competitive firm will not normally sell a quantity in a market at which marginal revenue is negative. Hence with any demand function which intersects the horizontal quantity axis, $\bar{Q}_i$ must be between zero and the point at which marginal revenue is zero. We call this latter quantity Q_i^* . We then choose $\bar{Q}_i$'s by dividing the demand curve into equal size segments between zero and Q_i^*. We now turn to some examples of problems that were solved using the above solution procedure. Each of the problems involves U-shaped cost functions.

Each problem contemplates a firm that faces a downward sloping demand function so that prices charged and quantities sold in different spatial or temporal markets may be decision variables as well as rates of output. Our first sample problem involves spatial considerations alone.

Section 1: The Pure Spatial Model

The pure spatial model is a tool used to allocate output from spatially distinct plants to spatially distinct markets in a profit-maximizing manner, given the cost functions for each plant, the demand functions for each market, and the transportation costs between any two points. Let x_{ij} represent the amount of good x produced at plant i and sold in market j. We can denote the total amount produced at plant i as

$$\mathop{}_{j=1}^{M} x_{ij} \text{ , and the total amount sold in market j as}$$

$$\sum_{j=1}^{N} x_{ij} \text{ ,}$$

where N is the number of plants and M is the number of markets. Note that x_{ij} is the amount of x shipped from i to j, or, the "flow." The final result of the model gives us the profit maximizing flows of goods, as well as the corresponding system-wide profit, cost, revenue (prices and quantities sold), and output. Our pure spatial model may be thought of as relevant for the case of an imperfectly competitive firm that can achieve all the cost economies required to maximize profits by only massing production in space. Profits are not increased by also massing production in time, perhaps because the product is very difficult and expensive to store. Hence the firm carries little or no product inventory.

In one of our sample problems we assumed three plants and six markets. The cost functions of the three plants are cubic and different from one another. They are:

Plant 1: $C(x_1) = 166.666x_1 - 2.66x_1^2 + .0166x_1^3$

$x_1^* = 53.333 \quad x_1^{**} = 80$

Plant 2: $C(x_2) = 380x_2 - 7.5x_2^2 + .05x_2^3$

$x_2^* = 50 \quad x_2^{**} = 75$

Plant 3: $C(x_3) = 325x_3 - 6x_3^2 + .04x_3^3$

$x_3^* = 50 \quad x_3^{**} = 75$

where x_1^* is the output level where marginal cost reaches its minimum, and x_i^{**} is the output level where average cost reaches its minimum.

The six different markets were assumed to have linear demand functions, different from one another.

Market 1: $P(Q_1) = 1500 - 15Q_1 \qquad Q_1^* = 50$

Market 2: $P(Q_2) = 1000 - 100_2 \qquad Q_2^* = 50$

Market 3: $P(Q_3) = 1250 - 12.5Q_3 \qquad Q_3^* = 50$

Market 4: $P(Q_4) = 1800 - 18Q_4$ $\quad Q_4^* = 50$

Market 5: $P(Q_5) = 2000 - 20Q_5$ $\quad Q_5^* = 50$

Market 6: $P(Q_6) = 2500 - 25Q_6$ $\quad Q_6^* = 50$

where Q_i^* is the point at which marginal revenue is zero.

In the present problem we assumed a single mode of transportation and also assumed that transport cost functions were linear in the amount shipped. However, just as our solution procedures are capable of handling non-linearities in costs of production they can handle cost functions for transport that also involve non-linearities. The model is capable of handling several modes of transport, so that one of the decision variables can be choice of mode of transport.

In the present problem the unit transport costs are:

from/to	1	2	3	4	5	6
1	0	7	12	5	6	20
2	7	0	11	10	4	7
3	12	11	0	6	16	8

For the purpose of generating a profit hill we chose our $\bar{Q}_i$'s to be 50, 45, 40, 35, and 30. 50 was chosen because it is the upper limit on the amount a monopolist will sell, i.e. marginal revenue equals zero at an output of 50. 30 seemed to be a good lower limit in view of the shape of the cost curves. Selling 30 in each market would put production close to the minimum of marginal cost at each plant if all three plants were producing. Finally, in view of the relatively low cost of running the two-stage procedure, we decided upon increments of 5. Less than 5, we felt, would have been too small and repetitive, but larger increments, e.g. 10, would have been too big and we possibly would have missed something.

