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Algebraic Invariants of Symbolic Systems

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Kitty Yang

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ABSTRACT

Algebraic Invariants of Symbolic Systems

Kitty Yang

We study the symmetry groups with respect to various equivalence relations defined on subshifts, and more generally, on Cantor systems. Two basic notions of equivalence for dynamical systems are conjugacy and flow equivalence. In this dissertation, we focus on the well-studied automorphism group, which is the group of self-conjugacies, and the mapping class group, the group of self-flow equivalences up to isotopy, of different classes of subshifts.

In Chapter 1, the introduction, we go into more detail about the historical context and motivation for the results that appear in this thesis. In Chapter 2, we give precise definitions and the background. We also include some illuminating examples of subshifts, automorphism groups, and mapping class groups.

Chapter 3 extends Ryan's Theorem, which computes the center of the automorphism group for an important class of subshifts of subshifts, shifts of finite type. We strengthen the result to show that any normal amenable subgroup of the automorphism group, which

includes the center, must be contained in the subgroup generated by the shift map. We also generalize to the class of sofic shifts, which include shifts of finite type.

In Chapter 4, we work in the setting of minimal subshifts of linear complexity. The condition of linear complexity leads to restrictions on dynamical properties of the subshift, and this corresponds to constraints on the automorphism group and mapping class group. We show that for the special class of substitution subshifts, the mapping class group is a finite extension of $\mathbb{Z}$. More generally, if a minimal subshift of linear complexity satisfies a technical condition of trivial infinitesimals, then the mapping class group is the finite extension of an abelian group.

Chapter 5 builds on the work in Chapter 4 in the more general context of Cantor systems. We can associate a C^*-algebra to any Cantor system, and we study Morita equivalences of these C^*-algebras. The group of self-Morita equivalences is called the Picard group. We show that there is a well-defined map from the mapping class group to the Picard group of a Cantor system. Furthermore, if the system is minimal, the mapping class group embeds in the Picard group.

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CHAPTER 1

Introduction

A *subshift* or *symbolic system* consists of a pair (X, σ), where X is a compact set of bi-infinite sequences on a finite alphabet, and σ is the left-shift. Symbolic systems arise as models of dynamical systems, and for example can be obtained by discretizing the underlying state space and coding orbits. Subshifts in general are a good source of examples which exhibit a wide range of dynamical behavior. A fundamental question in symbolic dynamics is how to determine when two systems are the same, and two natural notions to consider are conjugacy and flow equivalence. While there has been much interest in algebraic invariants of irreducible shifts of finite type (SFTs), which have positive entropy, there has been recent progress made on the other end of the spectrum for zero-entropy subshifts.

A conjugacy between dynamical systems is an isomorphism of the underlying spaces which respects the dynamics. Curtis, Hedlund, and Lyndon [**45**] proved in the 60s that any conjugacy between two subshifts is finitary: we only have to know how the map acts on a finite set of blocks, and that determines the map on all bi-infinite sequences. However, it remains remarkably difficult to find explicit conjugacies (or automorphisms when it is a self-conjugacy). There is essentially only one construction that allows us to produce automorphisms of shifts of finite type: the marker construction, originally also due to Hedlund [**45**] for the full shift, and later expanded by Boyle, Lind, and Rudolph [**18**] to SFTs in 1988.

As a corollary of the Curtis-Hedlund-Lyndon Theorem, the automorphism group of any subshift is countable, though it can be still quite complicated. The marker method explicitly constructs an abundance of finite order marker automorphisms, and many teams [**17, 18, 45, 50**] exploited this construction to show that the automorphism group of SFTs is very complicated; for example, it contains isomorphic copies of any finite group and a copy of the free group on two generators.

While there are many results pertaining to the subgroup structure of automorphism groups of SFTs, far less is known about the group structure. One of the few results in this direction is Ryan's Theorem [**76, 77**], which states that for an irreducible SFT, any element in the center of the automorphism group must be a power of the shift, and thus the center is precisely the subgroup generated by the shift map. One of the main theorems of this dissertation generalizes Ryan's Theorem in two ways: we weaken the hypothesis to the larger class of sofic shifts (which includes SFTs); and we strengthen the conclusion to compute the amenable radical (a subgroup which contains the center).

While the proof of Ryan's Theorem does not directly appeal to positive entropy, the construction relies on the shift containing marker words of arbitrary length. For subshifts, entropy is the exponential growth rate of the associated complexity function $p : \mathbb{N} \to \mathbb{N}$, where $p(n)$ counts the number of distinct words of length n that appear in the shift. Transitive sofic shifts also contain the necessary markers for the proof, and have positive entropy. It may be possible to adapt the proof to special subshifts that are not sofic, but the number of markers required may force the complexity function to grow exponentially.

More recently, there has been a flurry of activity on the other end of the spectrum, concerning the automorphism group of zero-entropy subshifts. Any subshift whose complexity function grows subexponentially has zero entropy, and we can further refine this class by looking at the growth rate of the complexity function. Cyr and Kra [**24**], Donoso, Durand, Maass, and Petite [**29**], and Coven, Quas, and Yassawi [**22**] (in the case of substitutions) independently showed that for transitive subshifts whose complexity function grows linearly, the automorphism group is virtually $\mathbb{Z}$; that is, the automorphism group has $\mathbb{Z}$ as a finite index subgroup. The results use different methods, and achieve different results for different classes of linear complexity subshifts. There are now results in various directions about the automorphism group of low complexity subshifts: computing the automorphism groups of subshifts of varying growth rates [**26, 25, 28**]; showing obstructions in embedding subgroups in automorphism groups [**23**]; and constructing a probability measure for any zero-entropy subshift which is invariant under its automorphism group [**38**].

The history of results related to flow equivalence between SFTs mirrors that for results about conjugacy. Using the matrix presentation of SFTs, Bowen and Franks [**11**] introduced the Bowen-Franks group, which is a homology group of the dimension group of the SFT. It was the first invariant of flow equivalence, rather than of conjugacy, defined in the context of subshifts. Franks [**36**] then showed that the Bowen-Franks group, along with another matrix invariant introduced by Parry and Sullivan [**68**] completely classifies flow equivalence for irreducible SFTs.

In the same paper, Parry and Sullivan [**68**] proved an analogous result to Curtis et al. to show that any flow equivalence between subshifts is isotopic to a flow code. A flow code is induced by a conjugacy between return systems, and one can think of flow codes as a generalization of block codes. As a corollary, the mapping class group, which is the group of self-flow equivalences up to isotopy, is countable.

The mapping class group of a subshift was first defined by Boyle and Chuysurichay [**15**] and studied for SFTs. The name is borrowed from a similar definition for surfaces, though since the base space is a Cantor system, the two mapping class groups behave quite differently. Since flow equivalence is weaker than conjugacy, there is a map from the automorphism group to the mapping class group, which we denote Ψ. By the construction of the suspension space, the shift map induces the time-one flow on the suspension. It follows that the subgroup generated by the shift is always contained in the kernel of Ψ. Under mild conditions, the kernel is precisely the subgroup generated by the shift. A natural question is to ask how much bigger the mapping class group is, relative to the automorphism group. Boyle and Chuysurichay obtain various results, including an analogue of Ryan's Theorem to show that the center of the mapping class group is trivial. However, the mapping class group of an irreducible SFT contains many more elements, and they construct a map which does is not induced by any automorphism of a flow equivalent SFT.

This thesis continues the research in these two directions: it generalizes Ryan's Theorem to sofic shifts and strengthens the result, and studies the mapping class group for linear complexity subshifts.

1.1. The automorphism group

Let $\mathcal{A}$ be a finite collection of symbols. For $x \in \mathcal{A}^{\mathbb{Z}}$, which we write as $x = (x_i)_{i\in\mathbb{Z}}$, define the left shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ to be $(\sigma x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. The dynamical system $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is the *full $\mathcal{A}$-shift*, and given a closed, σ-invariant set X, (X, σ) is a *subshift* (see Section 2.1.1 for precise definitions of the topology). To avoid trivial cases, we always assume that X is infinite. A *word* w is a finite concatenation of symbols of the form $w \in \mathcal{A}^n$ for some n, and we say the word w of length n *appears in* X if there is some $x \in X$ and $i \in \mathbb{Z}$ such that $w = x_i x_{i+1} \cdots x_{i+n-1}$. The *language* of X is the set of all finite words that appear in X, denoted $\mathcal{L}(X)$.

Let $\mathcal{A}^*$ be the monoid generated by $\mathcal{A}$. Note that $\mathcal{A}^*$ consists of all finite words over the alphabet $\mathcal{A}$. A subshift X is an SFT if there exists a finite set $\mathcal{F} \subset \mathcal{A}^*$ where a sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is an element of X if and only if no element of $\mathcal{F}$ appears as a subword of x. A subshift X is *irreducible* or *transitive* if for any pair of words $u, w \in \mathcal{L}(X)$, there is another word v such that $uvw \in \mathcal{L}(X)$.

Given a subshift (X, σ), a homeomorphism $g : X \to X$ is an *automorphism* if it commutes with σ. An automorphism is a topological conjugacy between X and itself. The *automorphism group*, denoted Aut(X), is the group of automorphisms under composition. By definition, any power of the shift is always in the center of $\mathrm{Aut}(X)$. Ryan proved that for the full shift, and later for transitive SFTs, the converse is true. It follows that the center of the automorphism group is precisely the subgroup generated by σ.

The group theoretic implications of Ryan's Theorem allow us to distinguish certain automorphism groups. Boyle, Lind, and Rudolph in [**18**] observed that the shift map of the full 2-shift does not have a root in the automorphism group, while the shift map for the full 4-shift does. Since any isomorphism of groups must preserve the center, the automorphism groups of the full 2-shift and full 4-shift cannot be isomorphic.

Recently, Frisch, Schlank, and Tamuz [**37**] generalized Ryan's Theorem to show that any normal amenable subgroup of the automorphism group of the full shift must be contained in the subgroup generated by the shift. A countable group G is *amenable* if it admits a *Følner sequence*, which is a sequence of nested finite subsets $F_i \subset G$ whose union is G, and for all $g \in G$ we have

$$\lim_{i \to \infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0.$$

Examples of amenable groups include finite groups and (countable) abelian groups, while the free group is not amenable. The center of a group is always a normal amenable subgroup.

However, Ryan's Theorem also holds for SFTs, so it is natural to ask whether the result of Frisch, Tamuz, and Schlank from [**37**] is true beyond the full shift. We extend the result to the even larger class of sofic shifts, which in particular includes SFTs. A subshift X is *sofic* if there exists factor map from an SFT to X (see Section 2.1 for precise definitions). In Chapter 3, we prove the following theorem [**83**]:

Theorem (Theorem 3.1)**.** *Let* (X, σ) *be a transitive sofic shift. Any normal amenable subgroup of* $\mathrm{Aut}(X)$ *is contained in* $\langle \sigma \rangle$*.*

The *amenable radical*, which is the subgroup generated by all normal amenable subgroups, is also a characteristic subgroup. Theorem 3.1 extends both [**76, 77**] and [**37**] to show that the normal amenable radical of Aut(X) for irreducible sofic shifts is $\langle\sigma\rangle$. This allows us to compute both the amenable radical and center of $\mathrm{Aut}(X)$.

The key method of the proof of Theorem 3.1 involves constructing a compact metric space Ω together with an action of Aut(X) which is minimal and strongly proximal. This makes Ω, as an Aut(X)-space, a topological boundary, and we show that the kernel of this action is exactly the center $\langle\sigma\rangle$. By a characterization of topological boundaries due to Furman [**40**], we can conclude that the amenable radical is contained in the kernel, $\langle\sigma\rangle$, of this action. The topological boundary is built from *left-periodic points*, points which are left periodic, but not fully periodic.

To show that the action is a topological boundary, we explicitly construct marker automorphisms, which are finite order automorphisms. More precisely, a *marker automorphism* acts on X by permuting a finite set of words satisfying a non-overlapping property. This ensures that the resulting map is a well-defined homeomorphism of X. Hedlund [**45**] constructed these markers for the full shift, and Boyle, Lind, and Ruldoph [**18**] showed that such markers also exist in SFTs.

The construction also carries over to sofic shifts. However, the main obstacle for *strictly sofic* shifts, which are sofic but not SFTs, is the existence of periodic points of arbitrarily high period which are fixed by all marker automorphisms. This is not the case for SFTs, as Boyle and Krieger [**17**] construct marker automorphisms which permute periodic points of sufficiently high period. It remains open whether there are non-marker automorphisms which permute these fixed points in sofic shifts. Instead, we restrict the action only to those periodic points which are in the orbit of synchronizing points under Aut(X). A word w is *synchronizing* if whenever uw and wv are allowable in X, then so is

uwv; a periodic point is *synchronizing* if all sufficiently long subwords are synchronizing. Note that this gives an alternate characterization of SFTs as shifts where all sufficiently long words are synchronizing.

We also give a short discussion about higher dimensional shifts and the difficulties that arise when trying to generalize Theorem 3.1 to dimension $d \geq 2$.

1.2. The mapping class group

Given a compact space X and a homeomorphism $T : X \to X$, the pair (X, T) is a topological system, and let $\Sigma_T X$ denote its suspension under a constant roof function. Two systems (X, T) and (Y, S) are *flow equivalent* if their suspensions are homeomorphic, under an orientation-preserving map which maps orbits to orbits. Two flow equivalences are *isotopic* if they are connected by a path, when viewed as elements of $\mathrm{Homeo(\Sigma_T X, \Sigma_S Y)}$. Algebraically, we can study the group of isotopy classes of self-flow equivalences of (X, T), which we call the *mapping class group* and denote it by $\mathcal{M}(T)$.

In joint work with Schmieding [**78**], we obtain results for low complexity minimal subshifts. A subshift (X, σ) is *minimal* if the σ-orbit of every point is dense. We work in two settings: substitution subshifts, and more generally subshifts of linear complexity (with a technical condition). A *substitution* is a map $\xi : \mathcal{A} \to \mathcal{A}^*$, where $\mathcal{A}^*$ is the monoid generated by $\mathcal{A}$. We can iterate ξ by applying ξ to each symbol and concatenating the image words. A substitution is *primitive* if for any two symbols $a, b \in \mathcal{A}$, a appears in some iterate $\xi^n(b)$. Since $|\mathcal{A}| \geq 2$, this condition implies that $\lim_{n\to\infty} |\xi^n(a)| = \infty$ for all $a \in \mathcal{A}$. By iterating ξ, we obtain a bi-infinite sequence x_0 which is fixed by ξ. The subshift associated to ξ is the orbit closure of x_0 under the shift map.

The rigid structure of substitutions allows us to compute the mapping class group explicitly in Chapter 4:

Theorem (Theorem 4.17 in the text; joint with Schmieding [**78**])**.** *Suppose* (X, σ) *is a minimal subshift associated to a primitive substitution* ξ*. Then* $\mathcal{M}(\sigma)$ *fits into an exact sequence*

$$1 \to \mathcal{F} \to \mathcal{M}(\sigma) \to \mathbb{Z} \to 1$$

where $\mathcal{F}$ *is a finite group. If* ξ *is of type CR, then* $\mathcal{F}$ *is isomorphic to* $\mathrm{Aut}(\sigma)/\langle\sigma\rangle$*.*

This implies that the mapping class group of a substitution system is the semi-direct product of a finite group and $\mathbb{Z}$. Substitutions of type CR have suspensions which are topologically conjugate to the tiling space associated to the substitution (see Section 4.3 for details).

