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Stochastic Production and Cost/Production Duality*

#425-04

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November 1978

*I wish to thank D. Baron, R. Chandrasekaran, J. Ledyard, J. Roberts and K. Smith for their suggestions without necessarily implicating them in any remaining errors. Research was supported in part by U.S.D.O.T. Contract # DOT-OS-70061.

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ABSTRACT

This paper considers production processes that are stochastic: inputs or parameters follow a distribution resulting in stochastic output that the entrepreneur must account for when choosing the optimal level of deterministic inputs to use. We examine the implications of noisy production in terms of functional structure and the duality between production and cost functions. This results in an extended notion of duality as well as the conditions under which noise can be ignored, which severely restrict the admissible class of production functions.

1. Introduction

Most production processes are stochastic in nature. In general while entrepreneurs may plan to produce some specified level of output, they often do not exactly fulfill their plans: shortfalls or overages occur. Railroads and airlines do not precisely meet their schedules, peanut-butter makers attempt to control the quantity of lice and rat hairs in their product, and most of us would try to avoid buying a car made on a Monday (at least if they were so labelled). This is especially true since the firm's output is often broader than the simple quantity of units they produce: customer services are performed in terms of delivery, quality of merchandise, etc. This paper is concerned with some of the implications of the presence of random variables in the production process for analyses of cost and production.

This is certainly not a new topic. Zellner, Kmenta and Dreze [14] resurrected the direct estimation of the Cobb-Douglas model by considering a multiplicative term and firms that maximize expected profit. Feldstein [4] considers a more general Cobb-Douglas case and derives results for the estimation procedure and implications for factor shares of returns from production. While there have been a number of papers concerned with stochastic input prices, Rothenberg and Smith [8] appear to be the first to trace out some of the general equilibrium resource allocation effects of assuming the noise to be in input variables rather than the prices.

What will be new will be the direction of the inquiry. While for the most part the literature has been concerned with the econometrics of analyzing cost and production, our interest will concern the models of cost and production themselves. Specifically we will study some of the implications for analysis of technology when there is noise (i.e. random disturbances) in the production process. It will be shown that ignoring the noise is equivalent to severely restricting the admissible class of production processes. This will also shed light on how such stochastic effects must be accounted for in analyses of production and cost.

Again, it should be stressed that the randomness being analyzed is not the type that is usually addressed (except as noted above) by econometric analysis. We will not be concerned for example, with problems of measurement error or differences amongst firms. We will consider noise that enters the production process that the entrepreneur must acknowledge, plan for and act upon: failures of machines, variability of labor quality, imperfections in quality control schemes (as well as quality control itself), theft, accidents, etc. Often this noise is modeled in some sort of multiplicative or otherwise convenient way. We will examine the issue for functional structure that noise in the process implies.

The approach will be to pose a maximum expected utility model for the firm. Most of the analysis will consider the risk-neutral case on the basis that such a case provides an upper bound on fortuitous circumstances: problems of accounting for risk aversion only makes matters worse. We attempt, however, to relate the two situations in our basic development.

In the third section of the paper we consider the implications of ignoring the presence of noise in the production process. Ignorance is bliss for the following cases:

1) In the case of direct analyses of the production process, ignorance is bliss if and only if the random disturbances enter the production process in a separable way.

2) In the case of indirect analyses of the production process (i.e. via the cost function and duality), ignorance is bliss if and only if the production function is constant returns-to-scale.

The above results prompt an extension of the duality between cost and production so as to allow for general production processes. The fourth section provides a summary and conclusions.

2. A Model of Stochastic Production.

We consider a firm producing a single output y using an input vector* x, $x \in R^n_+$, which it purchases at given prices $q \in R^n_{++}$. Production follows a production function $f: R^n_+ \times D \to R_+$ represented as $f(x,\omega)$ with the following properties

1. $f(0,\omega) = 0 \quad \forall_{\omega \in D}$
2. $D \subset R$, typically an interval $[\cdot,\cdot]$
3. $f(x,\omega)$ is finite for bounded x and all ω
4. f is continuous with continuous first and second derivatives (in x and ω)
5. $\nabla_x f(x,\omega) > 0 \quad \forall_\omega$
6. $\omega \sim G(\gamma)$, $\gamma \in R^m$ (γ is the vector of parameters for the distribution G)