The initial conditions in the cost minimization stages were as follows:

from/to	1	2	3	4	5	6
1	$\bar{Q}$	0	0	$\bar{Q}$	0	0
2	0	$\bar{Q}$	0	0	$\bar{Q}$	0
3	0	0	$\bar{Q}$	0	0	$\bar{Q}$

The choice of letting plants 1, 2, and 3 serve their home markets arises from the fact that it is the likely solution in view of the transportation costs. There is no rationale, however, for the choice in markets 4, 5, and 6. Concerned that the initial conditions for markets 4, 5, and 6 affected the results, we ran one two-stage costs minimization-profit maximization procedure using the following as initial conditions:

from/to	1	2	3	4	5	6
1	45	0	0	15	15	15
2	0	45	0	15	15	15
3	0	0	45	15	15	15

We found the initial conditions to have no effect on the results.

As explained earlier, the purpose of segmenting the demand curves and selecting different starting places and initial conditions grew out of our understanding that the profit surfaces involved in our problems are complex and can involve numerous local maxima. However, in the present case the flow configurations that resulted from each starting place was essentially the same. The profit that resulted from the second, unconstrained profit maximization stage was $223,950 in each case. The consistency of the results achieved with quite different initial conditions made us feel confident that we had found a profit maximizing solution.

A summary of the results follows. Note the sales totals for each market. In steps III - IV, sales were originally constrained to be 40, 35, or 30 in the cost minimization stage. In the unconstrained profit maximization stage, however, the sales in each market exceed the original constraint. To conserve space we report on only two of the profit explorations. These are the extremes. One is the largest possible sale in each market, and hence the largest possible system output. The largest possible sale in a normal monopoly model is one in which marginal revenue is zero. The low sales and output extreme is that in which sales in each market are 30 units. The initial conditions for this solution have each plant operating on the falling portion of its marginal cost function. As already noted, the unconstrained profit maximizing stage of the two experiments converge to essentially identically the same flows, outputs, sales and prices.

As to prices, we note that the profit maximizing solution of the firm in question involves price discrimination. Thus, the equilibrium solution of our problem has prices between markets that are not equal to differences in the transport costs of serving them. Indeed, markets served by the same plant have prices that differ by more than the transport costs between the plant and markets in question. The prices in markets 1, 4, and 5, each served by plant 1 provide an example. In a perfectly competitive spatial model the equilibrium condition is that prices between markets cannot differ by more than the cost of shipping the product between them.

I. Constrain sales to 50 at each market in the cost minimization stage.

A. Constrained Cost Minimizaton

1. Flows:

x_{ij}	1	2	3	4	5	6	total
1	50	0	0	50	18.566	0	118.566
2	0	50	0	0	31.434	7.9738	89.408
3	0	0	50	0	0	42.026	92.026
tot	50	50	50	50	50	50	

B. Unconstrained Profit Maximization

1. Flows:

x_{ij}	1	2	3	4	5	6	total
1	44.855		0	45.573	14.019	0	104.477
2	0	42.189	0	0	31.972	7.5933	81.754
3	0	0	43.792	0	0	39.141	82.933
tot	44.855	42.189	43.792	45.573	45.991	46.734	

2. System output: 269.14

3. Cost: 24,871

4. Profit: 223,950

5. Prices:

$P_1 = 828$

$P_2 = 578$

$P_3 = 702$

$P_4 = 979$

$P_5 = 1080$

$P_6 = 1332$

II. Constrain sales to 30 at each market in the cost minimization stage.

A. Constrained Cost Minimization

1. Flows:

x_{ij}	1	2	3	4	5	6	total
1	30	0	0	30	1.4898	0	61.490
2	0	30	0	0	28.510	4.3543	62.864
3	0	0	30	0	0	25.646	55.646
tot	30	30	30	30	30	30	

2. System output: 180

3. Cost: 17,628

4. Profit: 193,420

B. Unconstrained Profit Maximization

1. Flows:

x_{ij}	1	2	3	4	5	6	total
1	44.862	0	0	45.580	13.969	0	104.411
2	0	42.181	0	0	32.021	7.5508	81.753
3	0	0	43.782	0	0	39.184	82.966
tot	44.862	42.181	43.782	45.580	45.90	46.735	

2. System output: 269.13

3. Cost: 24,871

4. Profit: 223,950

5. Prices as in the previous solution

We conclude our discussion of the pure spatial model with some summary comments.