The substitution map ξ induces an element of $\mathcal{M}(\sigma)$, which we denote $\widetilde{\xi}$. The map $\widetilde{\xi}$ stretches along the flow direction, while shrinking the cross sections (the Cantor direction). Since a flow equivalence induced by an automorphism must preserve cross sections, this implies $\widetilde{\xi}$ cannot be isotopic to such a map. We show that $\widetilde{\xi}$, along with its powers, are the only elements in the mapping class group which can stretch (or shrink) along the flow. To prove this, we work on the suspension of X defined by a roof function given by the Perron-Frobenius eigenvector associated to ξ, denoted by Ω_ξ. The suspension Ω_ξ is homeomorphic to the constant suspension, but the two dynamical systems may not be conjugate (under the $\mathbb{R}$ action). When they are conjugate, we obtain the stronger result that $\mathcal{F} \cong \mathrm{Aut(X)}/\langle\sigma\rangle$. The inflation substitution acts as a renormalization map on Ω_ξ, which allows us to compute elements of the form $\widetilde{\xi}^{-1} g \widetilde{\xi}$, where $g \in \mathcal{M}(\sigma)$. We show that g

must either be a power of $\widetilde{\xi}$, or must preserve cross sections. The inflation map $\widetilde{\xi}$ generates a copy of $\mathbb{Z}$, and otherwise we are left with a finite extension of the automorphism group.

The second setting we work in consists of minimal subshifts of linear complexity. The *complexity function* $P_X(n) : \mathbb{N} \to \mathbb{N}$ counts the number of distinct words of length n in the language $\mathcal{L}(X)$. Because we are working with infinite subshifts, complexity must grow at least linearly with n. The subshifts of lowest complexity are *Sturmians*, defined to be those subshifts that are not ultimately periodic with complexity function satisfying $n+1$ for all $n \in \mathbb{N}$.

We can show that with some technical conditions, the mapping class group of a minimal shift of linear complexity is constrained. We say that a continuous map $\gamma : X \to \mathbb{Z}$ is an *infinitesimal* if it integrates to zero with respect to all ergodic measures. We say γ is a *coboundary* if $\gamma = \eta - \eta \circ \sigma$ for some continuous $\eta : X \to \mathbb{Z}$.

A group is *virtually abelian* if it contains an abelian group which has finite index.

Theorem (Theorem 4.25 in the text; joint with Schmieding [**78**])**.** *Let* (X, σ) *be a minimal subshift for which* $\operatorname{Inf} \mathcal{G}_\sigma = 0$ *and*

$$\liminf_n \frac{P_X(n)}{n} < \infty.$$

Then $\mathcal{M}(\sigma)$ *is virtually abelian.*

The main tool in the study of minimal low complexity subshifts is the group of coinvariants $\mathcal{G}_\sigma$, which is an algebraic invariant of flow equivalence for minimal Cantor systems (see Section 2.4 for definitions). The assumption of minimality is key in the construction of $\mathcal{G}_\sigma$, an ordered group we can build from continuous maps from X to $\mathbb{Z}$, modulo the subgroup generated by coboundaries. The group of coinvariants plays a vital role because an element of the mapping class group induces an automorphism of $\mathcal{G}_\sigma$ which preserves the order structure [**16, 41**]. For a probability measure μ, we get a trace map $\tau_\mu : \mathcal{G}_\sigma \to \mathbb{R}$, given by integration. Note that τ maps positive elements of $\mathcal{G}_\sigma$ to $\mathbb{R}_{>0}$. In the case of linear complexity, (X, σ) supports only finitely many non-atomic ergodic measures [**10**], so the trace space is finite dimensional. For any element $f \in \mathcal{M}(\sigma)$, some f^k must leave the state space invariant.

We require the technical condition that all infinitesimals are coboundaries. Note that in the case of linear complexity, (X, σ) supports only finitely many non-atomic ergodic measures [**10**]. This condition is crucial because it gives us a finite criterion to determine when an element of $\mathcal{G}_\sigma$ is trivial, which is difficult in the general setting.

As an application of Theorems 4.17 and 4.25, we obtain a classification of the mapping class group of Sturmians, which have no infinitesimals [**71**]. There are two cases: if (X, σ) is conjugate to a substitution, then $\mathcal{M}(\sigma) \cong \mathbb{Z}$; otherwise, $\mathcal{M}(\sigma)$ is trivial.

1.3. The Picard group

Let X be a Cantor set and $T : X \to X$ be a homeomorphism. We say that (X, T) is a Cantor system. We can associate to (X, T) a non-commutative C^*-algebra which reflects the dynamics. The crossed product algebra $C(X) \rtimes_T \mathbb{Z}$ is generated by continuous functions on X and $u \in \mathbb{Z}$ a generator. Conjugation by u mimics the action of T:

$$ufu^* := f \circ T \text{ for all } f \in C(X).$$

The crossed product algebra is a very complicated algebraic object, but passing to invariants can shed light on the topological system. For example, $K_0(C(X) \rtimes_T \mathbb{Z})$ is isomorphic to $\mathcal{G}_T$, the group of coinvariants we saw in the previous section.

Given two C^*-algebras A and B, we say they are *Morita-equivalent* if there exists a bimodule Z, which is a Hilbert space over A and B, where the inner products of A and B satisfy certain identities (see Chapter 5 for precise definitions). The bimodule Z is called an $A - B$ imprimitivity bimodule. The Picard group of A, denoted Pic$A$, is the group of $A - A$ imprimitivity bimodules, under $\otimes_A$. We can interpret the Picard group as the symmetry group with respect to Morita equivalence.

Let $A = C(X) \rtimes_T \mathbb{Z}$. In Chapter 5, given a flow code, we produce an $A - A$ imprimitivity bimodule. The main theorem shows the relationship between the mapping class group and the Picard group:

Theorem (Theorem 5.1 in the text; joint with Schmieding [**78**])**.** *Suppose* (X,T) *is a Cantor system. Then there is a homomorphism*

$$\Theta : \mathcal{M}(T) \to \mathrm{Pic}A.$$

If (X,T) *is minimal, then* Θ *is injective.*

It is quite difficult to compute the Picard group of Cantor systems, but we include an example of irrational rotations due to Kodaka [**52**]. A Sturmian is the symbolic coding of an irrational rotation, and it is an almost-everywhere 1-1 extension. Similar to Sturmians, there is a dichotomy when the irrational rotation is quadratic and thus a factor of a substitution, and when it is not.

CHAPTER 2

Background

First, in Section 2.1, we give background information on symbolic dynamics, including many examples of subshifts and automorphism groups. Next in Section 2.2, we give an analogous treatment to the suspensions and mapping class groups. In the final two sections, we give the technical background necessary for Chapters 3 and 4. This chapter is a survey of results in the literature and does not comprise original research.

2.1. Symbolic dynamics

A *topological system* is a pair (X, T), where X is a nonempty compact space and $T : X \to X$ is a continuous map.

Given two topological systems (X, T) and (Y, S), let $\varphi : X \to Y$ be a continuous map which respects the dynamics: $\varphi \circ T = S \circ \varphi$. We say that φ is a *factor map* if it is surjective. In this case, X is an *extension* of Y, and Y is a *factor* of X. If φ is a homeomorphism, then we call φ a *conjugacy*, and we say that X and Y are *conjugate*. It is easy to check that conjugacy is an equivalence relation.

The topological systems in this dissertation are *Cantor systems*, where the base space X is a Cantor set, and T is a homeomorphism. Of particular interest are symbolic systems, or subshifts.

Let $\mathcal{A}$ be a finite alphabet endowed with the discrete topology and equip $\mathcal{A}^{\mathbb{Z}}$ with the product topology. For $x = (x_i \colon i \in \mathbb{Z}) \in \mathcal{A}^{\mathbb{Z}}$, let $x_n \in \mathcal{A}$ denote the value of x at $n \in \mathbb{Z}$.

Define the shift map $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by setting $(\sigma x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the *full $\mathcal{A}$-shift*, or simply the *full shift*.

The full shift is a compact metric space; let the metric d be defined as follows: for any $x \neq y \in \mathcal{A}^{\mathbb{Z}}$, let i_0 denote the index closest to 0 where x and y differ; that is, let $i_0 = \min(|i| : x_i \neq y_i)$. We set

$$d(x, y) := 2^{-i_0}.$$

Under this metric, two points are close if they agree on large interval centered around the origin.

If $X \subset \mathcal{A}^{\mathbb{Z}}$ is a closed and shift-invariant subset, we call (X, σ_X) a *subshift*, where σ_X is the restriction of σ to X. When the context is clear, we simply write (X, σ). To avoid trivial cases, we assume X is infinite. Throughout this section, let (X, σ) denote a subshift.

2.1.1. Basic properties

Given $x \in X$, let

$$\mathcal{O}(x) := \{\sigma^i x : i \in \mathbb{Z}\}$$

be the *orbit of x under the shift map*, and $\overline{\mathcal{O}(x)}$ denote its closure in X.

Given an interval $[i, i+n-1] \subset \mathbb{Z}$, let $x_{[i,i+n-1]}$ be the word w in $\mathcal{A}^n$ given by $w_j = x_{i+j}$ for $j = 0, 1, \ldots, n-1$. A word w in $\mathcal{A}^n$ is *allowable in X* if there exists $x \in X$ and $i \in \mathbb{Z}$ such that $w = x_{[i,i+n-1]}$; we say that w *occurs in x at i*. For any word $w \in \mathcal{A}^n$, let $|w| = n$ be the length of the word. We denote the collection of allowable words of length

n in X by $\mathcal{L}_n(X)$, and the *language of* X, $\mathcal{L}(X) = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(X)$, is the set of all finite words that occur in X. Given two words u and w, uw is the word in $\mathcal{A}^{|u|+|w|}$ obtained by concatenating u and w; when we concatenate a word with itself, we simplify notation by writing w^2.

The language of a subshift encodes a wealth of information about the system. We can associate a subset of X with a word $w \in \mathcal{L}(X)$ as follows: define the *cylinder set* $[w] \subset X$ to be

$$[w] := \{x \in X : x_n = w_n \text{ for } 0 \leq n < |w|\}.$$

Such cylinder sets are clopen and, together with their translates, form a basis for the subspace topology on X. Thus, we can describe topological properties of a subshift in terms of its language. Just the growth rate of the language captures some dynamical properties of the subshift. Let $P_X(n) : \mathbb{N} \to \mathbb{N}$ be the *complexity function* given by

$$P_X(n) := |\mathcal{L}_n(X)|,$$

where $P_X(n)$ counts the number of allowable words of length n in X.

We note that the language is highly dependent on the presentation of the subshift, so the complexity function itself is not invariant under conjugacy; however, the growth rate is. If two subshifts X and Y are conjugate, then there exists a constant C such that

$$P_X(n - C) \leq P_Y(n) \leq P_X(n + C)$$

for all $n \in \mathbb{N}$ (see [**33**]).

By passing between words and cylinder sets, we see that the exponential growth rate of $P_X(n)$ is equal to the topological entropy of the system:

$$h_{top}(X) = \lim_{n\to\infty} \frac{1}{n} \log P_X(n).$$

Thus, if $P_X(n)$ grows subexponentially, then X has zero entropy. It is useful to further distinguish zero entropy subshifts by the growth rate of the complexity function. A subshift has *linear complexity* when the complexity function satisfies

$$\lim_{n\to\infty} \frac{P_X(n)}{n} < \infty.$$

Sometimes, we can replace the limit with liminf, and the results still hold. This is non-trivial; Donoso et al. [**29**, Ex 4.1] produce an example of a subshift whose complexity function $P(n)$ satisfies $\liminf_n P(n)/n$ is finite but $\limsup_n P(n)/n$ is infinite.

A subshift (X, σ) is *transitive* if for any pair of words u and $w \in \mathcal{L}(X)$, there is some word v such that $uvw \in \mathcal{L}(X)$; (X, σ) is *mixing* if for any $u, w \in \mathcal{L}(X)$, there exists an N such that for any $n \geq N$, there is a word $v \in \mathcal{L}_n(X)$ such that uvw is again allowable. Note that mixing implies transitivity.

A word $w \in \mathcal{L}(X)$ is *synchronizing* if whenever uw and $wv \in \mathcal{L}(X)$, then uwv is again allowable in X. It follows that if w is synchronizing, then any word that contains w must also be synchronizing.

A point $x \in X$ is *periodic* if there exists $p \in \mathbb{N}$ such that $x_i = x_{i-p}$ for all $n \in \mathbb{Z}$. We say $x \in X$ is *left-periodic up to* $N \in \mathbb{Z}$ if there exists $p \in \mathbb{N}$ such that $x_i = x_{i-p}$ for all $i < N$ and $x_N \neq x_{N-p}$. If the index N is not important, then we simply say x is a *left-periodic point*. Note that a left-periodic point is not periodic, and that the index N is independent of the choice of p. When the periodic index p is minimal, we say that x is (left-)p-periodic. Denote the set of points in X of period p by $\mathrm{Per_p}$. If X contains a left-periodic point, then it must also contain a periodic point, but the converse is not true.

We say a dynamical system (X, T), where X is a compact metric space with metric d, is *expansive* if there exists a constant ℓ such that for $x, y \in X$, if $d(T^i x, T^i y) < \ell$ for all $i \in \mathbb{Z}$, then $x = y$. Here, ℓ is called the *expansivity constant*.

Any subshift (X, σ) is expansive. Recall that the metric d is defined as follows: is, let $i_0 = \min(|i| : x_i \neq y_i)$.

$$d(x, y) := 2^{-i_0}$$

where i_0 is index closest to 0 where x and y differ. If $x_0 \neq y_0$, then $d(x, y) = 1$. Thus, if $d(\sigma^n x, \sigma^n y) < 1$ for all $n \in \mathbb{Z}$, then $x_n = y_n$ for all n.

We say that two distinct points x, y are *forward asymptotic*, or simply asymptotic, if $\lim_{i\to\infty} d(T^i x, T^i y) = 0$. Any expansive system contains asymptotic points. We refer to asymptotic points as *asymptotic pairs*, and the set of all distinct points asymptotic to x as the *asymptotic class* containing x.

Before giving examples of subshifts, it will be useful to characterize maps between shift spaces.

Theorem 2.1 (Curtis-Hedlund-Lyndon [**45**]). *Let (X, σ_X) and (Y, σ_Y) be subshifts. Suppose X is defined over the alphabet $\mathcal{A}$ and Y is defined over $\mathcal{B}$. Let $\varphi : X \to Y$*

be a continuous map satisfying $\varphi \circ \sigma_X = \sigma_Y \circ \varphi$. *Then there exists* $R \in \mathbb{N}$ *and a map* $\hat{\varphi} \colon \mathcal{L}_{2R+1}(X) \to \mathcal{B}$ *such that* $(\varphi x)_i = \hat{\varphi}(x_{i-R,\cdots,i+R})$.

We say that $\hat{\varphi}$ is a *sliding block code*, and R is a *range* for g.

2.1.2. Examples of subshifts

Example 2.2 (Shifts of finite type)**.** A subshift is of *finite type*, or an SFT, if it can be described by a finite set of forbidden words; that is, X is an SFT if there exists a finite set of words $\mathcal{F}$ such that $x \in X$ if and only if no element of $\mathcal{F}$ appears as a subword of x. An SFT is *j-step* if $\mathcal{F}$ consists of words of length $j+1$.

A natural way of constructing conjugate subshifts is by recoding to a higher block representation. Given $X \subset \mathcal{A}^{\mathbb{Z}}$, the *higher block representation*, or *k-block representation* is the system (Y, σ_Y) defined over the alphabet $\mathcal{B} = \mathcal{A}^k$. The symbols are words of length k. The set Y is

$$Y := \{(y)_i \in \mathbb{Z} \in \mathcal{B}^{\mathbb{Z}} : \text{there exists } x \in X \text{ with } y_i = x_{[i,i+k-1]} \text{ for all } i \in \mathbb{Z}\}.$$

For example, consider the full 2-shift $\{0,1\}^{\mathbb{Z}}$. Its 2-block representation is a subshift defined over the alphabet $\{(00),(01),(10),(11)\}$. The full shift is a SFT (with no forbidden words), so the higher block representation must also be an SFT. By the construction of the higher block representation, a symbol b can only be followed by c if the second letter of b is the first letter of c. This is also the only restriction, so the forbidden words are $\{(00)(10),(00)(11),(01)(00),(01,01),(10)(10),(10)(11),(11)(00),(11)(01)\}$. Up to recoding to a higher block representation, any SFT is conjugate to a 1-step SFT.