Thus if G is the normal distribution then $\gamma = (\mu,\sigma^2)'$. Let p be the price of output; thus profits will be taken to be $\pi = py - q'x$, $y = f(x,\omega)$. We further assume that the firm maximizes expected utility of profits and thus assume a function $U: R \to R$ on profits with $U' > 0$, $U'' \leq 0$. If $U'' = 0$, we are considering the risk-neutral case which provides the expected profit maximizing results. We take γ to be given and thus the firms problem is to select x so as to maximize expected utility, i.e. it wishes to

$$\max \int_D U(pf(x,\omega) - q'x)dG$$

where we have suppressed γ, since it is fixed.

The first order conditions are straightforward:

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* $R^n_+ = \{x : x \in R^n,\ x \geq 0\}$, $R^n_{++} = \{x : x \in R^n,\ x > 0\}$

$$(1) \qquad p\int_D U' \, f_i dG = q_i \int_D U' dG \qquad i = 1,\dots,n$$

where $f_i \equiv \partial f(x,\omega)/\partial x_i$ and $U' \equiv dU(\pi)/d\pi$. Another way to write (1) is as follows

$$(2) \qquad \frac{\int_D U' f_i dG}{\int_D U' f_j dG} = \frac{q_i}{q_j} \qquad \forall_{i,j}$$

which is the familiar marginal rate of substitution set equal to the ratio of prices. Notice that if the firm is risk-neutral then (2) becomes

$$(2') \qquad \frac{E(f_i)}{E(f_j)} = \frac{q_i}{q_j} \qquad \forall_{i,j}$$

where $E(f_i) = \int_D f_i dG$, $\forall_i$. Thus (2′) is a generalized version of the usual rate-of-technical-substitution condition for firms that are maximizing expected profits. Note that, in general, $E(f_i) \neq f_i(x,E(\omega))$.

In the case of risk-neutrality we can readily examine the relationship between the production process and the cost function. Consider the cost minimization problem (CMP):

$$\text{(CMP)} \qquad \min \quad q'x$$
$$\text{S.T.} \quad E(f(x,\omega)) = u$$

with q given and u parametric ($u \equiv E(y)$). In the language of duality (see, for example, [13]) we have a (expected) production possibility set $Y \subset R^{n+1}$ and an inputs requirement set $V(E(y))$:

$$V(E(y)) = \{x : x \in R^n_+, (E(y), -x) \in Y\}$$

We define expected isoquants $Q(E(y))$:

$$Q(E(y)) = \{x : x \in R^n_+,\ x \in V(E(y)), x \notin V(E(y) + \alpha) \qquad \forall_\alpha > 0\}$$

and the result of (CMP) is a cost function $C(E(y),q)$. Finally, we consider the cost function-implied technology $V^*(E(y))$:

$$V^*(E(y)) = \{x : x \in R^n_+, q'x \geq C(E(y),q) \quad \forall_q \geq 0\}.$$

Since the first order conditions imply that we may work with $E(f(x,\omega)$ (i.e., $E(y)$) then the above sets and functions are direct extensions of their non-stochastic counterparts in the literature (see [3], [7], [9], and [13]). $C(E(y),q)$, $V(E(y))$ and $V^*(E(y))$ have the usual regularity, monotonicity, etc., properties which will not be repeated here (see [13], chapter 1), except to note that the above functions and sets are well-defined and come from a specified technology $\{f(x,\omega),G\}$ and a given set of input prices q.

3. Implications for Functional Structure

Noise in the production process implies restrictions on the functional structure of the production function. The theorems developed in this section are based on results on functional structure (see [2], [5], [10]) and on cost/production duality (see [3], [7], [9]). We will show that the duality between cost and production that allows investigation of the production function via the cost function will necessitate very strong requirements on the production function structure in order to be operative.