We are now quite certain that the solution procedures we have developed allow us to solve quite large multi-plant, multi-market models in which outputs at different plants as well as prices and sales in different markets are decision variables. The existence of complex cost functions and different downward sloping demand functions do not appear to pose insuperable difficulties to finding profit maximizing solutions. Up to now we have used linear transport costs. We see no problem in introducing transport cost functions that exhibit shipment size economies. Indeed, alternative functions for different modes and even some service variations can be introduced so that choice of mode of transport also becomes a part of the optimal solution. However, at this stage we are limited to transport cost and service functions that have no discontinuities in them. Typical rate structures do involve discontinuities. In trucking, for example, there is one rate for shipments of less than 5000 pounds, a lower rate for shipments between 5000 and 15,000 pounds, etc. Future research will include methods of introducing such realistic rate structures into our spatial models.

Section 2: Pure Temporal Problems

The firm of Section 1, above, was characterized as one that did not need to aggregate across time as well as space in order to maximize profits. Alternatively, it was suggested that the firm's product was so expensive to store that it behaved statically. The firm of this present section finds that it operates over time but each plant serves a single market, perhaps because the product is difficult and expensive to ship.

At one stage in our efforts to model and develop solution procedures for different types of real world logistical problems we became overly impressed with the similarities between the economies of carrying goods through space and the economies of carrying them through time. We failed to note an important difference between a variety of temporal problems and the short run spatial problem.

It will be recalled that the short run was earlier described as that period of time in which the firm cannot add to its productive capacity by building new plants at new locations. Where production can take place is fixed in the short run, but is fully variable in the long run. It is in this sense that our pure temporal differ from the pure spatial problems. Even in the short run, when productive capacity is fixed, the firm can alter when it produces, i.e., it can alter the timing of production and length of run. In this sense, all of our temporal problems are like the long run locational problems referred to briefly in Part 3 above. Therefor, the problem of when and how much to produce in a short run temporal problem is not the pure analog of the short run locational problem. Rather, it is the analog of the long run locational problem, one in which the choice of where to produce and how much to produce at alternative locations is determined. This characteristic of temporal problems makes our multiple cost-min-profit-max optimization procedure somewhat more complicated than the one used to solve short run spatial problems. We now explain the nature of the procedure used to solve temporal problems.

We take the short run planning horizon of the firm as given and begin by determining whether profit maximization even requires a dynamic strategy. If the demand function of each period intersects the marginal cost function of each period above minimum average cost and if the cost function is invariant, then a static strategy is profit maximizing. If the above conditions are not satisfied, the solution procedure described below is followed. A sample problem is used to illustrate the procedure.

In this problem, demand in each period of time was:

$P(Q) = 1000 - 10Q$.

This equation has a value of Q^*, the sales at which marginal revenue is zero, equal to 50. The production cost function for each period is:

$$C(x) = 500x - 8x^2 + .05x^3 .$$

Minimum marginal and average costs occur at outputs that respectively are 53 and 80. The storage cost function is assumed to be linear in both quantity and time and equal to \$20 per unit product stored per day.[3] The problem was run with different assumptions as to the per period interest rate. A 3% interest rate was assumed in the problem reported on below. The firm was assumed to have an 8 period planning horizon.

The reader will recall that in the spatial problem, a $\bar{Q}$ was assumed for each market. A cost minimizing problem was then run to determine the optimal allocation of output to the various plants. This minimum cost solution provided the initial conditions for the unconstrained profit maximization storage. The $\bar{Q}$'s were then changed and the two stage procedure was repeated. The solution procedure for the pure temporal problem does not have a formal cost minimizing stage, though it is still a multiple two stage procedure. Instead of obtaining the initial conditions for the unconstrained profit maximizing step from a cost minimizing step, initial conditions in the temporal problem are obtained by specifying a time pattern of production. This is the way in which the temporal problem more closely resembles a long run than a short run locational problem. The step-by-step testing of how many and where among a set of potential sites the firm should locate its plants is the spatial equivalent of testing out different lengths of production-sales runs in a temporal problem.