We can also characterize SFTs using synchronizing words: a subshift X is an SFT if and only if all words of sufficient length are synchronizing.

When (X, σ) is a transitive SFT, it is a classical result that X can be decomposed into disjoint mixing components which are cyclically permuted. More precisely, there exists a period k and a partition $\{E_i\}_{1\leq i\leq k}$ of X such that each (E_i, σ^k) is mixing, and $\sigma E_i = E_{i+1 \mod k}$ (see [**1**, §3],[**12**, p. 543]). Here, k refers to the greatest common divisor of all p with $\mathrm{Per_p} \neq \emptyset$. This extra structure says that in the case of a transitive SFT, given w and u, we can extend u on the left to $\widetilde{u}$ and for sufficiently large n, there is a word v of length np with $wv\widetilde{u} \in X$. Transitivity also implies that periodic points are dense for SFTs.

Let $X \subset \{0,1\}^{\mathbb{Z}}$ be the 1-step SFT defined by the forbidden word 11; (X, σ) is the golden mean shift, so called because its entropy is precisely the golden mean. An bi-infinite sequence $x \in \{0,1\}^{\mathbb{Z}}$ belongs to X if and only if $x_{[i,i+1]} \neq 11$ for all i.

Example 2.3 (Sofic shifts)**.** The class of SFTs is not closed under factors, and we can consider the larger natural class which is. We say that a subshift is *sofic* if it is a factor of an SFT. There are many equivalent definitions of sofic shifts, and we refer the reader to [**57**, Theorem 3.2.1] for more details. The class of sofic shifts is the smallest class of subshifts that is closed under taking factors and contain SFTs. Note that transitivity and mixing are each preserved under factors, and a transitive (cf. mixing) sofic shift is a factor of a transitive (cf. mixing) SFT. It follows that in a transitive sofic shift, periodic points are also dense. By recoding to a higher block presentation, we can assume that this factor map is a 0-block map. This presentation is convenient as it allows us to lift words in the sofic shift to words of the same length in the SFT. In transitive sofic shifts, as with transitive SFTs, between any two allowable words, we can insert arbitrary spacer words whose lengths form an arithmetic progression.

Let $Y \subset \{0,1\}^{\mathbb{Z}}$ be the set of bi-infinite sequences which only contain an even number of consecutive 1s. The subshift (Y, σ), called the even shift, is a factor of the golden mean shift from the previous example (Example 2.2). To define the factor map from X to Y, it suffices to define a sliding block code from the language of X to Y. In this case, we define the map $\hat{\pi}$ on $\mathcal{L}_2(X) = \{00, 01, 10\}$:

$$00 \mapsto 0 \qquad 01 \mapsto 1 \qquad 10 \mapsto 1.$$

This sliding block code induces a map $\pi : X \to Y$ as follows: for any $x \in X$, we set

$$(\pi(x))_i = \hat{\pi}(x_{[i,i+1]}).$$

For two consecutive sliding blocks w and u, $w_1 = u_0$, so 01 must be followed by 10. This implies that the image of any point in X cannot contain an odd number of consecutive 1s. A similar argument shows that π is surjective.

The word 1^{2r+1} for any $r \in \mathbb{N}$ is allowable (as a subword of $01^{2r+2}0$) and $01^{2r+1}, 1^{2r+1}0$ are also allowable. However, $01^{2r+1}0$ is not in the language of Y, so 1^{2^r+1} is not a synchronizing word. We have generated arbitrarily long non-synchronizing words; thus, Y cannot be an SFT. We call such shifts *strictly sofic*.

Non-trivial (that is, infinite and transitive) SFTs and sofic shifts have positive entropy and are far from minimal. Here are some examples of minimal zero-entropy subshifts.

Example 2.4 (Sturmians)**.** As in the definition of complexity function for a subshift, we can define the complexity function of a sequence x by letting $P_x(n)$ count the number distinct words of length n that appear in x The Morse-Hedlund Theorem gives a lower bound on the complexity function of a subshift:

Theorem 2.5 (Morse-Hedlund [**62**])**.** *Let $x \in \mathcal{A}^{\mathbb{Z}}$. Then x is periodic if and only if there exists $n \in \mathcal{N}$ such that $P_x(n) \leq n$.*

It follows that the lowest possible complexity function for an infinite shift is $P_X(n) = n + 1$. Recall that a sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is left-periodic if there if there exists an integer p and index N such that for all $i < N$, $x_i = x_{n-p}$. We can similarly define a right-periodic point. If a subshift does not contain any left- or right-periodic points and has minimal complexity, we call it a *Sturmian subshift*. Since $P_X(1) = 2$, this implies that Sturmians are defined over a two-letter alphabet.

Consider the infinite Fibonacci word, which is obtained by recursively concatenating words: Let $w_0 = 0, w_1 = 01$, and define $w_n = w_{n-1}w_{n-2}$. Let $x = \lim_{n\to\infty} w_n$. The point x is not eventually periodic, and there are precisely $n + 1$ subwords of length n.

Remark 2.6. Note that x is technically a one-sided sequence, and thus an element of $\mathcal{A}^{\mathbb{N}}$. We can extend this to the left to a bi-infinite sequence x' which still satisfies the characteristics above. The orbit closure of x' under σ is a Sturmian subshift. This particular subshift is called the *Fibonacci subshift* (it is also an example of a substitution shift, see Example 2.8).

As the least complicated shifts, quite a bit is known about Sturmians. They are minimal, uniquely ergodic, and almost-everywhere 1-1 extensions of irrational circle rotations. For more background on Sturmian shifts, see [**35, 73**].

Example 2.7 (Codings of interval exchange transformations)**.** We generalize Sturmian subshifts by considering a generalization of irrational rotations, and taking a symbolic coding.

Fix $d \geq 2$. Let $0 < \lambda_j < 1$, $j = 1, \cdots, d$ be a collection of lengths which sum to 1, and let $\pi \in S_d$ be a permutation on d elements. The collection $\{\lambda_j\}$ gives rise to a partition of the interval $[0, 1)$ by considering the subintervals $\Delta_j = \big[\sum_{k=0}^{j-1} \lambda_k, \sum_{k=0}^{j} \lambda_k\big)$, where we set $\lambda_0 = 0$. Define a transformation T from the interval to itself as follows: apply the permutation π to the collection of lengths, and again partition $[0, 1)$ by $\lambda_{\pi(j)}$. The transformation T is a piecewise linear function which translates the interval Δ_j to $\Delta_{\pi(j)}$. We call such a map an *interval exchange transformation*, or an *IET*. If the number of intervals is minimal (by choosing appropriate permutations), then T is a d-IET. This map is not continuous at the iterates endpoints of the subintervals, but this only forms a countable collection of points, and we can discard them without issue. We can think of rotations as IETs with $d = 2$.

Under mild conditions, IETs are minimal systems, and Veech [**79**] and Masur [**59**] showed that almost every IET is uniquely ergodic

To obtain a subshift from T, we take the symbolic coding using the partition $\{\Delta_i\}$. The alphabet $\mathcal{A} = \{\Delta)i\}$ is precisely the partition. Given a point $y \in [0, 1)$, define $x \in \mathcal{A}^{\mathbb{Z}}$ to be $x_i = \Delta_i$, where $T^i y \in \Delta_i$. The resulting subshift is the *coding* or *cover* of an IET.

The coding is not topologically conjugate to the IET, but it is an almost everywhere finite-to-one extension. Thus, it enjoys the many of the same properties: the coding is minimal if and only if the IET is minimal. The same is true for the number of ergodic measures. The coding of an IET also has linear complexity, and for a large class of IETs, we can compute the complexity function exactly.

For more details on the ergodic theory of IETs, see [**80**], and for background on the combinatorics of codings of IETs, see [**34**].

Example 2.8 (Substitutions)**.** A *substitution* ξ on the alphabet $\mathcal{A}$ is a map $\xi : \mathcal{A} \to \mathcal{A}^*$. It sends a symbol $a \in \mathcal{A}$ to a word $\xi(a)$. By applying the substitution to each symbol in a word or sequence and concatenating the images, we can extend ξ to a map on $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{Z}}$, and by abuse of notation, we also refer to these maps as ξ when the context is clear.

To rule out trivial cases, we require that for all $a \in \mathcal{A}$, $|\xi^n(a)| \to \infty$. We define $x \in \mathcal{A}^{\mathbb{Z}}$ to be a *fixed point of* ξ if $\xi(x) = x$, and a *periodic point of* ξ if $\xi^p(x) = x$ for some p. Note that periodic points of ξ are not periodic points under the shift map σ. Since there are only finitely many letters, there always exists a periodic point of ξ. The subshift obtained by by taking the orbit closure under the shift of a ξ-periodic point is called a *substitution subshift*.

We say that ξ is *primitive* if there exists some positive integer n such that for all $a, b \in \mathcal{A}$, the symbol b appears in $\xi^n(a)$. If ξ is primitive, then the resulting subshift is minimal.

The Fibnoacci Sturmian sequence from Example 2.4 is a substitution sequence, given by the substitution

$$0 \mapsto 01 \qquad 1 \mapsto 0.$$

The fixed point of the substitution begins with

$$0100101001001...$$

Note that this is a one-sided sequence. As the in the case with Sturmians, there is a method for extending into a two-sided sequence which is fixed under the substitution. Iterating the substitution twice sends 1 to 01, so the substitution is primitive, so the corresponding subshift is minimal.

Another example of a primitive substitution is the *Thue-Morse substitution*, given by the map

$$0 \mapsto 01 \qquad 1 \mapsto 10.$$

The fixed point of the substitution obtained by iterating 0 is

$$0110100110010110...$$

Here, the image of the substitution consists of only words of length 2, so it is a *constant length substitution*.

We can associate an incidence matrix A_ξ to ξ, which keeps track of the number of times each symbols shows up in the substitution. Suppose ξ is defined over the alphabet $\mathcal{A} = \{a_1, \cdots, a_m\}$. Then A_ξ is a non-negative $m \times m$ matrix where $(A_\xi)_{ij}$ is the number of times a_i appears in the word $\xi(a_j)$. For any $n \in \mathbb{N}$, $(A_\xi^n)_{ij}$ counts the number of times a_i appears in $\xi^n(a_j)$.

A matrix A is *primitive* if there exists some n such that A^n is a positive matrix. Applied to the incidence matrix A_ξ, this is precisely the condition that the substitution is primitive.

We can apply Perron-Frobenius Theory to a primitive incidence matrix A_ξ, which gives a lot of information about the substitution associated to ξ.

Theorem 2.9 (Perron[**70**], Frobenius [**39**])**.** *Let A be a non-negative primitive matrix. Then A admits a strictly positive eigenvalue λ which strictly dominates the other eigenvalues α: $|\lambda| > |\alpha|$. The eigenvalue λ is simple and there exists a positive eigenvector associated to λ.*

We refer to λ_ξ as the *Perron-Frobenius* eigenvalue. We call v^r_{PF} and v^ℓ_{PF} the positive right and left eigenvectors, respectively, associated to λ_ξ, renormalized so that the sum of entries is 1.

Substitution systems always satisfy linear complexity, and are uniquely ergodic. A comprehensive reference for substitution subshifts is [**73**].

Example 2.10 (Toeplitz)**.** We say that $x \in \mathcal{A}^{\mathbb{Z}}$ is *quasi-periodic* if for any word w that appears in x at i, there exists a period p such that w appears in x at $i + np$ for all $n \in \mathbb{Z}$. Any word that appears in x appears periodically, though the period may depend on the word. A point x is a *Toeplitz sequence* if it is quasi-periodic and not periodic. The orbit closure of a Toeplitz sequence under the shift map is a *Toeplitz system*. By construction, Toeplitz systems are minimal.

A process for generating a Topelitz sequence is to build skeletons in steps by filling in coordinates periodically. For example, in the first step, fill in all the even coordinates with 0s, and leave odd coordinates blank, marked by *s. In the second step, fill in every other * with 1s. In the third step, fill in every other * with 0s, and continue indefinitely. The first few skeletons are provided below:

0 * 0 * 0 * 0 * 0 * 0 * 0 * 0 * 0 * 0 * 0 * 0 * 0 * ...

0 1 0 * 0 1 0 * 0 1 0 * 0 1 0 * 0 1 0 * 0 1 0 * ...

0 1 0 0 0 1 0 * 0 1 0 0 0 1 0 * 0 1 0 0 0 1 0 * ...

Toeplitz systems are almost 1-1 extensions of odometers (see Example 2.23), and the periodic structure building Topelitz sequences is related to the periodic structure of its odometer factor.

Unlike the previous minimal examples, we can build Toeplitz systems with positive entropy, and in fact arbitrarily high entropy, due to Williams [**82**]. For a primer on Toeplitz sequences and systems, see [**48**].

2.1.3. Basic ergodic theory

Given a topological system (X, T), a probability measure μ on X is an *invariant measure* if it preserves measures under T: for any μ-measurable set $A \subset X$, $\mu(A) = \mu(T^{-1}A)$.

It is a classical result that any topological dynamical system supports an invariant measure [**81**, §6.2]. Note that given two invariant measures, any linear combination is again an invariant measure. We say μ is *ergodic* if it cannot be written as a nontrivial linear combination of invariant measures. If (X, T) has only one invariant measure, which is necessarily ergodic, then we say (X, T) is *uniquely ergodic*.

An important result on ergodicity is the Pointwise Ergodic Theorem, due to Birkhoff in 1931.

Theorem 2.11 (Pointwise Ergodic Theorem [**9**]). *Let (X,T) be a topological system and μ an ergodic probability measure. Then for all continuous functions $f \in C(X)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int_X f d\mu$$

for μ-almost every $x \in X$.

Remark 2.12. This theorem applies more broadly to measurable dynamical systems, where we can relax the requirement that T be continuous and consider measurable maps. Interval exchange transformations (see Example 2.7) are examples of measurable dynamical systems.

When (X,T) is uniquely ergodic, then the limit in the statement of the Pointwise Ergodic Theorem exists for all $x \in X$, and equality holds. In the context of subshifts, the Ergodic Theorem translates to frequencies of words that appear in sequences.

Since cylinder sets are clopen sets, indicator functions on cylinder sets are continuous. Given a word w and a point x, let $\mathbb{1}_{[w]}$ be the indicator function of $[w]$. The sum $\sum_{j=0}^{n-1} \mathbb{1}_{[w]}(x)$ counts the number of times w appears in $x_{[0,n-1]}$. So, the Pointwise Ergodic Theorem states that for μ-almost every point $x \in X$, the frequency of w appearing in X exists and is equal to the measure of $[w]$. When (X,σ) is uniquely ergodic, the frequency of any word w exists in every point.

2.1.4. Conjugacy and automorphism groups

An *automorphism of* (X, σ) is a self-conjugacy; that is, an automorphism is a homeomorphism from X to itself that commutes with the shift map.

Recall that the Curtis-Hedlund-Lyndon Theorem (Theorem 2.1) characterizes continuous maps between shift spaces. This applies to automorphisms. The set of automorphisms of X under composition forms a group $\mathrm{Aut}(X, \sigma)$, or simply $\mathrm{Aut}(X)$ when σ is clear from context. Since only finitely many automorphisms can have a given range, $\mathrm{Aut}(X)$ is countable. Given two automorphisms $g_1, g_2 \in \mathrm{Aut}(X)$, let $g_1 g_2$ denote the composition $g_1 \circ g_2$. We can think of the automorphism group as the symmetry group of (X, σ) with respect to conjugacy.

Given an arbitrary subshift, it is difficult to construct automorphisms or compute its automorphism group. However, some progress has been made for special classes of subshifts with more structure.

If a shift contains synchronizing words, there are finite order automorphisms called *marker automorphisms* originally defined by Hedlund [**45**] for full shifts, and later for SFTs by Boyle, Lind, and Rudolph [**18**]. We now define marker automorphisms more generally, making slight modifications to conventions introduced by Frisch, Schlank, and Tamuz [**37**].

We say that two words w and u *overlap* if we can write $w = w'v$ and $u = vu'$ (or vice versa). When needed, we specify the length of overlap, and we say that w and u overlap with length i, where $i = |v|$.