3.1. Stochastic Separability and the Expansion Path

Consider the extension of the Leontief-Sono separability condition [2,5,10] to the problem of separability of the random variable from the non-random inputs in the production process:

Definition. $f(x,\omega)$ is stochastically separable (s.s.) if

$$\frac{\partial(f_i(x,\omega)/f_j(x,\omega))}{\partial\omega} = 0 \qquad \forall_{i,j}$$

The importance of $f(x,\omega)$ being s.s. is that this implies (and is implied by the condition) that there exists functions $k:R^n \to R$ and $\tilde{f}:R^2 \to R$ (with $\tilde{f}_1 > 0$) such that ([2]):

$$f(x,\omega) = \tilde{f}(k(x),\omega)$$

Thus

$$f_i(x,\omega) = k_i(x)\tilde{f}_1(k(x),\omega)$$

$$\Rightarrow E(f_i(x,\omega)) = k_i(x)E(\tilde{f}_1(k(x),\omega))$$

and

$$E(U'f_i) = k_i(x)E(U'\tilde{f}_1(k(x),\omega)).$$

Thus, the first order conditions become

$$\frac{k_i(x)}{k_j(x)} = \frac{q_i}{q_j} \qquad \forall_{i,j}$$

Thus, if $f(x,\omega)$ is s.s. then both U and G are irrelevant: we may proceed as if the process was deterministic.

Clearly, the reverse issue is more important: what is the structure of $f(x,\omega)$ such that the form of U and G can be ignored? The answer is that $f(x,\omega)$ must be s.s. To see this we require only that there be functions $T_{ij}: R^n_+ \to R_{++}$ such that

$$T_{ij}(x) = \frac{\int U' f_i}{\int U' f_j} \qquad \forall_{i,j}$$

for arbitrary G and U (such that $U' > 0$). An obvious candidate is k(x) with $T_{ij}(x) \equiv k_i(x)/k_j(x)$.

The condition above is equivalent to

$$\int U' (f_i(x,\omega) - T_{ij}(x) f_j(x,\omega)) dG = 0$$

Since this must hold for arbitrary G then we must have:

$$\frac{f_i(x,\omega)}{f_j(x,\omega)} = T_{ij}(x)$$

which implies s.s. Thus, s.s. is a necessary and sufficient condition for using a deterministic production function to model first order conditions for a stochastic production process:

Theorem 1. $f(x,\omega)$ is s.s. if and only if $\exists T_{ij}: R^n_+ \to R_{++} \quad \forall_{i,j}$ such that

$$T_{ij}(x) = \frac{\int U'(\pi(x,\omega)) f_i(x,\omega) dG}{\int U'(\pi(x,\omega)) f_j(x,\omega) dG} \qquad \forall_G$$

where $\pi(x,\omega) = pf(x,\omega) - q'x$

To illustrate the above, we consider the following four Cobb-Douglas production functions:

$$f^A = x_1^{\alpha} x_2^{\beta} \omega$$

$$f^B = x_1^{\omega} x_2^{\beta}$$

$$f^C = (x_1\omega)^{\alpha} x_2^{\beta}$$

$$f^D = (x_1 + \omega)^{\alpha} x_2^{\beta}$$

It is easy to confirm that only f^A and f^C are s.s. Certainly, a priori, there is no theoretical reason to prefer one to the others. Thus, stochastic separability is a property that should be tested (by using a model sufficiently general to accept or reject it) rather than assumed, since the cost of not testing is to misrepresent the expansion path.

It should be emphasized that the theorem holds for both risk-neutral and risk-averse scenarios: nowhere did we actually rely upon properties of U (other than $U' > 0$). In other words, if we are analyzing technology via the production function (instead of the cost function) then:

1) We can ignore considerations of underlying "noiseiness" if and only if we can demonstrate that the production process is s.s.

2) If the process is s.s., the utility function is irrelevant to input decisions and we can therefore ignore considerations of risk-aversion.