In the temporal problem currently being discussed we began, as in the spatial problem, by specifying the $\bar{Q}_i$'s as the sales in the (temporal) market at which marginal revenue is zero. This is 50 units. We then chose the first temporal pattern to be tested, i.e. produce all output in the first period and then shut down. Below we call this Case A. In Case B, production takes place twice, once in period 1 and once in period 5. In Case C production takes place four times, in periods 1, 3, 5, and 7. The data of the three cases are shown below. We will shortly explain why these are the only relevant cases for the present problem.

Case A, produce 1 time

period	1	2	3	4	5	6	7	8
prod'n	400	0	0	0	0	0	0	0
inv'y	350	300	250	200	150	100	50	0
sales	50	50	50	50	50	50	50	50

Case B, produce 2 times

period	1	2	3	4	5	6	7	8
prod'n	200	0	0	0	200	0	0	0
inv'y	150	100	50	0	150	100	50	0
sales	50	50	50	50	50	50	50	50

Case C, produce 4 times

period	1	2	3	4	5	6	7	8
prod'n	100	0	100	0	100	0	100	0
inv'y	50	0	50	0	50	0	50	0
sales	50	50	50	50	50	50	50	50

Three unconstrained profit maximization problems were then run, with the above providing the initial conditions. The results are reported below, again as Cases A, B, and C. Among the things that our profit maximizing procedure can do is to shift to a different pattern of production than was entered initially. Thus Case A begins with production only in period 1, but profit maximization yields a much more complex pattern. The firm acts dynamically in period 1, producing in that period for the first three periods. However, beginning in period 4, it acts statically. Only in Case C is the initial pattern maintained. The pattern of on for a period, and off for a period provides the maximum profit. The three cases yield quite different patterns of sales, profits, outputs, demonstrating very clearly that the objective functions we work with are very complex and have numerous local maxima. The solution obtained in any stage is highly conditioned by the initial conditions. It is for this reason that temporal problems, even more than spatial problems, require experimentation based on segmentation of the demand functions, i.e. varying the $\bar{Q}_i$'s. Our last comments on the above temporal problem pertain to the pattern of prices and sales when the firm is operating dynamically. Any of the examples of dynamic behavior shown in the three cases can be used to illustrate the point we wish to make.

Results

Case A

System output: 32,469

Cost: 65,257

Revenue: 165,310

Profit: 100,050

period	1	2	3	4	5	6	7	8
prod'n	95.1	0	0	45.9	45.9	45.9	45.9	45.9
inv'y	61.8	30.1	0	0	0	0	0	0
sales	33.2	31.7	30.1	45.9	45.9	45.9	45.9	45.9

Case B

System output: 270,96

Cost: 51,884

Revenue: 156,720

Profit: 104,840

period	1	2	3	4	5	6	7	8
prod'n	95.1	0	0	86.6	89.3	0	0	0
inv'y	61.8	30.1	0	48.6	101.2	66.0	32.2	0
sales	33.3	31.7	30.1	38.0	36.6	35.2	33.7	32.2

Case C

System output: 321.52

Cost: 54.352

Revenue: 168.720

Profit: 114,370

period	1	2	3	4	5	6	7	8
prod'n	80.4	0	80.4	0	80.4	0	80.4	0
inv'y	39.6	0	39.6	0	39.6	0	39.6	0
sales	40.8	39.5	40.8	39.5	40.8	39.5	40.8	39.5

Consider the profit maximizing solution yielded under the initial conditions of Case A. Here the firm only acts dynamically in the first three periods, production in period 1 being used to serve demand in the three periods. We observe the observed pattern of falling sales over the three periods, which means that prices must be rising since we are dealing with a time invariant demand function. The equilibrium prices are: P_1 = \$668; P_2 = 683; P_3 = 699. Prices rise, but less than the cost of storage because our demand function is linear. In this sense the dynamic strategy involves discrimination in favor of future as against present consumers. One might ask what prices in periods 2 and 3 would have to be if price in period 1 were \$668 and expected to remain there. If the market were perfectly competitive, with storage cost of \$20 per unit per period and an interest rate of 3%, P_2 and P_3 would have to be equal to \$729 and \$769 in the two periods in order for a group of perfectly competitive sellers to be in temporal equilibrium, i.e. to be indifferent as to which market period they dispose of their existing output.