Let (X, σ) be a shift and M_ℓ and $M_r \in \mathcal{L}(X)$ be synchronizing words. Let $\mathcal{D} \subset \mathcal{L}_n$ be a set of words of length n appearing in X such that words of the form $M_\ell d M_r$ are allowable for all $d \in \mathcal{D}$. Suppose these words satisfy the following overlap condition: for any d and $d' \in \mathcal{D}$, if $M_\ell d M_r$ and $M_\ell d' M_r$ overlap nontrivially with length i, then $i \leq \min(|M_\ell|, |M_r|)$. Then any permutation τ of $\mathcal{D}$ induces an automorphism g_τ on X by sending words of the form $M_\ell d M_r$ to $M_\ell \tau(d) M_r$ and leaving other words unchanged. Such an automorphism is called a *marker automorphism*, and we refer to M_ℓ and M_r as the *left and right markers*, respectively, and $d \in \mathcal{D}$ as *data words*. We note that as originally defined for a j-step SFT, the length of marker words have to be greater than j. The key is that such words are synchronizing, which is the necessary condition to ensure that applying the map does not introduce forbidden words.

Example 2.13 (Shifts of finite type)**.** Recall the golden mean shift from Example 2.2, defined by the forbidden word 11. Let $M_\ell = 100$ and $M_r = 0101$ be start and end markers, and $\mathcal{D} = \{0, 1\}$ be data words. Given special blocks of the form $M_\ell d M_r$ and $M_\ell d' M_r$, for $d, d' \in \mathcal{D}$, $M_\ell d M_r$ and $M_\ell d' M_r$ can only overlap nontrivially by length 1.

Let $g \in \mathrm{Aut}(X)$ be the marker automorphism induced by the nontrivial permutation on $\mathcal{D}$. It permutes blocks of the form

$$\cdots 100\mathbf{1}0101 \cdots$$

$$\cdots 100\mathbf{0}0101 \cdots$$

and leaves other blocks unchanged.

For any transitive SFT, there are an abundance of marker automorphisms. By constructing specific marker automorphisms, Boyle, Lind, and Rudolph [**18**] showed that

Aut(X) contains isomorphic copies of any finite group and a copy of the free group on two generators.

Also using marker automorphisms, Kim and Roush [**50**] embedded the automorphism group of the full shift into the automorphism group of any mixing SFT, using markers to encode data words to act as symbols in a full shift. As a corollary, the automorphism groups of the full two-shift and the full three-shift contain the same subgroups up to isomorphism, but it remains an open question whether these automorphism groups are isomorphic. We note that the Kim and Roush construction does not embed the automorphism group of a mixing SFT into the automorphism group of another mixing SFT, as the embedding relies heavily on the lack of forbidden words in the full shift.

On the other hand, a method to distinguish automorphism groups follows from Ryan's Theorem [**76, 77**].

Theorem 2.14 (Ryan [**76, 77**])**.** *Let (X, σ) be a transitive SFT, and let $g \in \mathrm{Aut}(X)$ be an element which commutes with all automorphisms. Then $g = \sigma^k$ for some $k \in \mathbb{Z}$.*

In the full four-shift, the shift map has a root, while the shift map for the full two-shift does not. Using Ryan's Theorem, Boyle, Lind, and Rudolph [**18**] observed that these automorphism groups cannot be isomorphic, as the automorphism group of the full four-shift contains an element not in the center whose square is in the center, while the automorphism group of the full two-shift has no such element.

Example 2.15 (Sofic shifts)**.** Recall the even shift from Example 2.3, where points consist of only consecutive even 1s.

Any word which contains 0 is a synchronizing word, so we can define marker automorphisms with markers that contain 0.

Example 2.16 (Sturmians)**.** Sturmians are the subsuhifts with minimal complexity $P(n) = n+1$ (see Example 2.4). We say that a word $w \in \mathcal{L}(X)$ is *(right) special* if it can be extended in more than one way: there exists distinct $a, b \in \mathcal{A}$ such that $wa, wb \in \mathcal{L}(X)$. In a Sturmian system, for each n, there is precisely one special word of length n. This implies that there is exactly one pair of asymptotic points.

Because any automorphism comes from a sliding block code, it must map pairs of asymptotic points to pairs of asymptotic points. For Sturmians, any automorphism preserves the single asymptotic class, and in fact preserves the orbits of each asymptotic point. Since the automorphism acts as a power of the shift on one point, by minimality it is a power of the shift. Thus, the automorphism group of a Sturmian system is isomorphic to $\mathbb{Z}$, generated by σ.

Example 2.17 (Linear complexity)**.** Extending the argument from the previous example, Cyr and Kra [**24**] and Donoso, Durand, Maass, and Petite [**29**] showed that the automorphism group of transitive shifts of linear complexity contain $\mathbb{Z}$ as a finite index subgroup, or is *virtually* $\mathbb{Z}$.

Example 2.18 (Toeplitz)**.** By exploiting the fact that Toeplitz systems are almost 1-1 extensions of odometers, Donoso, Durand, Maass, and Petite [**30**] showed that the automorphism group of a Toeplitz system embeds into the automorphism of its odometer factor. Since the automorphism group of an odometer is itself, it is abelian, so the automorphism group of any Toeplitz is also abelian.

2.2. Suspensions of topological systems

Given a topological system (X, T), the *suspension* $\Sigma_T X$ is the space $X \times \mathbb{R}/\sim$, where $(x, t+1) \sim (T(x), t)$. Elements in $\Sigma_T X$ are equivalence classes $[(x, t)]$, where $x \in X$ and $t \in \mathbb{R}$; since every class in $\Sigma_T X$ has a unique representative of the form $[(x, t)]$ where $0 \le t < 1$, we often refer to points in $\Sigma_T X$ as simply (x, t) where $0 \le t < 1$. We often suppress the ordered pair notation and simply denote points by $z \in \Sigma_T X$, with the understanding that such a point is given by $z = (x, t)$ for some $x \in X, 0 \le t < 1$. The space $\Sigma_T X$ is compact, is locally the product of a totally disconnected set with an arc, and comes with an $\mathbb{R}$-action defined by

$$\Upsilon\colon \mathbb{R} \times \Sigma_T X \to \Sigma_T X, \quad \Upsilon(s, [(x, t)]) = [(x, t+s)]. \tag{2.2.1}$$

We almost exclusively use the simpler notation

$$\Upsilon(s, z) = z + s.$$

In particular, for $k \in \mathbb{Z}$ we have $\Upsilon(k, (x, t)) = (x, t) + k = (T^k x, t)$. We refer to the Υ-orbit of a point z as the *leaf* containing z, and observe that the leaf containing z coincides with the path component in $\Sigma_T X$ which contains z. There is a distinguished cross section to the flow that we denote by

$$\Gamma = \big\{[(x, 0)] \mid x \in X\big\} \subset \Sigma_T X.$$

At times we make use of suspensions of (X, T) over more general roof functions. For a positive locally constant function $r\colon X \to \mathbb{R}$ (called a *roof function*) we define the *suspension with roof function* r by

$$\Sigma_T^r X = X \times \mathbb{R} / \sim, \quad (x,t) \sim (T(x), t - r(x)).$$

The space $\Sigma_T^r X$ also carries a flow defined analogously to that of (2.2.1).

The suspension shares many dynamical and ergodic properties with the base space.

Points $z_1, z_2 \in \Sigma_T X$ are said to be *asymptotic* (under the flow) if $\lim_{t\to\infty} d(z_1+t, z_2+t) = 0$. We say two leaves ℓ_1, ℓ_2 are *asymptotic* if there are points $z_1 \in \ell_1, z_2 \in \ell_2$ such that z_1 and z_2 are asymptotic. Note that two leaves ℓ_1, ℓ_2 are asymptotic if and only if there exists points $x_1, x_2 \in X$ such that $(x_1, 0) \in \ell_1 \cap \Gamma, (x_2, 0) \in \ell_2 \cap \Gamma$, and x_1, x_2 are asymptotic under T (so $\lim_{n\to\infty} d(T^n x_1, T^n x_2) = 0$). The relation defined by

$$\ell_1 \underset{as}{\sim} \ell_2 \quad \text{if } \ell_1 \text{ and } \ell_2 \text{ are asymptotic}$$

defines an equivalence relation on leaves. We call a leaf ℓ *asymptotic* if its equivalence class $[\ell]_{as}$ under $\underset{as}{\sim}$ contains more than one element, and denote the set of equivalence classes of asymptotic leaves by $\mathrm{as}(T)$.

2.2.1. Flow equivalence and mapping class groups

A *flow equivalence* between Cantor systems (X,T) and (X',T') is a homeomorphism $f\colon \Sigma_T X \to \Sigma_{T'} X'$ such that f takes Υ-orbits onto Υ-orbits in an orientation preserving way. Two systems $(X,T), (X',T')$ are *flow equivalent* if there exists a flow equivalence between them. The group of self-flow equivalences $\mathrm{Homeo}^+\Sigma_T X$ is a subgroup of the topological group of homeomorphisms of $\Sigma_T X$, and we say $f \in \mathrm{Homeo}^+\Sigma_T X$ is isotopic to the identity if f is in the path component of the identity map in $\text{Homeo}^+\Sigma_T X$. Equivalently, $f \in \text{Homeo}^+\Sigma_T X$ is isotopic to the identity if and only if there exists a continuous $\eta\colon \Sigma_T X \to \mathbb{R}$ such that $f(z) = z + \eta(x)$ for all $z \in \Sigma_T X$ (see [**14**, Theorem 3.1], together with [**13**], for a proof of this). The collection

$$\text{Homeo}_0^+\Sigma_T X = \{f \in \text{Homeo}^+\Sigma_T X \mid f \text{ is isotopic to the identity}\}$$

forms a normal subgroup, and we define the mapping class group of (X, T) to be the quotient group

$$\mathcal{M}(T) = \text{Homeo}^+\Sigma_T X \big/ \text{Homeo}_0^+\Sigma_T X.$$

Thus $\mathcal{M}(T)$ is the group of isotopy classes of self-flow equivalences of the system (X, T).

Any flow equivalence must take asymptotic leaves to asymptotic leaves. As a result, there is a homomorphism

$$\mathcal{M}(T) \xrightarrow{\pi_{as}} P(\text{as}(T)) \tag{2.2.2}$$

where $P(\text{as}(T))$ denotes the permutation group on the set $\text{as}(T)$.

2.2.2. Examples

Example 2.19 (SFTs)**.** The mapping class group of for subshifts was originally defined by Boyle and Chuysurichay [**15**] as a way to study the algebraic structure of flow equivalences between SFTs. The name comes from an analogous definition for surfaces, though since the base system is a Cantor set, the two mapping class groups behave quite differently.

Let (X, σ) be a mixing SFT. The main tool to study mapping class group of (X, σ) is the density of periodic points, which correspond to circles in the suspension. By appealing to flow codes, there is a version of Ryan's Theorem for mapping class groups: the center of $\mathcal{M}(\sigma)$ is trivial. Note that since σ is isotopic to the identity, the center is actually trivial here, not just $\langle\sigma\rangle$.

One significant difference is that while $\mathrm{Aut}(X)$ is residually finite, $\mathcal{M}(\sigma)$ is not. A group G is *residually finite* if for every non-identity $g \in G$, there is a homomorphism φ from G to a finite group such that $\varphi(g) \neq e$.

Flow equivalent spaces have isomorphic mapping class groups, and thus if (X, σ_X) and (Y, σ_Y) are flow equivalent, then $\mathrm{Aut}(Y)/\langle\sigma\rangle$ must embed into $\mathcal{M}(\sigma)$. However, unless the two systems are conjugate, the automorphism groups may not be isomorphic, so this embedding is nontrivial. If X and Y are mixing SFTs, elements of $\mathcal{M}(\sigma_X)$ which are induced by automorphisms of flow equivalent SFTs have invariant cross sections. See [**15**] for an explicit example of a map which does not arise from this construction.

2.2.3. Flow Codes

Given a topological system (X, T), we call (X, T) a *Cantor system* when X is a Cantor set. As a consequence of the work of Parry and Sullivan in [**68**], any flow equivalence between Cantor systems is isotopic to one given by a conjugacy of return systems. We briefly outline here the parts from this theory relevant for our purposes; our presentation follows closely the work of Boyle and Chuysurichay [**14**], which consolidates, and greatly expands upon, results from [**68**]. We further refer the reader to [**14**] for details and more background on flow equivalence and isotopy for Cantor systems.

A clopen subset $C \subset X$ is called a *discrete cross section* if there is a continuous function $r\colon C \to \mathbb{N}$ defined by $r(x) = \min\{i \in \mathbb{N} \mid T^i(x) \in C\}$ for which $X = \{T^i(x) \mid x \in C, i \in \mathbb{N}\}$. Any discrete cross section $C \subset X$ determines a return system (C, T_C) by defining

$$T_C\colon C \to C, \quad T_C\colon x \mapsto T^{r_C(x)}x.$$

The systems $(X, T), (C, T_C)$ are always flow equivalent. Note that if (X, T) is minimal, then any clopen subset is a discrete cross section, and any return system (C, T_C) is also minimal.

For Cantor systems $(X, T), (X', T')$, an $(X, T), (X', T')$-*flow code* is a triple (φ, C, D) where $C \subset X, D \subset X'$ are discrete cross sections and $\varphi\colon (C, T_C) \to (D, T_D)$ is a conjugacy. Any $(X, T), (X', T')$-flow code (φ, C, D) induces a flow equivalence $S_\varphi\colon \Sigma_T X \to \Sigma_{T'} X'$ as follows: since C is a cross section, we can represent any point $z \in \Sigma_T X$ by $z = (x, t)$ with $x \in C$ (with t possibly larger than 1). The flow code maps the segment with endpoints $(x, 0)$ and $(x, r_C(x)) \sim (T_C x, 0)$ linearly to the segment in $\Sigma_{T'} X'$ with endpoints $(\varphi x, 0)$ and $(\varphi x, r_D(\varphi x)) \sim (T_D(\varphi x), 0)$. Since φ is a conjugacy, the map is well-defined at endpoints. As C is a discrete cross section, any leaf is a union of such segments, and the flow code defines a flow equivalence from $\Sigma_T X$ to $\Sigma_{T'} X'$. The following is a fundamental result in the study of flow equivalence of zero-dimensional systems. While originally due to Parry and Sullivan in [**68**], the version below is stated differently, and we refer the reader to [**14**] for details.

Theorem 2.20. [**14**, Theorem 4.1] *Let* $(X, T), (X', T')$ *be Cantor systems. For any flow equivalence* $f\colon \Sigma_T X \to \Sigma_{T'} X'$ *there exists a flow code* (φ, C, D) *such that* f *is isotopic to* $S_\varphi\colon \Sigma_T X \to \Sigma_{T'} X'$.

To simplify notation, we refer to an (X, X)-flow code simply as a flow code. The following is a consequence of Theorem 2.20.

Proposition 2.21. [**15**, Corollary 2.3] *If* (X, σ) *is a subshift then* $\mathcal{M}(\sigma)$ *is countable.*

Remark 2.22. The subshift hypothesis in Proposition 2.21 cannot be dropped. For odometers, the group $\mathcal{M}(T)$ is uncountable (see Example 2.23).

2.3. Topological boundaries of compact spaces

Throughout this section, let G be a locally compact group and let Ω be a compact metric space with a continuous G action $G \times \Omega \to \Omega$: for any $g \in G$ and $\omega \in \Omega$,

$$(g, \omega) = g \cdot \omega.$$

We call Ω a *G-space.* Given $\omega \in \Omega$, let $G\omega$ denote the G-orbit of ω:

$$G\omega = \{g \cdot \omega : g \in G\} \subset \Omega$$

and $\overline{G\omega}$ its closure in Ω.

Let $\mathrm{Prob}(\Omega)$ be the set of Borel probability measures on Ω, equipped with the weak-* topology. Since Ω is compact, $\mathrm{Prob}(\Omega)$ is also a compact metric space. Given $\omega \in \Omega$, let δ_ω denote the Dirac measure concentrated at ω. The mapping $\omega \mapsto \delta_\omega$ gives an embedding of Ω into $\mathrm{Prob}(\Omega)$.