3.2. A Non-S.S. Example

To illustrate the implications of s.s., we will examine a simple example of a stochastic production process which is not stochastically separable. Consider the production process represented by:

$$f(x,\omega) = x_1^{\omega} x_2^{\beta}$$

with $0 \le \omega \le a$, $a > 0$, $\omega \sim U[0,a]$. $f(x,\omega)$ is Cobb-Douglas in (x_1,x_2) with ω distributed uniformly on $D = [0,a]$ (i.e., $dG = d\omega/a$) and β non-stochastic. The expected production function $E(f(x,\omega))$ is simply:

$$E(f(x,\omega)) = \int_0^a x_1^{\omega} x_2^{\beta} \frac{d\omega}{a} = \frac{(x_1^a - 1)x_2^{\beta}}{\ell n x_1^a} \qquad x_1 \neq 1$$

For convenience, we take $U(Z) = Z$ so that we are considering the risk-neutral case. Finally, it is trivial to show that $\lim_{x_1 \to 1} E(f(x,\omega)) = x_2^{\beta}$ from both the left and right and thus $E(f(x,\omega))$ is continuous and given by

$$E(f(x,\omega)) = \begin{cases} \dfrac{(x_1^a - 1)x_2^{\beta}}{\ell n x_1^a} & x_1 \neq 1 \\ x_2^{\beta} & x_1 = 1 \end{cases}$$

Now if $f(x,\omega)$ were estimated directly (by, say, taking logs and estimating the random coefficients regression model (see [11], [12]) along with input demand equations), then the resulting estimated function would be $f(x,E(\omega))$. Thus, we will compare $E(f(x,\omega))$ and $f(x,E(\omega))$ for our example. To see this we express the first order conditions as follows:

$$\frac{E(f_1)}{E(f_2)} = \frac{ax_2}{x_1^{\beta}} \; \frac{x_1^a \ell n x_1^a - x_1^a + 1}{(x_1^a - 1)/\ell n x_1^a} = \frac{q_1}{q_2}$$

$$= \frac{f_1(x,E(\omega))}{f_2(x,E(\omega))} \; Z(x_1,a)$$

where $f_1(x,E(\omega)) = af(x,E(\omega))/2x_1$, $f_2(x,E(\omega)) = \beta f(x,E(\omega))/x_2$ and

$$Z(x_1,a) = \begin{cases} 2\,\dfrac{x_1^a \ell n x_1^a - x_1^a + 1}{(x_1^a - 1)\ell n\, x_1^a} & x_1 \neq 1 \\ 1 & x_1 = 1 \end{cases}$$

It can be shown that

$$\lim_{x_1 \to 1} Z(x_1,a) = Z(1,a) \qquad \forall_a$$

$$\lim_{x_1 \to +\infty} Z(x_1,a) = 2 \qquad \forall_a$$

$$\lim_{x_1 \downarrow 0} Z(x_1,a) = 0 \qquad \forall_a$$

$$Z(x_1,a) \begin{cases} > 1 & x_1 > 1 \\ < 1 & x_1 < 1 \end{cases} \qquad \forall_a$$

$$\frac{\partial Z(x_1,a)}{\partial x_1} > 0 \qquad x_1 \neq 1 \qquad \forall_a$$

Consider now the expansion path of the firm, i.e. let

$$\mathcal{E} = \{x : x \in R^2_+,\ q_2 E(f_1) = q_1 E(f_2)\}$$

for a given $q = (q_1, q_2)'$ and consider the following two rays from the origin:

$$A = \{x : x \in R^2_+,\ q_1 \beta x_1 = q_2 \tfrac{a}{2} x_2\}$$

$$B = \{x : x \in R^2_+,\ q_1 \beta x_1 = q_2\, a\, x_2\}$$

A is the "expansion path" that would be associated with $f(x,E(\omega))$, i.e. associated with the following cost minimization problem

$$\min \quad q'x$$

$$\text{S.T.} \quad f(x,E(\omega)) = u$$

with u parametric. On the other hand B is the asymptote for $\mathcal{E}$. It can be viewed as the "expansion path" associated with a slightly different version of the same problem:

$$\min \quad \tilde{q}'x$$
$$\text{S.T.} \quad f(x,E(\omega)) = u$$

with $\tilde{q} = (q_1/2, q_2)'$ and u parametric. From the analysis of $Z(x_1,a)$ we see that $\mathcal{S}$ represents a continuous function that starts at the origin, travels initially above A, cuts A at $x_1 = 1$, and asymptotically approaches B. This is illustrated in Figure 1 with $\bar{f} = (2\beta q_1/aq_2)^\beta$ providing the isoquant of $f(x,E(\omega))$ where $\mathcal{S}$ crosses A.