Our solution procedure is perfectly capable of being applied to situations in which costs and demands change over time. Experiments were performed for a case in which costs varied in a systematic way over time. We do not report on this example because the model reviewed in the next section incorporates cost changes as well as other things.

Section 3: A Product Switching Model

Economies of long production runs, as we saw in our theoretical model, contribute to a situation in which a firm produces in excess of current sales and carries inventory of finished product. Beyond a certain point it is not economical to produce "today" for future markets becauses storage costs become an uneconomic alternative to a resumption of production as a way of satisfying future sales. Thus we have a pattern in which production is switched on and off over time. However, there is no need for the plant to actually shut down if there are alternative products that can be produced by some or all of the fixed factors of production found in the plant. When production of one product is discontinued, a run of another product is begun.

The product switching example presented below considers a situation in which two products are produced in one facility. Only one good can be produced at a time, but both goods can be sold simultaneously. The product switching model determines how daily demand for the two products can be satisfied given the constraint that the two goods cannot be produced simultaneously. This is achieved by alternating production runs and carrying inventory.

The product switching model is solved by a procedure that is very similar to the one employed in the pure spatial model. We work with a sequence of sets of $\bar{Q}$'s , one set for each product in each period of time. The initial set of $\bar{Q}$'s are those at which the marginal revenue of each product goes to zero in each period. By starting with these sales for each period, we assure ourselves that we have logically bounded the system, since an imperfectly competitive firm will not normally sell more in any period than the quantity at which marginal revenue is zero.

As in the pure spatial model, we first run a constrained cost minimization problem. The $\bar{Q}$'s of each product in each period provide the linear constraints used in the cost minimization stage of the solution procedure. What is obtained is the least costly way of satisfying any given set of $\bar{Q}$'s over time. The solution specifies a certain switching pattern in which only one good is produced on any given day. The excess of production over sales is stored to be sold on days on which the good is not produced. There are two kinds of days of this kind in the example reported on below. First, there are days when the alternate product is being produced. Second, in our example of a monthly planning model, there are Saturdays and Sundays. Costs of production of both products on those days carry a penalty. They are higher than the costs of producing the products during the week. As a result, production does not take place on the weekend. The penalty costs for weekend production tend to have the effect on assuring that plant-down time occurs on weekend days.

Above we were discussing the cost minimization stage of each step in the solution procedure. Each cost minimization solutions provides the initial conditions for an unconstrained profit maximization. The objective function for the profit maximization stage has the following elements in it for each period of the planning horizon: (1) the revenues obtained from selling each of the products; (2) the cost of producing whichever product is being produced in each period; (3) the cost of storing each product in each period.

The solution procedure involves a sequence of two stage (cost min and profit max) solutions. In each stage a different set of $\bar{Q}$'s is selected. By varying the $\bar{Q}$'s , we investigate the possibility that the profit surface is complex and has local optima. As in the pure spatial model we establish a profit hill. We turn now to our example.

The short run planning horizon assumed in the problem is 30 days. The cost of producing good 1 is:

$$C(X_1) = 500x_1 - 8x_1^2 + .05x_1^3.$$

In this cost function, marginal and average cost reach their minima at 53.33 and 80. The cost of producing good 2 is:

$$C(X_2) = 350x_2 - 2x_2^2 + .005x_2^3.$$

Minimum marginal and average cost occur at 133.33 and 200. All of the above costs are in hundreds of dollars. In our solution procedure, we formally permit the two products to be produced simutaneously, but we impose a very high penalty on such production. There is also a penalty for production on weekends. Costs are 1.1 times the above figures. The weekend days are 7, 8, 13, 14, 20, 21, 27, and 28.

Our problem assumes linear storage costs, but there is no reason why U-shaped storage cost functions could not have been used. Storage cost for good 1 is $1.00 per day and for good 2 is $15.

A time invariant linear demand function is assumed for each product, though there would have been no great difficulty in allowing demands to vary over time. The two demand functions are:

$$P(Q_1) = 1000 - 10Q_1$$

$$P(Q_2) = 1425 - 15Q_2 .$$

Q_1^* and Q_2^*, the two sales rates at which marginal revenue equals zero are 50 and 47.5 respectively.