The G-action on Ω induces an action on $\mathrm{Prob}(\Omega)$ by viewing elements of G as self-homeomorphisms of Ω: for any $g \in G$ and $\omega \in \Omega$,

$$g \cdot \mu = \mu \circ g^{-1}.$$

We say that the G-action on Ω is *minimal* if for any $\omega \in \Omega$, the G-orbit closure $\overline{G\omega} = \Omega$. The G-action on Ω is *strongly proximal* if for all $\mu \in \mathrm{Prob}(\Omega)$, the G-orbit closure $\overline{G\mu} \subset \mathrm{Prob}(\Omega)$ contains a Dirac measure δ_ω for some $\omega \in \Omega$. A G-space Ω is a *topological boundary* if the G-action on Ω is minimal and strongly proximal.

The G-action on Ω is *extremely proximal* if $|\Omega| \geq 2$ and for any proper closed set $C \subsetneq \Omega$ and any open set $U \subset \Omega$, there is some $g \in G$ with $gC \subset U$.

It is known that extreme proximality implies strong proximality [**43**, §3] and the product of strongly proximal actions is again strongly proximal [**44**, §3].

A group G is *amenable* if for every compact G-space Ω, the G-action on $\mathrm{Prob}(\Omega)$ has a fixed point. Examples of amenable groups include abelian groups and finite groups, while the free group is not amenable.

Given two G-spaces Ω_1 and Ω_2, with a continuous map $\varphi : \Omega_1 \to \Omega_2$, φ is *(G)-equivariant* if for every $g \in G$, the action of g on Ω_1 and Ω_2 is consistent with φ, that is, $\varphi(g \cdot \omega) = g \cdot (\varphi\omega)$, where $\omega \in \Omega_1$, and $\cdot$ refers to correct G-action.

2.4. Group of coinvariants of a Cantor system

Let (X, T) be a Cantor system, and let $C(X, \mathbb{Z})$ denote the abelian group of continuous integer valued functions on X. The map $\partial\colon \gamma \mapsto \gamma - \gamma \circ T^{-1}$ defines a homomorphism

$\partial\colon C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$, and we define the group of coinvariants to be the abelian group

$$\mathcal{G}_T = C(X,\mathbb{Z})/\mathrm{Image}(\partial).$$

The group $\mathcal{G}_T$ contains a positive cone

$$\mathcal{G}_T^+ = \{[\gamma] \in \mathcal{G}_T \mid \text{ there exists non-negative } \gamma' \in C(X,\mathbb{Z}) \text{ such that } [\gamma] = [\gamma']\},$$

and together with the distinguished order unit [1] (the class of the constant function $1 \in C(X,\mathbb{Z})$), the triple $(\mathcal{G}_T, \mathcal{G}_T^+, [1])$ becomes a unital preordered group. A *preordered group* is a pair (G, G^+), where G is an abelian group and the positive cone G^+ is a submonoid of G which generates G (see [**16**, Section 1.2]). The preordered group becomes an *ordered group* if $G^+ \cap -G^+ = \{0\}$. When (X,T) is minimal, $\mathcal{G}_T$ is a simple dimension group (see [**41**]). By an isomorphism of ordered groups we mean an isomorphism taking the positive cone onto the positive cone; if the isomorphism in addition takes the distinguished order unit to the distinguished order unit, we say it is an isomorphism of unital ordered groups. The pair $(\mathcal{G}_T, \mathcal{G}_T^+)$ is an invariant of flow equivalence: if (X,T) and (X',T') are flow equivalent, then $(\mathcal{G}_T, \mathcal{G}_T^+)$ and $(\mathcal{G}_{T'}, \mathcal{G}_{T'}^+)$ are isomorphic. A proof of this can be found in [**16**, Theorem 1.5]. Note that the triple $(\mathcal{G}_T, \mathcal{G}_T^+, [1])$ is not in general an invariant of flow equivalence; there are flow equivalences that do not preserve the distinguished order unit. For example, let $C \subset X$ be a return system. Then (X,T) and (C, T_C) are always flow equivalent. However, the order unit $[1_X]$ gets mapped to $[r_C]$ the return map, which is not in general cohomologous to $[1_C]$.

Let $\pi^1(Y)$ denote the group $[Y, S^1]$ of homotopy classes of maps from Y to S^1 (the group structure being given by pointwise multiplication $[f]+[g]=[fg]$). For a compact

Hausdorff space Y, the group $\pi^1(Y)$ is isomorphic to the first integral Čech cohomology group $\check{H}^1(Y, \mathbb{Z})$ (see [**53**]).

Example 2.23 (Odometers)**.** Given an infinite sequence of prime numbers $P = \{p_i\}_{i=1}^{\infty}$, let $Q = \{q_j = \prod_{i=1}^{j} p_i\}_{j=1}^{\infty}$. The compact abelian group

$$\mathcal{O}_P = \varprojlim\{\mathbb{Z}/q_k\mathbb{Z}, \pi_k\}\pi_k \colon \mathbb{Z}/q_k\mathbb{Z} \to \mathbb{Z}/q_{k-1}\mathbb{Z}\pi_k \colon a \bmod q_k \mapsto a \bmod q_{k-1}.$$

together with the translation map $T_P \colon \mathcal{O}_P \to \mathcal{O}_P$, where

$$T_P \colon x \mapsto x + (1, 1, \ldots),$$

form a minimal equicontinuous Cantor system $(\mathcal{O}_P, T_P)$ which we refer to as the P-odometer. The suspension $\Sigma_{T_P}\mathcal{O}_P$ is homeomorphic to the P-adic solenoid

$$\mathcal{S}_P = \varprojlim\{S^1, w_{p_k}\}w_{p_k} \colon S^1 \to S^1, \quad w_{p_k} \colon z \mapsto z^{p_k}$$

We give here a presentation of $\mathcal{M}(T_P)$ based on [**54**, Theorem 1].

We have a short exact sequence

$$1 \to \mathrm{Aut}(T_P)/\langle T_P \rangle \to \mathcal{M}(T_P) \to \mathrm{Aut}(\mathcal{G}_{T_P}, \mathcal{G}_{T_P}^{+}) \to 1 \tag{2.4.1}$$

where $\mathrm{Aut}(\mathcal{G}_{T_P}, \mathcal{G}_{T_P}^{+})$ denotes the group of automorphisms of the abelian group

$$\mathcal{G}_{T_P} = \{v \in \mathbb{Q} \mid q_k \cdot v \in \mathbb{Z} \text{ for some } k\}$$

preserving the positive cone

$$\mathcal{G}_{T_P}^{+} = \{v \in \mathbb{Q} \mid q_k \cdot v \in \mathbb{Z}_+ \text{ for some } k\}.$$

The ordered group $(\mathcal{G}_{T_P}, \mathcal{G}_{T_P}^{+})$ is isomorphic to the ordered group of coinvariants defined in Section 2.4. When p is a prime which appears infinitely often in P the map

$$\varphi_p \colon \mathcal{O}_P \to \mathcal{O}_P \varphi_p \colon (x_1, x_2, \ldots) \longmapsto (p \cdot x_1, p \cdot x_2, \ldots)$$

is injective and $\varphi_p \colon \mathcal{O}_p \to \operatorname{Image} \varphi_p$ defines a flow code, giving a self-flow equivalence $S_{\varphi_p} \in \operatorname{Homeo}^{+} \Sigma_T X$. The map S_{φ_p} agrees with the Frobenius automorphisms defined in [**54**].

As a group $\operatorname{Aut}(\mathcal{G}_{T_P}, \mathcal{G}_{T_P}^{+})$ is generated by the set of automorphisms

$$\{m_p \colon x \longmapsto p \cdot x \mid p \text{ appears infinitely often in } P\}$$

and the map

$$\operatorname{Aut}(\mathcal{G}_{T_P}, \mathcal{G}_{T_P}^{+}) \to \mathcal{M}(T_P) m_p \longmapsto S_{\varphi_p}$$

defines a right-splitting map for the sequence (2.4.1). The automorphism group of the odometer is the odometer: $\operatorname{Aut}(T_P)$ is known to be isomorphic to $\mathcal{O}_P$ (see [**29**, Lemma 5.9]), so we have a semidirect product presentation for the mapping class group

$$\mathcal{M}(T_P) \cong \mathcal{O}_P / \langle (1, 1, \ldots) \rangle \rtimes \operatorname{Aut}(\mathcal{G}_{T_P}, \mathcal{G}_{T_P}^{+}).$$

As an explicit example, for the set of primes $P_{2,3} = \{p_i\}$ defined by

$$\begin{cases} p_i = 2 & \text{if } i \text{ is even} \\ p_i = 3 & \text{if } i \text{ is odd} \end{cases}$$

we have $\mathrm{Aut}(\mathcal{G}_{T_P}, \mathcal{G}_{T_P}^+) \cong \mathrm{Aut}(\mathbb{Z}[\frac{1}{6}], \mathbb{Z}_+[\frac{1}{6}]) \cong \mathbb{Z}^2$ and

$$\mathcal{M}(T_{P_{2,3}}) \cong \mathcal{O}_{P_{2,3}}/\langle(1,1,\ldots)\rangle \rtimes \mathbb{Z}^2.$$

2.4.1. Alternate definitions

Given a roof function $r\colon X \to \mathbb{R}$ and a map $\eta\colon \Sigma_T^r X \to S^1$, given $z \in \Sigma_T^r X$ there exists a continuous function $g_z\colon \mathbb{R} \to \mathbb{R}$ satisfying $\eta(z+t) = \eta(z)e^{2\pi i g_z(t)}$ for all $t \in \mathbb{R}$. Let $C_+(\Sigma_T^r X, S^1)$ denote the set of maps $\eta\colon \Sigma_T^r X \to S^1$ such that for all $z \in \Sigma_T^r X$, g_z is non-decreasing. The group $\pi^1(\Sigma_T^r X)$ becomes a pre-ordered group by defining the positive cone associated to the 'winding order'

$$\pi^1_+(\Sigma_T^r X) = \{[\eta] \mid \eta \in C_+(\Sigma_T^r X, S^1)\} \subset \pi^1(\Sigma_T^r X).$$

To check that this indeed defines a pre-order, within a given homotopy class $[\eta]$ one can choose a representative such that the function g defined above is the difference of non-decreasing functions; see [**16**, Section 4] for details. Given $\gamma \in C(X, \mathbb{Z})$, define $\eta_\gamma \in C(\Sigma_T X, S^1)$ by

$$\eta_\gamma(x,t) = e^{2\pi i t\gamma(x)},\ \ 0 \le t < 1.$$

The following result relating the groups $\mathcal{G}_T$ and $\pi^1(\Sigma_T X)$ is classical, though the claim regarding the order structures is less so. A proof may be found in [**16**, Proposition 4.5].

Proposition 2.24. *For a Cantor system (X,T) the map*

$$\mathcal{G}_T \to \pi^1(\Sigma_T X)[\gamma] \longmapsto [\eta_\gamma] \tag{2.4.2}$$

is an isomorphism of groups. When (X,T) is minimal, the isomorphism respects the order structures, giving an isomorphism of ordered groups

$$(\mathcal{G}_T, \mathcal{G}_T^+) \cong (\pi^1(\Sigma_T X), \pi^1_+(\Sigma_T X)).$$

In general, the group isomorphism $\mathcal{G}_T \to \pi^1(\Sigma_T X)$ in Proposition 4.1 may not be an order isomorphism (see [**16**, Example 4.7]). A map $f \in \mathrm{Homeo}^+\Sigma_T X$ induces a map on $\pi^1(\Sigma_T X)$ via $[\eta] \mapsto [\eta \circ f]$, and two maps $f, g \in \mathrm{Homeo}^+(\Sigma_T X)$ which are isotopic induce the same map on $\pi^1(\Sigma_T X)$. It follows from Proposition 2.24 that any $[f] \in \mathcal{M}(T)$ induces an automorphism $f^* \in \mathrm{Aut}(\mathcal{G}_T)$; the map f^* also preserves the positive cone $\mathcal{G}_T^*$, and there is a well-defined homomorphism

$$\pi_T \colon \mathcal{M}(T) \to \mathrm{Aut}(\mathcal{G}_T, \mathcal{G}_T^+)\pi_T \colon [f] \longmapsto f^*. \tag{2.4.3}$$

Note that in general, π_T does not take $\mathcal{M}(T)$ to $\mathrm{Aut}(\mathcal{G}_T, \mathcal{G}_T^+, [1])$; the induced map f^* may not preserve the order unit [1]. Let us briefly describe how flow codes provide a convenient way to explicitly compute the map f^* on $\mathcal{G}_T$ induced by some $[f] \in \mathcal{M}(T)$. Let $C \subset X$ be a discrete cross section. Given $[\gamma] \in C(X,\mathbb{Z})$, define $\gamma_C(x) \in C(C,\mathbb{Z})$ by

$$\gamma_C(x) = \begin{cases} \sum_{i=0}^{r_C(x)-1} \gamma(T^i x) & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases} \tag{2.4.4}$$

The map $res_C\colon [\gamma] \mapsto [\gamma_C]$ (the name *res* coming from re-stacking) induces an isomorphism $res_C\colon \mathcal{G}_T \xrightarrow{\cong} \mathcal{G}_{T_C}$ (a proof of this can be extracted from the proof of Theorem 1.5 in [**16**]). Now given $[f] \in \mathcal{M}(T)$, let (φ, C, D) be a flow code representing $[f]$. Then the map $f^*\colon \mathcal{G}_T \to \mathcal{G}_T$ coincides with the composition

$$\mathcal{G}_T \xrightarrow{res_D} \mathcal{G}_{T_D} \xrightarrow{S_\varphi^*} \mathcal{G}_{T_C} \xrightarrow{res_C^{-1}} \mathcal{G}_T. \tag{2.4.5}$$

where the map $S_\varphi^*\colon \mathcal{G}_{T_D} \to \mathcal{G}_{T_C}$ is given by $S_\varphi^*([\gamma]) = [\gamma \circ \varphi]_{\mathcal{G}_{T_C}}$.

Remark 2.25. There are minimal Cantor systems (X, T) which admit automorphisms $\varphi \in \mathrm{Aut}(X)$ such that $\pi_T(\varphi)$ acts non-trivially on $(\mathcal{G}_T, \mathcal{G}_T^+)$. Many such examples may be found in [**60**, Section 4]. For a particular example (coming from [**60**]), for the substitution defined on $\{0, 1, 2, 3\}$ by

$$\xi\colon \quad 0 \mapsto 012230, \quad 1 \mapsto 123301, \quad 2 \mapsto 230012, \quad 3 \mapsto 301123 \tag{2.4.6}$$

the map $\varphi\colon X \to X$ given by $\varphi(x_j) = x_j + 1 \bmod 4$ defines an automorphism, and one check directly that it acts non-trivially on the dimension group (for details, we refer the reader to [**60**, Section 4]).