Thus, if $f(x,\omega)$ were directly estimated in the usual way we might be mislead into assuming the existence of a bias on the part of the firm. Since the firm will be operating on $\mathcal{S}$ (not A), observations of input combinations will occur on (or about) $\mathcal{S}$, not on (or about) A. This becomes more acute the greater the range of operation of the firm. Since the observations lie off the path described by A, these observations may be taken to imply inefficient operation of the firm, when in fact the firm is acting efficiently.

3.3. Cost/Production Duality and Stochastic Production Functional Structure

The previous two subsections have briefly examined a problem (stochastic separability) associated with analyzing production functions directly. This subsection will consider the implications of indirect assessment of technology via the cost function. Specifically, we will employ the now familiar apparatus of cost/production duality theory to explore the implications of noise in the production process for the proper representation of the associated cost function.

As was shown in section two above the cost function described by (CMP) can be written as $C(E(y),q)$. Consider, however, the result of observing

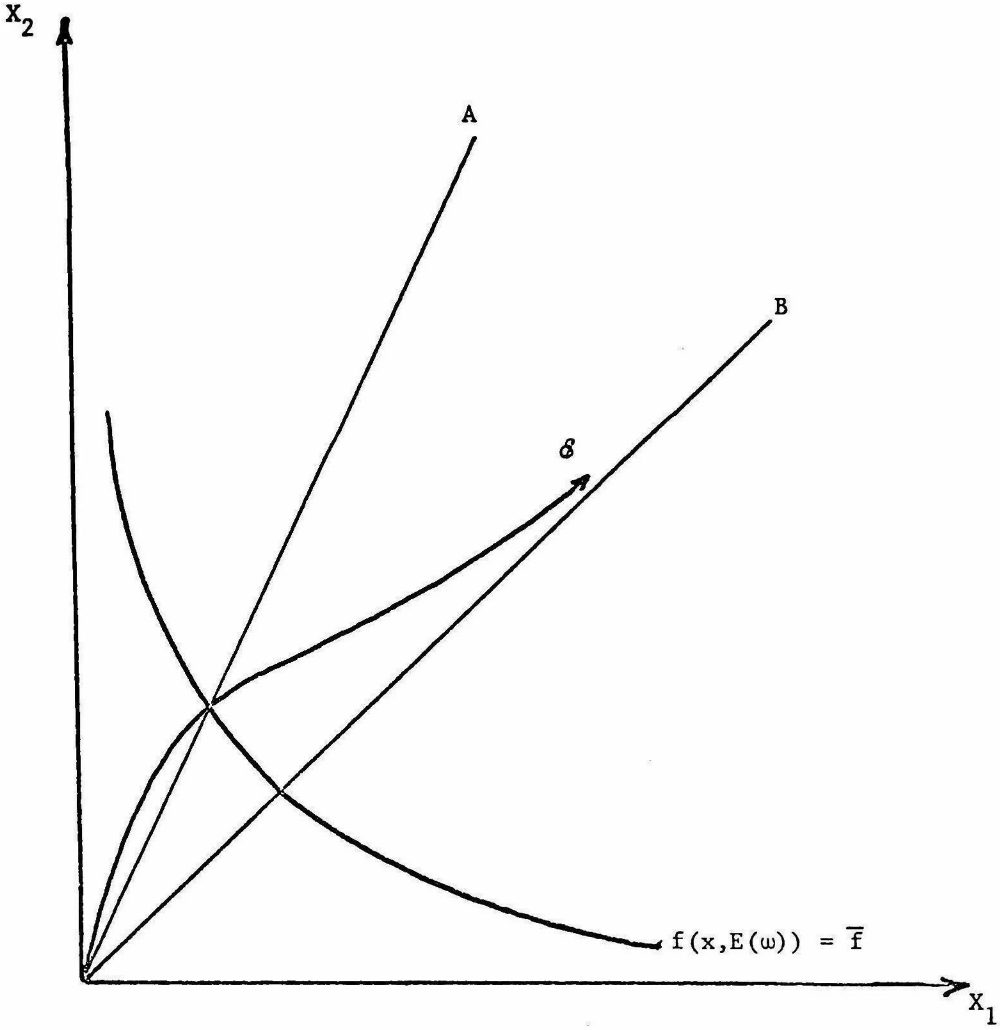

Figure 1

Expansion Path Relationships

output y and prices q and estimating some specified form for $C(y,q)$, the cost function for the firm. As has been shown elsewhere, $C(y,q)$ should have as general a form as possible (consistent with it being a cost function) so that we do not make unnecessary assumptions on the underlying technology. Thus, the result of estimating $C(y,q)$ via, say, a regression model would be $E(C(y,q))$ while the appropriate cost function is $C(E(y),q)$. The two cost functions provide the same model if and only if:*

$$C(y,q) = y\ell(q) + d(q)$$

where $\ell:R^n_{++} \to R_+$ and $d:R^n_{++} \to R_+$. Furthermore since all inputs are variable and $C(E(y),q)$ must be a cost function, $C(0,q) = 0$ and therefore $d(q) = 0$. Thus