In the product switching model, we assume an initial inventory for each product. This is not an essential aspect of the model. An alternative to initial inventory is to allow buildup time. This is time in which production can take place but in which no sales are made of one or the other product. The initial inventories assumed in our present formulation were:

$$I_1 = 55 \text{ and } I_2 = 25.$$

Three tables appear below. Table 1 contains the data that were entered into the first cost minimization problem. This is the problem in which $\bar{Q}_1$ and $\bar{Q}_2$ are equal to 50 and 47.5, the quantities at which the two marginal revenues are zero. Table 2 contains the cost minimizing solution. In this problem, sales are constrained to be greater than or equal to the above two quantities in each period. Columns (2) and (3) show the temporal pattern of production of each product, i.e. the switching back and forth and the down days. Columns (6) and (7) show the end of period inventory of each product. The results shown in Table 2 are entered as the initial conditions for the first unconstrained profit maximization problem. As expected, sales in each period of each product are less than the sales at which marginal revenue is zero. Total output of each product over the 30 day period is of course less than that of the cost minimization stage. The time pattern of production of each good, and the down days for the plant as a whole are the same in the unconstrained profit maximization as in the cost minimization problem. In the next stage of the solution procedure, a reduced set of $\bar{Q}$'s is selected and new cost minimization and profit maximization problems run. We do not bother to report on the additional steps since they all yielded the same results as those shown in Table 3.

Table 1
Initial Conditions

period	Initial production condition X_1	X_2	Initial inventory condition I_1	I_2
1	0	142.5	5	120
2	150	0	105	72.5
3	50	47.5	105	72.5
4	0	142.5	55	167.5
5	150	0	155	120
6	0	0	105	72.5
7	0	0	55	25
8	0	142.5	5	120
9	150	0	105	72.5
10	50	47.5	105	72.5
11	0	142.5	55	167.5
12	150	0	155	120
13	0	0	105	72.5
14	0	0	55	25
15	0	142.5	5	120
16	150	0	105	72.5
17	50	47.5	105	72.5
18	0	142.5	55	167.5
19	150	0	155	120
20	0	0	105	72.5
21	0	0	55	25
22	0	142.5	5	120
23	150	0	105	72.5
24	50	47.5	105	72.5
25	0	142.5	55	167.5
26	150	0	155	120
27	0	0	105	72.5
28	0	0	55	25
29	0	70	5	47.5
30	45	0	0	0

Table 2
Cost Minimization Results

(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Production		Sales		End of period inventory	
period	X_1	X_2	Q_1	Q_2	I_1	I_2
1	0.0	117.5	50.0	47.5	5.0	95.0
2	104.3	0.0	50.0	47.5	59.3	47.5
3	104.3	0.0	50.0	47.5	113.6	0.0
4	0.0	190.0	50.0	47.5	63.6	142.5
5	104.5	0.0	50.0	47.5	118.1	95.0
6	0.0	0.0	50.0	47.5	68.1	47.5
7	101.7	0.0	50.0	47.5	119.8	0.0
8	0.0	143.5	50.0	47.5	69.8	96.0
9	104.7	0.0	50.0	47.5	124.5	48.5
10	104.8	0.0	50.0	47.5	179.3	1.0
11	0.0	189.0	50.0	47.5	129.3	142.5
12	104.9	0.0	50.0	47.5	184.2	95.0
13	0.0	0.0	50.0	47.5	134.2	47.5
14	0.0	0.0	50.0	47.5	84.2	0.0
15	0.0	143.5	50.0	47.5	34.2	96.0
16	105.2	0.0	50.0	47.5	89.4	48.5
17	105.2	0.0	50.0	47.5	144.6	1.0
18	0.0	189.0	50.0	47.5	94.6	142.5
19	105.4	0.0	50.0	47.5	150.0	95.0
20	0.0	0.0	50.0	47.5	100.0	47.5
21	0.0	0.0	50.0	47.5	50.0	0.0
22	0.0	143.5	50.0	47.5	0.0	96.0
23	87.9	0.0	50.0	47.5	37.9	48.5
24	88.0	0.0	50.0	47.5	75.8	1.0
25	0.0	189.0	50.0	47.5	25.8	142.5
26	88.1	0.0	50.0	47.5	64.0	95.0
27	0.0	0.0	50.0	47.5	14.0	47.5
28	86.0	0.0	50.0	47.5	50.0	0.0
29	0.0	95.0	50.0	47.5	0.0	47.5
30	50.0	0.0	50.0	47.5	0.0	0.0