2.4.2. States

We call a homomorphism $\tau\colon \mathcal{G}_T \to \mathbb{R}$ a *positive homomorphism* if it satisfies $\tau(\mathcal{G}_T^+) \subset \mathbb{R}_+$. A positive homomorphism τ is called a *state* if $\tau([1]) = 1$. Let $p(T)$ denote the monoid of all positive homomorphisms $\tau\colon \mathcal{G}_T \to \mathbb{R}$, and let $\mathcal{S}(T) \subset p(T)$ denote the set of states. Any T-invariant Borel probability measure μ on X gives rise to a state τ_μ given by

$\tau_\mu([\gamma]) = \int_X \gamma d\mu$, and there is a bijection between the set $m(T)$ of all T-invariant Borel probability measures on X and $\mathcal{S}(T)$ given by

$$\mu \longmapsto \tau_\mu. \tag{2.4.7}$$

(see [**16**, Section 1.6] for details regarding this bijection). Under the correspondence in (2.4.7) an ergodic measure μ corresponds to an extremal state τ_μ. If (X, T) has exactly $k < \infty$ T-invariant ergodic probability measures $\{\mu_i\}_{i=1}^k$, then $p(T)$ may be identified with the positive orthant in $\mathbb{R}^k$ given by $\{v = \sum_{i=1}^k c_i \tau_{\mu_i} \mid c_i \in \mathbb{R}_+\}$. The subgroup of infinitesimals of $\mathcal{G}_T$ is defined to be

$$\mathrm{Inf}\,(\mathcal{G}_T) = \{[\gamma] \in \mathcal{G}_T \mid n[\gamma] \leq [1] \text{ for all } n \in \mathbb{Z}\}.$$

Equivalently, when $\mathcal{G}_T$ is simple (as is always be the case for us), we have

$$\mathrm{Inf}\,(\mathcal{G}_T) = \{[\gamma] \in \mathcal{G}_T \mid \tau([\gamma]) = 0 \text{ for all } \tau \in \mathcal{S}(T)\}. \tag{2.4.8}$$

An automorphism $\alpha \in \mathrm{Aut}(\mathcal{G}_T, \mathcal{G}_T^+)$ induces a dual map $p(\alpha)\colon p(T) \to p(T)$ given by $p(\alpha)(\tau)([\gamma]) = \tau(\alpha([\gamma]))$. Using the homomorphism (2.4.3), we get an action of $\mathcal{M}(T)$ on the space of positive homomorphisms, giving a representation

$$L_T\colon \mathcal{M}(T) \to \mathrm{Aut}(p(T)). \tag{2.4.9}$$

For $[f] \in \mathcal{M}(T)$, we denote $L_T([f])$ by f_*.

CHAPTER 3

The automorphism group of sofic subshifts

In this chapter, we prove the main result about the automorphism group of a sofic shift, which is published in [**83**]:

Theorem 3.1. *Let* (X, σ) *be a transitive sofic shift. Any normal amenable subgroup of* $\mathrm{Aut}(X)$ *is contained in* $\langle \sigma \rangle$*.*

First, we extend Ryan's Theorem to sofic shifts in Section 3.1. In Section 3.2, we construct an $\mathrm{Aut}(X)$-space using left-periodic points. Next, in Section 3.3, we show that space is actually a topological boundary. The proof of the result appears in Section 3.4. Finally, we discuss issues that arise when extending to higher dimensions in Section 3.5, and present open questions in Section 3.6.

3.1. Generalized Ryan's Theorem

By generalizing the definition of marker automorphisms, we can adapt the proof of Ryan's Theorem to show that for a transitive sofic shift, the center of the automorphism group must be the subgroup generated by the shift. We show that for a transitive sofic shift, the automorphism group contains enough markers so that Ryan's Theorem still holds. The key proposition needed, which is a well-known result we state without proof, is that a transitive shift contains infinitely many synchronizing words

Proposition 3.2. [**57**, Proposition 3.3.16] *Suppose* (X, σ) *is a transitive sofic shift. Then any word* $w \in \mathcal{L}(X)$ *can be extended on the right to a synchronizing word* wu*.*

Remark 3.3. There is an alternate definition of sofic shifts and SFTs as the set of bi-infinite paths on a labeled graph. The graph presentation allows us to associate to the shift an incidence matrix, which is necessarily non-negative and integral. This has been instrumental in the development of the theory of SFTs. We note that the definition of synchronizing word in [**57**] is dependent on the graph, while they use the term *intrinsically synchronizing* to denote words we call synchronizing. However, if one chooses the minimal graph presentation for the sofic shift, these definitions coincide. Since we do not appeal to the graph presentation, we refer the reader to [**57**] for complete details.

To prove Ryan's Theorem for transitive sofic shifts, it suffices to show that there exist infinitely many synchronizing words which do not overlap themselves (see Section 2.1.4 for definitions). To prove this lemma, we rely on the following theorem in the field of combinatorics on words, translated in the language of symbolic dynamics.

We say that a word u is a *root* of w if we can write $w = u^k$ for some $k \in \mathbb{N}$. If w has no proper roots, then w is *primitive*.

Two words u, u' are *conjugate* if we can write $u = wv$ and $u' = vw$ for some v, w (possibly empty) words.

Theorem 3.4. [**32**, Theorem 3.1][**58**] *Let* $w \in \mathcal{A}^n$ *be a primitive word. Then there exists* w' *conjugate to* w *which does not overlap itself.*

Lemma 3.5. *Let* (X, σ) *be a transitive sofic shift. Then for any* $n \in \mathbb{N}$*, there is a synchronizing word* M *of at least length* n *which does not overlap itself non-trivially.*

Proof. Let w be a synchronizing word of at least length n. By transitivity, there exists $u \in \mathcal{L}(X)$ such that $wuw \in \mathcal{L}(X)$. Note that wuw is again a synchronizing word.

Since periodic points are dense, there is a periodic point x of minimal period $k \geq |wuw|$ such that wuw appears in x. By Theorem 3.4, x must contain a subword M of length k which does not overlap itself. Since $|wuw| \leq k$, w must appear in M, and thus M is a synchronizing word. □

We give a proof of the generalized Ryan's Theorem, and we follow the proof for SFTs, due to Kitchens [**51**, Theorem 3.3.22].

Theorem 3.6. *Let (X, σ) be a transitive sofic shift. The center of* Aut(X) *is* $\langle \sigma \rangle$.

Proof. Let (X, σ) be a transitive sofic shift, and let $\varphi \in$ Aut(X) commute with all automorphisms. Suppose φ has range R. Recall that for transitive sofic shifts, between any two words we can always insert spacers of lengths that form an arithmetic progression, where the difference is p, the period of X. Using these spacers and the sufficiently long markers produced by Lemma 3.5, we can find a synchronizing $M \in \mathcal{L}(X)$ and $n \in \mathbb{N}$, with $2R + 1 \leq n \leq |M|$, such that for

$$\mathcal{D}(M, n) := \{d \in \mathcal{L}_n(X) : MdM \in \mathcal{L}(X)\},$$

every word of length $2R+1$ appears as a subword of some element of $\mathcal{D}(M, n)$. This can be done by applying the transitive property simultaneously to M and words of length $2R + 1$ so that the spacers are of the same length. If necessary we can extend M to the left. Repeat the process on the right to get words of the form MdM. For any permutation $\tau \in \mathrm{Sym}(\mathcal{D}(\mathrm{M}, \mathrm{n}))$, let g_τ denote the marker automorphism induced by τ.

Consider the periodic points of period $|M|+n$ obtained by concatenating Md with itself, for any $d \in \mathcal{D}(M,n)$. We denote such points $\mathrm{Per(M,n)} \subset \mathrm{Per_{|M|+n}}$. Let $\mathrm{Orb(M,n)}$ be the set of distinct σ-orbits in $\mathrm{Per(M,n)}$. Note that $|\mathrm{Orb(M,n)}| \geq 2$, as each word of length $2R+1$ appears in some $d \in \mathcal{D}(M,n)$.

For any permutation of $\mathrm{Orb(M,n)}$, there is a g_τ, for some $\tau \in \mathrm{Sym}(\mathcal{D}(\mathrm{M,n}))$ whose action on $\mathrm{Orb(M,n)}$ coincides with the given permutation. In addition, g_τ acts as the identity on periodic points of period $|M|+n$ which are not in $\mathrm{Per(M,n)}$.

We claim that φ acts on $\mathrm{Orb(M,n)}$. Suppose not. Then φ maps some $x \in \mathrm{Per(M,n)}$ to a periodic point y not in $\mathrm{Per(M,n)}$. Since φ commutes with all g_τ, this means that all points in $\mathrm{Per(M,n)}$ are mapped to the σ-orbit of y, which contradicts that φ permutes the periodic points of each period.

Now we show that φ acts as the identity permutation. Let x and $y \in \mathrm{Per(M,n)}$ be in distinct σ-orbits. We note that there exists $\varphi(x) = \sigma^j(x)$ for some $-R \leq j \leq R$. This equality holds for all points in the σ-orbit of x, and we show that it holds for y as well. Let g_τ be a permutation that takes $\mathcal{O}(x)$ to $\mathcal{O}(y)$. As φ commutes with g_τ, we have

$$\varphi(y) = g_\tau^{-1} \circ \varphi \circ g_\tau(y) = \sigma^j y.$$

As every block of length $2R+1$ appears in some $d \in \mathcal{D}(M,n)$, we conclude $\varphi = \sigma^j$. □

3.2. The action of the automorphism group on left-periodic points

Recall from Chapter 2.1.1 that left-periodic points are not periodic. For any shift (X,σ), we build a compact space equipped with an $\mathrm{Aut(X)}$ action. Given $k \in \mathbb{N}$, denote the set of left-k-periodic points up to k by Q_k. From the definition of left-periodic points

(Section 2.1.1), the period and the index are typically two different values. However, by density of periodic points, we can choose them to be the same.

Lemma 3.7. *Let (X, σ) be a shift. Suppose X contains a left-k-periodic point for some $k \in \mathbb{N}$. Then Q_k is an $\mathrm{Aut}(X)$-space, and σ acts trivially on Q_k. If X contains a left-k-periodic point which is transitive, then the kernel of the action is $\langle \sigma \rangle$.*

Proof. Since any automorphism is a block map, the set of left-k-periodic points is invariant under $\mathrm{Aut}(X)$. The set of all left-k-periodic points is precisely $\bigcup_{i \in \mathbb{Z}} \sigma^i Q_k$. Thus, for any $g \in \mathrm{Aut}(X)$ and $x \in Q_k$,

$$gx \in \sigma^i Q_k, \tag{3.2.1}$$

for some unique i, since the shifts of Q_k are pairwise disjoint. Define a cocycle $\alpha \colon \mathrm{Aut}(X) \times Q_k \to \mathbb{Z}$ to be:

$$\alpha(g, x) = -i \tag{3.2.2}$$

where i is obtained from equation (3.2.1). The cocycle condition ensures that the induced $\mathrm{Aut}(X)$-action on Q_k is well-defined, where for $g \in \mathrm{Aut}(X), x \in Q_k$:

$$g \cdot x = \sigma^{\alpha(g,x)} \circ gx. \tag{3.2.3}$$

We note here that the action of Aut(X) on X is different from the action on Q_k, so we use different notation to make clear which action we are referencing. For any $x \in Q_k$, $\alpha(\sigma, x) = -1$, so $\sigma \cdot x = x$.

Suppose in addition $x \in X$ is a transitive left-k-periodic point. Let $g \notin \langle\sigma\rangle$. For each $n \in \mathbb{N}$, let R_n denote the maximum of n and the range of g. Let $\hat{g}_n$ and $\hat{\sigma}^n$ denote the sliding block codes of range R_n that induce g and σ^n, respectively. Then there exists a word w_n of length $2R_n + 1$ such that $\hat{g}_n(w_n) \neq \hat{\sigma}^n(w_n)$. Since every w_n appears in x, $g \cdot x \neq x$. □

The set of k-periodic points $\mathrm{Per_k}$ is invariant under $\mathrm{Aut}(X)$. We can decompose $\mathrm{Per_k}$ into a disjoint union of distinct σ-orbits:

$$\mathrm{Per_k} = \mathcal{O}(\mathrm{x}^1) \amalg \cdots \amalg \mathcal{O}(\mathrm{x}^\mathrm{j}). \tag{3.2.4}$$

Thus, the action of $\mathrm{Aut}(X)$ on X descends to an action on $\mathrm{Per_k}/\langle\sigma\rangle$.

Lemma 3.8. *Let (X, σ) be a shift that contains a left-k-periodic point, and Q_k be the set of left-k-periodic points up to k. There is a map*

$$\pi \colon Q_k \to \mathrm{Per_k}/\langle\sigma\rangle \tag{3.2.5}$$

which is $\mathrm{Aut}(X)$*-equivariant. When (X, σ) is a transitive sofic shift, π is a projection.*

Proof. Given $x \in Q_k$, there exists a unique k-periodic point such that $y_n = x_n$ for all $n < k$. Define the projection π which sends x to the σ-orbit in $\mathrm{Per_k}$ containing y. Since any automorphism is a sliding block code, for any $x \in Q_k$ and $g \in \mathrm{Aut}(X)$, $\pi(gx) = g\pi(x)$.

Suppose (X, σ) is a transitive sofic shift, and let $y \in \mathrm{Per_k}$. Consider the word $y_{[k-N,k)}$, which we can extend to the right to a synchronizing word w. Since X is transitive, there is a word $u \in \mathcal{L}(X)$ of the form $u = ww'a$, where $a \neq y_{k+p-1}$ (where p refers to the greatest common divisor of the periods of all periodic points; see Section 2.1.1). Then we can extend u to a left-k periodic point in Q_k (after shifting appropriately so it lands in Q_k) that projects to the σ-orbit containing y. $\square$

A map $s\colon \mathrm{Per_k}/\langle\sigma\rangle \to \mathrm{Q_k}$ is a section of the projection π if s is a right inverse of π. Let Ω be the collection of all sections $s\colon \mathrm{Per_k}/\langle\sigma\rangle \to \mathrm{Q_k}$ of the projection π. Since π is equivariant, the action of $\mathrm{Aut}(X)$ on Q_k induces an action on Ω: for any $g \in G$ and $s \in \Omega$,

$$gs = g \cdot (s \circ g^{-1})$$

where we view g^{-1} as the permutation of $\mathrm{Per_k}/\langle\sigma\rangle$ induced by g^{-1}.

Let $\{\Omega^m\}$ denote the fibers of $\pi\colon Q_k \to \mathrm{Per_k}/\langle\sigma\rangle$. Given a periodic orbit $\mathcal{O}(x^m) \in \mathrm{Per_k}/\langle\sigma\rangle$,

$$\Omega^m = \{x \in Q_k : \exists i \in \mathbb{Z} \text{ with } x_n = x^m_{n-i} \text{ for all } n < k\} \tag{3.2.6}$$

Let $N \triangleleft \mathrm{Aut}(X)$ be the normal subgroup given by the kernel of π. Since N preserves the fibers $\{\Omega^m\}$, the restriction of the action on Ω to N is isomorphic to the diagonal action of N on the product of the fibers $\prod_{m=1}^{j} \Omega^m$, where j is the number of distinct k-periodic orbits defined in (3.2.4).

3.3. Extreme Proximality

Given a bi-infinite sequence $x \in X$, we say that x is *synchronizing* if all sufficiently long words that appear in x are synchronizing. If x is periodic and some synchronizing word appears in x, then x itself is synchronizing. Since the definition only depends on words that appear in x, a sequence is synchronizing if and only if any point in its orbit closure is synchronizing.

Every periodic orbit $\mathcal{O}(x)$ is exactly one of the three following types, and we will refer to them as Type (1), (2) or (3):

Type (1): x is synchronizing

Type (2): x is not synchronizing, but there exists an automorphism $h \in \mathrm{Aut(X)}$ such that hx is synchronizing

Type (3): x is not synchronizing, and for all automorphisms $h \in \mathrm{Aut(X)}$, hx is not synchronizing.

Remark 3.9. An equivalent characterization of SFT is a shift in which all sequences are synchronizing. In this case, $\mathrm{Per_k}$ consists only of synchronizing points for large enough k, and orbits of Type (2) and Type (3) do not exist. This is not the case for strictly sofic shifts: consider the even shift from Example 2.15. A word is synchronizing if and only if it contains 0. Thus, the fixed point 0^∞ is synchronizing, while 1^∞ is not.

In general, the action of Aut(X) on $\mathrm{Per_k}$ may not be transitive. In the case of SFTs, however, for sufficiently large k, Aut(X) does act on $\mathrm{Per_k}$ transitively (see [**18**]). The proof constructs a composition of marker automorphisms which permute periodic points with disjoint orbits, building on work by Boyle and Krieger [**17**] for the full shift. The same proof shows that for a transitive sofic shift, Aut(X) acts transitively on the synchronizing points in $\mathrm{Per_k}$. However, non-synchronizing points do not contain any synchronizing subwords, so they are fixed by all marker automorphisms.