$$C(E(y),q) = E(y)\ell(q)$$

Finally, since cost functions should be positively linearly homogeneous (PLH) in prices, $\ell(q)$ is a homogeneous function of degree one. Hence, using a well-known result of Shephard [9], $E(f(x,\omega))$ must be <u>homogeneous of degree one in the inputs (x)</u>. This is a somewhat startling result and is restated as the following theorem:

<u>Theorem 2</u>. In general, estimation of a cost function $C(y,q)$ of noisy out-output y and given input prices q is consistent only with a underlying production process $f(x,\omega)$ that is (PLH) in x.

$f(x,\omega)$ must be (PLH) since if $E(f(x,\omega))$ is required to be (PLH) then by Euler's theorem

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*This is true since we must allow for arbitrary distribution on y, which reflects the fact that we did not start with a specific technology description $\{f(x,\omega),G\}$.

$$E(f(x,\omega)) = \sum E(f_i(x,\omega))x_i$$

$$\Longleftrightarrow$$

$$\int (f(x,\omega) - \sum f_i(x,\omega)x_i)dG = 0$$

for arbitrary G. Thus f is to be positively linearly homogeneous also.

3.4. Implied Duality

Even though the preceding theorem seems to rule out the use of duality theory in analyzing technology via the cost function, in fact the reverse is true. The point is that unless we explicitly recognize and account for the effect of noise in the production process, we will generally produce erroneous analyses. The implied duality between cost and production is shown in Figure 2. On the left side of the figure we see the underlying production structure $\{f(x,\omega),G\}$ which gives rise to $E(f(x,\omega))$ which is the necessary production structure in the sense that it is the function that appears in the first order conditions and in (CMP). On the right side of the figure we have the underlying cost structure $\{C(y,q),H\}$ where $y \sim H$. If we were to estimate a function $C(E(y),q)$ where $E(y) = \int ydH$ then we would obtain the necessary cost