Table 3
Profit Maximization Results

	Production		Sales		End of Period Inventory	
period	X_1	X_2	Q_1	Q_2	I_1	I_2
1	0.0	106.6	40.41	44.36	14.6	87.2
2	81.5	0.0	40.36	43.86	55.8	43.4
3	81.7	0.0	40.31	43.36	97.1	0.0
4	0.0	172.8	40.26	43.94	56.9	128.8
5	81.9	0.0	40.21	43.44	98.5	85.4
6	0.0	0.0	40.16	42.94	58.4	42.4
7	80.0	0.0	40.11	42.44	98.2	0.0
8	0.0	132.7	40.06	44.72	58.2	87.9
9	82.4	0.0	40.01	44.22	100.5	43.7
10	82.5	0.0	39.96	43.72	143.0	0.0
11	0.0	172.8	39.91	43.94	103.1	128.8
12	82.7	0.0	39.86	43.44	146.0	85.4
13	0.0	0.0	39.81	42.94	106.2	42.4
14	0.0	0.0	39.76	42.44	66.4	0.0
15	0.0	132.7	39.71	44.72	26.7	87.9
16	83.2	0.0	39.66	44.22	70.2	43.7
17	83.3	0.0	39.61	43.72	113.8	0.0
18	0.0	172.8	39.56	43.94	74.3	128.8
19	83.5	0.0	39.51	43.44	118.2	85.4
20	0.0	0.0	39.46	42.94	78.8	42.4
21	0.0	0.0	39.41	42.44	39.4	0.0
22	0.0	132.7	39.36	44.72	0.0	87.9
23	74.9	0.0	42.84	44.22	32.1	43.7
24	75.1	0.0	42.79	43.72	64.3	0.0
25	0.0	172.8	42.74	43.94	21.6	128.8
26	75.4	0.0	42.69	43.44	54.2	85.4
27	0.0	0.0	42.64	42.94	11.6	42.4
28	73.5	0.0	42.59	42.44	42.5	0.0
29	0.0	86.8	42.54	43.64	0.0	43.1
30	45.9	0.0	45.92	43.14	0.0	0.0

Some additional experiments will be carried out with the product switching model in the future, including the introduction of additional products. However, there is one important problem about which we are unlikely to be able to do anything that is likely to prove to be analytically rigorous and satisfying. This is the problem of set-up costs.

Real world product switching situations entail some kind of cost of going from the production of one product to the production of another. A machine may, for example, be capable of producing a variety of metal fasteners, but not without its being adjusted. The adjustment entails down time and costs that we call set-up costs. Such costs are very difficult to introduce into the kind of solution we employ, because they are discrete. They are a kind of fixed cost in the sense that once the money is expended and the plant's machinery has been set up to produce a product, any amount of that product can be produced without additional set-up costs in any run that is uninterrupted except for plant down time. While set-up costs are fixed in the above sense, they are variable temporally in that they may be borne more than once for each product over the planning horizon. The solution procedure with which we have been working cannot handle lumpy costs. However, a procedure of introducing them through a heuristic device that seems promising will be tested in the next academic year. This completes this part of our report. Many other models and problems have been investigated. It was felt that the above three were enough to report on at present.

Footnotes

[1]This part of the report was written with Martha Weidner, Department of Economics, Northwestern University.

[2]In a temporal problem we determine the time of production used to satisfy demands in different temporal markets.

[3]We make the same comment about the linear storage costs of the temporal models as was made earlier about the assumption of linear transport costs. That is, we now see that there is no problem in introducing storage cost functions that exhibit cost economies with quantities stored, so long as the function are smooth.

Part 5: Future Research and Plans

The next stage in the Center's physical distribution research will largely be in the area of applications of our approach to actual problems. Members of the Business Advisory Committee will be approached and an effort made to work out arrangements in which they frame a physical distribution problem that is important to their firms and provide the data needed to analyze the problems. Center staff will then set up the analytical model, write the computer programs needed to solve the problems, and interpret the results.

More work will have to be done on the problems mentioned in the text if our approach is to be most useful in a real world setting. In particular, our models must be capable of dealing with realistic rate structures and their discontinuties. The models and computer programs should also be capable of incorporating transportation service variables so that problems involving mode choice, production and inventory planning, and product pricing can be solved. These goods will require the development of additional solution procedures.