Let $\mathrm{Syn_k} \subset \mathrm{Per_k}$ be the subset of periodic points of Type (1) and Type (2). Then for sufficiently large k, $\mathrm{Syn_k}$ is the $\mathrm{Aut}(X)$-orbit under $\mathrm{Aut}(X)$ of any synchronizing point, so $\mathrm{Aut}(X)$ must act transitively on $\mathrm{Syn_k}$.

Remark 3.10. It is possible that no periodic points of Type (2) exist. We do not know if there exist automorphisms which do not fix non-synchronizing points.

Define $\widetilde{\Omega}$ analogously to Ω in Section 3.2, but for the restricted action. For any section $s \in \Omega$, where $s : \mathrm{Per_k}/\langle\sigma\rangle \to \mathrm{Q_k}$, let $s|_{\mathrm{Syn_k}/\langle\sigma\rangle} : \mathrm{Syn_k}/\langle\sigma\rangle \to \mathrm{Q_k}$ be the corresponding element in $\widetilde{\Omega}$. Note that this is not injective and many $s \in \Omega$ project to the same map in $\widetilde{\Omega}$.

We now show that $\widetilde{\Omega}$ is a topological boundary for Aut(X). Recall that N, the kernel of the action of $\mathrm{Per_k}$, acts on each Ω^m. If we consider the action of N on $\widetilde{\Omega}$,

$$\widetilde{\Omega} \cong \prod_m \Omega^m$$

where the product is taken over values of m where x^m is of Type (1) or Type (2).

We show that the action of N on Ω^m is extremely proximal, and use this to prove that the full action of Aut(X) on $\widetilde{\Omega}$ is a topological boundary. The key step is constructing marker automorphisms in Ω^m, where x^m is a synchronizing point. Then for non-synchronizing points, we exploit the fact that such points are in the $\mathrm{Aut}(X)$-orbit of some synchronizing point to achieve the same result.

The set Ω^m is closed in X, so cylinder sets of the form

$$[w]^m := [w] \cap \Omega^m, \text{ where } w \in \mathcal{L}(X) \tag{3.3.1}$$

form a subbase that generates the subspace topology on Ω^m.

Proposition 3.11. *Let (X, σ) be a transitive sofic shift. Suppose X contains a left-k-periodic point for some $k \in \mathbb{N}$. Fix $x^m \in \mathrm{Syn_k}$, and define Ω^m as in* (3.2.6). *Let*

w, u be words in $\mathcal{L}(X)$ such that the corresponding cylinder sets $[w]^m$ and $[u]^m$ in Ω^m are nonempty and proper. Then there is an involution $g \in \mathrm{Aut}(X)$ which acts as the identity on $\mathrm{Per_k}/\langle\sigma\rangle$ and satisfies $g \cdot [w]^m \subset [u]^m$.

Proof. We fix $x^m \in \mathrm{Syn_k}$, so either x^m is of Type (1) or Type (2).

Case 1: x^m is synchronizing. There exists $c \in \mathbb{N}$ such that all subwords of length at least c are synchronizing.

Since points in Ω^m are left-k-periodic up to k, if $|w| \leq k$, we can replace it with a word $\widetilde{w}$ of length $k+1$ such that $[w]^m = [\widetilde{w}]^m$. So, we may assume that $|w|, |u| > k$. Write $a = w_{[0,\ldots,k-1]}$. If $w[0] \neq u[0]$, let $\widetilde{u}$ be the unique extension of u so that w and $\widetilde{u}$ begin with the same letter.

We can assume that $\widetilde{u}$ does not appear as the initial word of w, otherwise some power of the shift maps $[w]^m$ into $[u]^m$ and the identity automorphism satisfies the conclusion of lemma. We first deal with the case that w does not appear as the initial word of $\widetilde{u}$.

Let a^r be the word obtained by concatenating r copies of a. Since it is a word of at least length c which appears in x^m, it must be synchronizing. As $[w]^m \neq \emptyset$, $a^r w$ is allowable, and must also be synchronizing. Since X is transitive, choose $v \in \mathcal{L}(X)$ such that $\widetilde{u} v a^r$ is an allowable word. Since a is the initial word of $\widetilde{u}$, we can choose v with $|\widetilde{u}v| = rpk$ for some $r \geq \max\{|w|, c\}$, where p is the greatest common divisor of k with $\mathrm{Per_k} \neq \emptyset$. Set a^r to be the left marker, $a^r w$ to be the right marker, and $\mathcal{D} = \{a^{rp}, \widetilde{u}v\}$ to be data words.

To show these markers induce a well-defined marker automorphism, it suffices to check that special words of the form $a^r d a^r w$, for $d \in \mathcal{D}$, satisfy the overlap condition given in the definition of marker automorphism. The word a is a primitive word of length k which appears in x^m, a k-periodic point. While a may overlap itself non-trivially, a^r can only overlap itself by multiples of k, as k is the least period of x^m.

The initial word of w and $\widetilde{u}$ is a, while a does not occur at position k in w or $\widetilde{u}$. Thus, if the special words $a^r a^{rp} a^r w$ and $a^r \widetilde{u} v a^r w$ overlap nontrivially, the length of the overlap must be either less than $|w|$, or exactly $rk + |w|$. In the second case, the special blocks would overlap by a^r. However, this overlap would force $\widetilde{u}$ to be the initial word of w, which contradicts the assumption.

Because special words begin with a^r, a similar argument shows that a special word can only overlap with itself nontrivially by at most $|w|$. By the choice of r, $|a^r| \geq |w|$, so the marker automorphism g determined by the nontrivial permutation on $\mathcal{D}$ is well-defined, and g is an involution.

Let $x^i \in \mathrm{Per_k}$. Since no special words appear, g acts as the identity on x^i, so g is in the kernel of the action on $\mathrm{Per_k}/\langle\sigma\rangle$.

Lastly, we show that $g \cdot [w]^m \subset [u]^m$. Let $y \in [w]^m$. Since y is left-k-periodic, the first occurrence of a special word in y is $a^r a^{rp} a^r w$ at $-(2r + rp)k$. Thus,

$$gy = \cdots a\widetilde{u}v.w \cdots .$$

Applying the cocycle α gives

$$g \cdot y = \cdots \widetilde{a}.uvw \cdots \in [u]^m$$

where $\widetilde{a}$ is the initial k-block of u.

Suppose now w is the initial word of $\widetilde{u}$. We can partition $[w]^m$ by the finitely many allowable extensions of w given by wb, where each wb is of length $|\widetilde{u}|$. Applying the process above gives marker automorphisms g_b for each extension wb. Since special words for each g_b have the same length and are distinct, the involutions g_b all commute. The composition of $\{g_b\}$ is well-defined, and is an involution that maps $[w]^m$ into $[u]^m$.

Case 2: x^m is not synchronizing, and there is an $h \in \mathrm{Aut(X)}$ such that hx^m is synchronizing. Let $h \in \mathrm{Aut(X)}$ where $hx^m = x^i$ is synchronizing, and let $[w]^m, [u]^m$ satisfy the hypothesis. Consider the sets

$$h[w]^m \text{ and } h[u]^m.$$

We can partition them into finitely many cylinder sets, so by the previous construction, there is some $g \in N$ that maps

$$g \cdot h[w]^m \subset h[u]^m.$$

Then

$$h^{-1}gh \cdot [w]^m \subset [u]^m$$

with $h^{-1}gh \in N$, as desired. □

Recall that $N \triangleleft \mathrm{Aut}(X)$ is the kernel of $\pi : \Omega \to \mathrm{Per_k}/\langle\sigma\rangle$, so the automorphism produced above is contained in N.

Corollary 3.12. *Let (X, σ) be a transitive sofic. Suppose that X contains a left-k-periodic point for some $k \in \mathbb{N}$. Then the following hold:*

(1) The action of N on Ω^m, as defined in (3.2.6)*, is minimal.*

(2) The action of $\mathrm{Aut}(X)$ *on Ω, as defined in Section 3.2 is minimal.*

(3) The N-action on Ω^m is extremely proximal.

(4) The Aut(X)*-action on* Ω *is strongly proximal.*

Proof. (1) It suffices to show that the N-orbits of any nonempty open subset U covers all of Ω^m. Let $[u]^m \subset U$ be a nonempty cylinder set with $|u| \geq k$ and $[w]^m$ be an nonempty cylinder with $|w| \geq k$. By Proposition 3.11, there exists an involution $g \in N$ such that $g \cdot [w]^m \subset [u]^m$. Since $g = g^{-1}$, we have $[w]^m \subset g \cdot [u]^m$. As $[u]^m$ was arbitrary, this shows that $\bigcup_{g \in N} g \cdot U = \Omega^m$. (2) Let $U \subset \Omega$ be an open set. If the intersections $U \cap \Omega^m$ are all nonempty, then by part (1), the N-orbit of U covers Ω. Suppose U is contained in some Ω^m. The action of $\mathrm{Aut}(X)$ on $\mathrm{Syn_k}$ is transitive, so there exists $g \in \mathrm{Aut}(X)$ such that $gU \cap \Omega_n$ is nonempty for any Ω_n. By part (1), the $\mathrm{Aut}(X)$-orbit of U covers Ω. (3) Each Ω^m contains more than two points, and cylinder sets form a subbase that generates the topology on Ω^m. In addition, Ω^m is compact, so any closed set is covered by finitely many cylinder sets. By Proposition 3.11, the N action on each Ω^m is extremely proximal. (4) As extremely proximal actions are also strongly proximal, by part (3), the N action on each Ω^m is strongly proximal. Thus, the product action of N on $\prod_{m=1}^{j} \Omega^m$ is also strongly proximal. Since the diagonal action of N on the product space is isomorphic (as continuous group actions) on Ω, it follows that the action of $\mathrm{Aut}(X)$, which contains N, on Ω is also strongly proximal. □

We appeal to a proposition of Furman which relates the kernel of boundary actions and normal amenable subgroups:

Proposition 3.13 (Furman [**40**])**.** *Let G be a discrete group, and consider the following subgroups of G:*

(1) $N = \bigcap_{i\in I} Ker(G \to Homeo(X_i))$*, where* I *is the set of isomorphism classes of boundary actions on the set of* G*-spaces,*

(2) $\sqrt{G}$*, the group generated by all closed normal amenable subgroups in* G*.*

Then $N = \sqrt{G}$*.*

In particular, the kernel of a boundary action contains all normal amenable subgroups.

3.4. Proof of theorem

We have now assembled the ingredients to prove Theorem 3.1.

Proof of Theorem 3.1. Let (X, σ) be a transitive sofic shift. As periodic points are dense, there exists some k such that X contains k-periodic points and $\mathrm{Syn_k} \neq \emptyset$. Since X is not finite, X also contains left-k-periodic points.

Corollaries 3.12 (2) and 3.12 (4) show that Ω is an $\mathrm{Aut}(X)$-boundary. By Proposition 3.13, any normal amenable subgroup of $\mathrm{Aut}(X)$ is contained in the kernel of a boundary action.

An element in Ω is a section of the projection $\pi : \Omega \to \mathrm{Per_k}/\langle\sigma\rangle$, so the kernel of $\mathrm{Aut}(X)$ acting on Ω must be contained in the kernel of $\mathrm{Aut}(X)$ acting on Q_k, the set of left-k-periodic points up to k. Thus, it follows from Lemma 3.7 that the kernel of the $\mathrm{Aut}(X)$ action on Ω is precisely $\langle\sigma\rangle$, and we obtain the desired result. □

3.5. Higher dimensions

We show that the direct analogue of Theorem 3.1 in higher dimensions fails by giving a counterexample and explain why the methods of proof do not generalize even with stronger hypotheses. Consistent with the definition of one-dimensional shifts given in

Chapter 2.1, we define a $\mathbb{Z}^d$-shift to be a closed, translation-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$. A $\mathbb{Z}^d$-shift is an SFT if it can be described by forbidden patterns in $\mathcal{A}^{\mathcal{F}}$, for some finite set $\mathcal{F} \subset \mathbb{Z}^d$, and a $\mathbb{Z}^d$-sofic shift is a topological factor of a $\mathbb{Z}^d$-SFT. The automorphism group consists of self-homeomorphisms of the shift that commute with the shift maps, which can be identified with $\mathbb{Z}^d$.

Hochman [**46**] constructs a two-dimensional SFT $X \subset \mathcal{A}^{\mathbb{Z}^2}$, which is topologically mixing and has positive entropy. Hochman explicitly computes the automorphism group to be $\mathbb{Z}^2 \oplus \bigcup S_{i,j}$, where $\mathbb{Z}^2$ is generated by the shift maps and $\bigcup S_{i,j}$ is a directed union of infinitely many finite groups, arising from higher dimensional marker automorphisms. Amenability is closed under taking direct limits and sums; thus, the automorphism group is amenable. In higher dimensions, Ryan's Theorem holds [**46**], and the center is the subgroup generated by the shifts, $\mathbb{Z}^2$. In particular, $\mathrm{Aut}(X)$ has normal amenable subgroups that are not contained in the center. While this shift is topologically mixing, the set of periodic points is not dense, which suggests this may not be the right condition to impose.

There are various notions of uniform mixing in higher dimensions, (for example, strongly irreducible, uniform filling, and block gluing) each of which imply that periodic points are dense. In each case, if two allowable patterns are sufficiently far apart, there is another allowable pattern which agrees with the original patterns; the distinct notions of uniform mixing depends on the shape of patterns we consider. In contrast, for $d = 1$, these definitions of uniform mixing are equivalent to topological mixing.

However, even with dense periodic points, we cannot construct a topological boundary for uniformly mixing $\mathbb{Z}^d$-SFTs as we did in the one-dimensional case. Because there are now more directions of periodicity, we cannot construct a space on which the automorphism group acts in the same manner. More specifically, we cannot define a $\mathbb{Z}^d$-cocycle as we did in equation 3.2.1.

In the case of the higher dimensional full shift, Frisch, Schlank, and Tamuz [**37**] show that any normal amenable subgroup must be contained in the subgroup generated by the shifts; unfortunately, their methods do not generalize to uniformly mixing SFTs. They construct a class of automorphisms of $\mathcal{A}^{\mathbb{Z}^d}$, induced by automorphisms of $\mathcal{A}^{\mathbb{Z}}$, which act independently on bi-infinite sequences of a configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$. This relies strongly on the fact that in the full shift, there are no forbidden blocks. In a more general $\mathbb{Z}^d$-SFT, acting independently on lower dimensional subspaces may produce forbidden patterns. We note that higher dimensional marker automorphisms cannot arise from such a construction.

3.6. Open questions

(1) The construction of the $\mathrm{Aut}(X)$-space Q_k in Lemma 3.7 only requires the existence of a transitive left-periodic point. Proposition 3.11 is the main proposition showing that there are enough marker automorphisms which map arbitrary cylinder sets into other cylinder sets. The marker automorphisms require the existence of synchronizing words. This does not apply to any minimal subshifts, or subshifts of low complexity. A class of subshifts which do contain synchronizing words are *synchronized systems*, which is a special class of coded systems which contain a synchronizing word [**57**]. Can we adapt the proof of Proposition 3.11 to synchronized systems?

(2) The construction of Q_k for transitive sofic shifts works as long as k is sufficiently large. This produces countably many topological boundaries, which are pairwise non-homeomorphic. Each Q_k produces a different space on which $\mathrm{Aut}(X)$ acts in a rigid way (proximal and minimal). Can these representations give more insight to $\mathrm{Aut}(X)$?

(3) Ryan's Theorem has a higher dimensional analogue (see Section 3.5, due to Hochman [**46**]. Given a topologically mixing $\mathbb{Z}^d$-SFT with positive entropy, then the center of the automorphism group is isomorphic to $\mathbb{Z}^d$, generated by the translation maps. In the same paper, Hochman also constructs an example of a topological mixing $\mathbb{Z}^2$-SFT with positive entropy, whose automorphism group is amenable, and strictly larger than $\mathbb{Z}^2$. This is a counterexample to the direct analogue of Theorem 3.1. However, Frisch, Schlank, and Tamuz [**37**] prove that the amenable radical of the automorphism group of the full $\mathbb{Z}^d$ shift is precisely $\mathbb{Z}^d$. This suggests some intermediate notions of mixing may be the right hypothesis. What uniform mixing conditions can we produce an analogue for Theorem 3.1, or produce counterexamples?