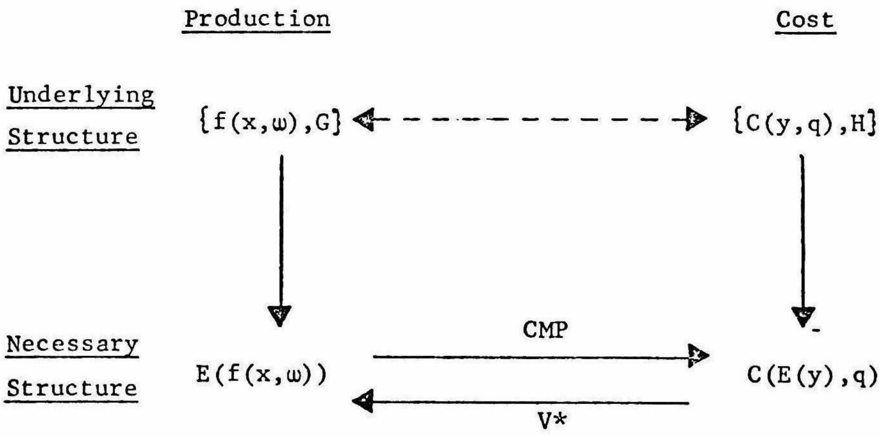

Figure 2
Cost/Production Duality

function $C(E(y),q)$. Notice, this is different then simply estimating $C(y,q)$, which we would generally do if attention were not paid to the inherent stochastic nature of y (i.e. H). The duality exists between $E(f(x,\omega))$ and $C(E(y),q)$ (via (CMP)) on the one hand, and between $C(E(y),q)$ and $E(f(x,\omega))$ (via $V^*(E(y))$) on the other hand. This duality between $E(f(x,\omega))$ and $C(E(y),q)$ implies a duality between $\{f(x,\omega),G\}$ and $\{C(y,q),H\}$. Again it should be stressed that the expectations in $E(f(x,\omega))$ and $C(E(y),q)$ are over G and H respectively.

In general $G \neq H$ (though they might be from the same class). If $f_\omega(x,\omega)$ is monotonic, and G and H have density functions $g(\omega)$ and $h(y)$ respectively, then we can use a change-of-variables relationship to find H given G or vice-versa. Since x^* is to be picked before ω is observed then

$$y = f(x^*,\omega),$$

and if f_ω is monotonic then there exists a function $t:R_+ \to D$ such that $\omega = t(y)$. Therefore

$$h(y) = \left|\frac{d}{dy} t(y)\right| g(t(y)) I_{R_+}(y)$$

where $I_{R_+}(y)$ is the indicator function for y on R_+. Similarly a reconstruction of $g(\omega)$ would be $\tilde{g}(\omega)$:

$$\tilde{g}(\omega) = \left|\frac{d}{d\omega} f(x^*,\omega)\right| h(f(x^*,\omega)) I_D(\omega)$$

with $I_D(\omega)$ is the indicator function for ω on D. Hence, we will call G and H dual distributions. In general, we would expect H to be bounded from above and below (no tails); perhaps allowing for H to be from the Beta distribution class would be a sufficiently general, yet constructive, assumption.

It would appear, then, that a constructive procedure would be to pose a cost function $C(E(y),q)$ and a distribution H and estimate the following system.

$$C(Z,q)$$

$$Z = E_H(y)$$

i.e. estimate parameters for $C(E(y),q)$ and H by computing $E(y) = \int ydH$ as the estimation is made. The resulting cost function will, via V*, provide the convex hull of $E(f(x,\omega))$.

4. Summary and Conclusions

In general, unless one has strong prior information to the contrary, we must assume production and cost structures as described in section 3.4 above. To do otherwise is to assume either stochastic separability (in the direct approach) or constant returns-to-scale (in the indirect approach). Both of these conditions should be tested rather than assumed.

This means that care must be taken to properly include stochastic elements in the production/cost model formulations (see, for example, [1], [6]). The extended duality described above clearly shows that explicit recognition of the appropriate distributions is necessary for us to overcome the limitations imposed by theorems one and two. In both the direct and indirect cases, parameters of the relevant distributions (G or H) must be estimated along with the production or cost functions.

It should be noted that the duality between $\{f(x,\omega),G\}$ and $\{C(y,q),H\}$ is weak in the following sense: many functions can give rise to the same expectation. Thus, posing $\{C(y,q),H\}$ and estimating $C(E(y),q)$ will provide $E(f(x,\omega))$. However, there may be many possible $f(x,\omega)$ consistent with $E(f(x,\omega))$. This is true even though we may have deduced a form for G from our estimate of H. This provides a number of related questions of interest for further research:

1) Is there a specifiable relationship amongst $C(y,q)$, $f(x,\omega)$, G and H that would allow us to deduce $f(x,\omega)$ from $C(E(y),q)$?
2) What is the nature of any direct duality relationship between $\{C(y,q),H\}$ and $\{f(x,\omega),G\}$? If one exists, is it of the usual variety (supporting hyperplanes) or is it of a more general nature (supporting hyper<u>surfaces</u>)?

3) What assumptions on $f(x,\omega)$ or $C(y,q)$ give rise to G and H being of the same type of distribution, i.e. when is there some form of self-duality for G and H?

These and many other questions need to be answered before a full, stochastic duality theory emerges. Such a theory is necessary for a complete description of cost/production relationships.

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