CHAPTER 4

Mapping class group of low complexity subshifts

This chapter is split into five sections, presenting results published in [**78**]. In Section 4.1, we investigate flow equivalences in more detail. Of note, we give sufficient conditions for when a flow equivalence is isotopic to the identity, and we also give sufficient conditions when an element of the mapping class group is induced by an automorphism. In Section 4.2, we define a representation of the mapping class group for uniquely ergodic minimal Cantor systems, which sets up the main result characterizing the mapping class group of substitution systems in Section 4.3. Section 4.4 contains results about the mapping class group of linear complexity subshifts more generally. Lastly, we pose some open questions in Section 4.5.

4.1. Flow equivalences

For a Cantor system (X, T), any $f \in \mathrm{Homeo}^+\Sigma_T X$ permutes the set of leaves. The permutation induced by f depends only on the isotopy class of f, giving an action of $\mathcal{M}(T)$ on the set of leaves. It can happen that for a topologically transitive subshift (X, σ), a map $f \in \mathrm{Homeo}^+\Sigma_\sigma X$ fixes every leaf but is not isotopic to the identity (see [**14**, Example 3.3]). The following result is a corollary of Petite and Maas (which can also be obtained from [**2**, Theorem 2.5]), and shows that when (X, T) is minimal, this representation of $\mathcal{M}(T)$ is faithful.

Proposition 4.1. *Let (X,T) be a minimal Cantor system and let $f \in \mathrm{Homeo}^+\Sigma_T X$. The following are equivalent:*

(1) f is isotopic to the identity.

(2) f is homotopic to the identity.

(3) f maps every leaf to itself.

Proof. That (1) implies (2) implies (3) is straightforward. For (3) implies (1), suppose $f \in \mathrm{Homeo}^+\Sigma_T X$ maps every leaf to itself. Then there exists some $\beta\colon \Sigma_T X \to \mathbb{R}$ satisfying

$$f(z) = z + \beta(z)$$

for all $z \in \Sigma_T X$, and Kwapicz [**55**] shows that, when (X,T) is minimal, the map β is continuous. One can verify that for all t, the map

$$f_t(z) = z + t \cdot \beta(z)$$

lies in $\mathrm{Homeo}^+\Sigma_T X$ (see for example [**14**, Prop. 3.1], together with [**13**]) and hence the family f_t provides an isotopy from f to the identity. □

Recall from Section 2.2.1 that the mapping class group $\mathcal{M}(T)$ is the group of self-flow equivalences of (X,T), up to isotopy. There is a map

$$\Psi\colon \mathrm{Aut}(X) \to \mathcal{M}(T).$$

Given an automorphism $g \in \mathrm{Aut}(X)$, $\Psi(g) = (g, \mathrm{Id})$. As a map on suspensions, $\Psi(g)$ acts as g on cross sections, and acts as the identity on the second coordinate. Note that

T maps to an element which is isotopic to the identity. We can now show, under mild conditions, that the kernel of this map is only powers of T.

Recall that a discrete cross section (see Section 2.2.3 is a clopen set $C \subset X$ such that for each point in X, $T^j x \in C$ for some j. When C is a discrete cross section, the return function r_C is well-defined and locally constant, and (C, T_C), where $T_C(x) = T^{r_C(x)}(x)$, is a well-defined return system.

Proposition 4.2. *Let (X,T) be an topologically transitive Cantor system without any periodic points and let $C \subset X$ be a discrete cross section. Then $\ker \Psi_C = \langle T_C \rangle$, the subgroup generated by T_C. In particular, there is an embedding*

$$\Psi \colon \mathrm{Aut}(X)/\langle T \rangle \hookrightarrow \mathcal{M}(T).$$

Proof. To see that $T_C \in \ker \Psi_C$, first note the map S_{T_C} sends each leaf to itself. Moreover, the function $\beta \colon \Sigma_T X \to \mathbb{R}$ satisfying

$$S_{T_C}(z) = z + \beta(z)$$

is continuous, from which it follows (see [**14**, Prop. 3.1], together with [**13**]) that S_{T_C} is isotopic to the identity. Conversely, suppose $\Psi_C(\varphi)$ is isotopic to the identity. Since (X,T) is topologically transitive, we may choose $x \in C$ such that the T_C-orbit of x is dense in C. Then $\varphi(x) = T^k(x)$ for some k, and since $\varphi(x) \in C$, this means $\varphi(x) = T_C^j(x)$ for some j. Thus the automorphism $T_C^{-j}\varphi \in \mathrm{Aut}(T_C)$ fixes x. Since x has a dense T_C-orbit in C, this implies $\varphi = T_C^j$. □

The following proposition, while unsurprising, is useful.

Proposition 4.3. *Let (X,T) be a minimal Cantor system and let $f \in \mathrm{Homeo}^+\Sigma_T X$. If $f(z+t) = f(z)+t$ for all $z \in \Sigma_T X, t \in \mathbb{R}$, then $[f] \in \mathrm{Image}\,\Psi$.*

Proof. The hypotheses imply there exists $0 \le s < 1$ such that $f(\Gamma) - s = \Gamma$. For any $x \in X$ there is then a unique point $y \in X$ such that $f(x,0) - s = (y,0)$, and the map $\varphi_f(x) = y$ defines an automorphism of (X,T) such that $\Psi(\varphi_f) = [f]$ in $\mathcal{M}(T)$. □

Example 4.4. Example 3.9 in [**18**] constructs a minimal subshift (Y,σ) for which $\mathrm{Aut}(\sigma) \cong \mathbb{Q}$ and the isomorphism takes σ to $1 \in \mathbb{Q}$. Proposition 4.2 then implies there is an embedding $\mathbb{Q}/\mathbb{Z} \longrightarrow \mathcal{M}(\sigma)$. Since any subgroup of a residually finite group must also be residually finite, and $\mathbb{Q}/\mathbb{Z}$ is not residually finite, the mapping class group $\mathcal{M}(\sigma)$ of (Y,σ) is not residually finite.

When (X,T) is a Cantor system without periodic points, we can restrict the action of a self-flow equivalence to a particular leaf, and view it as a function from $\mathbb{R}$ to $\mathbb{R}$. This point of view forgets the Cantor topology in the base space, but proves very fruitful in our analysis. In particular, for flow codes, we can describe this map in great detail.

Suppose (X,T) is an Cantor system without periodic points. Given a self-flow equivalence $f \in \mathrm{Homeo}^+\Sigma_T X$, the map

$$\alpha_f \colon \Sigma_T X \times \mathbb{R} \to \mathbb{R}$$

defined implicitly by

$$f(z+t) = f(z) + \alpha_f(z,t)$$

is well-defined, and satisfies the cocycle condition

$$\alpha_f(z, t_1 + t_2) = \alpha_f(z, t_1) + \alpha_f(z + t_1, t_2).$$

For a flow code (φ, C, D), the cocycle α_{S_φ} is piecewise linear, where the slopes are given by the ratios of the return times. Given $x_0 \in C$, for $0 < t < r_C(x_0)$ the slope of $\alpha_{S_\varphi}((x_0, 0), t)$ is given by

$$\frac{r_D(\varphi x_0)}{r_C(x_0)}.$$

More generally, for

$$\sum_{i=0}^{k-1} r_C(T_C^i x_0) < t < \sum_{i=0}^{k} r_C(T_C^i x_0), \ \ k \in \mathbb{N}$$

the slope of $\alpha_{S_\varphi}((x_0, 0), t)$ is given by

$$\frac{r_D(T_D^k \varphi x_0)}{r_C(T_C^k x_0)}. \tag{4.1.1}$$

while for $t < 0$, for times

$$\sum_{i=-k}^{-1} -r_C(T_C^i x_0) < t < \sum_{i=-k+1}^{-1} -r_C(T_C^i x_0), \ \ k \in \mathbb{N}$$

the slope is given by

$$\frac{r_D(T_D^{-k} \varphi x_0)}{r_C(T_C^{-k} x_0)}. \tag{4.1.2}$$

We now appeal to the coinvariant representation to give sufficient conditions when a flow equivalence is isotopic to a map induced by an automorphism of the base space. Recall the definitions of the $\mathcal{G}_T$ and its automorphism group from Chapter 2.4.

Theorem 4.5. *Suppose* (X,T) *is a minimal Cantor system. Then* $\operatorname{Image}\Psi$ *is precisely the set of* $[f] \in \mathcal{M}(T)$ *satisfying* $f^*[1] = [1]\}$*, where* f^* *refers to the action of* f *on* $\operatorname{Aut}(\mathcal{G}_T, \mathcal{G}_T^+)$ *(see Section 2.4. In particular, we have* $\ker \pi_T \subset \operatorname{Image}\Psi$*.*

Proof. It is clear that any $[\varphi] \in \operatorname{Image}\Psi$ satisfies $\varphi^*[1] = [1]$. Suppose then that $[f] \in \mathcal{M}(T)$ satisfies $f^*[1] = [1]$, and choose a flow code (φ, C, D) representing $[f]$. We claim there exists some $\gamma \in C(X,\mathbb{Z})$ such that $r_D \circ \varphi - r_C = \gamma - \gamma \circ T_C^{-1}$. By assumption we have $[f^*(1)]_{\mathcal{G}_T} = [1]_{\mathcal{G}_T}$ in $\mathcal{G}_T$. Using (2.4.5), this implies that in $\mathcal{G}_{T_C}$

$$[r_C]_{\mathcal{G}_{T_C}} = \operatorname{res}_C([1]_{\mathcal{G}_T}) = \operatorname{res}_C(f^*([1]_{\mathcal{G}_T})) = [r_D \circ \varphi]_{\mathcal{G}_{T_C}}$$

so there exists $\gamma \in C(C,\mathbb{Z})$ such that $r_D \circ \varphi - r_C = \gamma - \gamma \circ T_C^{-1}$. Let $x_0 \in C$. For $t_k = \sum_{i=0}^{k-1} r_C(T_C^i x_0)$ we have

$$\begin{aligned}
\alpha_f((x_0,0),t_k) - t_k &= \sum_{i=0}^{k-1} r_D(\varphi T_C^i x_0) - t_k = \sum_{i=0}^{k-1} r_D(\varphi T_C^i x_0) - \sum_{i=0}^{k-1} r_C(T_C^i x_0) \\
&= \sum_{i=0}^{k-1} (\gamma - \gamma \circ T_C^{-1})(T_C^i x_0) = \gamma(T_C^{k-1} x_0) - \gamma(T_C^{-1} x_0)
\end{aligned}$$

which is bounded for all k. Since $\alpha_f((x_0,0),t)$ is piecewise linear with uniformly bounded slopes between t_k and t_{k+1}, $\alpha_f((x_0,0),t) - t$ is bounded for $t \geq 0$. Using minimality of the $\mathbb{R}$-action, it follows that for all $z \in \Sigma_T X$, $\alpha_f(z,t) - t$ is bounded. An application of the Gottschalk-Hedlund Theorem(see e.g. [**61**, Theorem C]) then implies the cocycle $\alpha_f(z,t) - t$ is equal to a coboundary, and there exists $\eta \colon \Sigma_T X \to \mathbb{R}$ such that $\alpha_f(z,t) - t = \eta(z) - \eta(z+t)$ for all $z \in \Sigma_T X, t \in \mathbb{R}$. Given $z \in \Sigma_T X$, define $\beta \colon \Sigma_T X \to \Sigma_T X$ by

$\beta(z) = f(z) + \eta(z)$. The map β is continuous, and satisfies

$$\beta(z+t) = f(z+t) + \eta(z+t) = f(z) + \alpha_f(z,t) + t - \alpha_f(z,t) + \eta(z)$$
$$= f(z) + \eta(z) + t = \beta(z) + t$$

for all $z \in \Sigma_T X, t \in \mathbb{R}$. It is then straightforward to check that β is both injective and surjective on leaves, and hence $\beta \in \mathrm{Homeo}^+\Sigma_T X$. Thus β is a conjugacy of $(\Sigma_T X, \mathbb{R})$ to itself, so $[\beta] \in \mathrm{Image}\,\Psi$ by Proposition 4.3. Since f is isotopic to β, $[f] \in \mathrm{Image}\,\Psi$ as well. □

Remark 4.6. The example from Remark 2.25 shows that $\ker \pi_T$ can be a proper subgroup of $\mathrm{Image}\,\Psi$.

Remark 4.7. When (X, σ) is a non-trivial irreducible shift of finite type, it follows from [**15**, Corollary 3.3] that the map $\mathcal{M}(\sigma) \to \mathrm{Aut}(\mathcal{G}_\sigma, \mathcal{G}_\sigma^+)$ is injective.

As a consequence of Theorem 4.5, for a minimal Cantor system (X, T) the group $\mathcal{M}(T)$ fits into an exact sequence

$$1 \to K \to \mathcal{M}(T) \to \mathrm{Image}\,\pi_T \to 1 \tag{4.1.3}$$

where $K = \ker \pi_T$ can be identified with a subgroup of $\mathrm{Aut}(X)/\langle T \rangle$, and $\mathrm{Image}\,\pi_T$ is a subgroup of $\mathrm{Aut}(\mathcal{G}_T, \mathcal{G}_T^+)$. We remark that from a group-theoretic point of view, (4.1.3) does not provide much restriction on how 'large' $\mathcal{M}(T)$ can be. When the rank of the abelian group $\mathcal{G}_T$ is finite, the group $\mathrm{Aut}(\mathcal{G}_T, \mathcal{G}_T^+)$ embeds into $GL(k, \mathbb{R})$, where $k = \mathrm{rank}\mathcal{G}_T$. While the order-preserving condition does provide restrictions on $\mathrm{Aut}(\mathcal{G}_T, \mathcal{G}_T^+)$, even in the finite rank case the group $\mathrm{Aut}(\mathcal{G}_T, \mathcal{G}_T^+)$ need not be finitely-generated, nor amenable (see Examples 6.7 and 6.9 in [**18**]).

Corollary 4.8. *If $[f] \in \mathcal{M}(T)$ satisfies $f^*([1_C]) = [1_C]$ for some clopen $C \subset X$, then $[f] \in \mathrm{Image}\,\Psi_C$.*

Proof. Let $h\colon \Sigma_T X \to \Sigma_{T_C} C$ be a flow equivalence, giving an induced isomorphism $h_*\colon \mathcal{M}(T) \to \mathcal{M}(T_C)$. Since $res_C([1_C]) = [1] \in G_{T_C}$, the hypotheses imply the induced action of $h_*([f])$ on $\mathcal{G}_{T_C}$ sends [1] to [1]. By Theorem 4.5, this implies $h_*([f])$ is isotopic to an automorphism of (C, T_C), and hence $[f] \in \Psi_C$. □

In what follows, a *box* in $\Sigma_T X$ is a set of the form

$$\{C + t \mid t_0 - \epsilon \leq t \leq t_0 + \epsilon\}$$

for some $t_0 \in \mathbb{R}$, $\epsilon > 0$, and $C \subset X$ clopen.

Theorem 4.9. *Let (X, T) be a minimal Cantor system. If $[f] \in \mathcal{M}(T)$ is finite order then $[f] \in \mathrm{Image}\,\Psi_C$ for some clopen set $C \subset X$.*

Proof. Let $[f]$ be order $n > 1$. It follows from Proposition 4.1 that for some leaf ℓ, for $0 \leq i \leq n-1$ the leaves $f^i(\ell)$ are distinct. Choose $z_0 \in l \cap \Gamma$, and for $0 \leq i \leq n-1$ let $z_i = f^i(z_0)$. We first adjust f by an isotopy to place these points in Γ. Define $t_i = \min\{t \geq 0 \mid z_i + t_i \in \Gamma\}$, and let $y_i = z_i + t_i$. Note that since the leaves $f^i(\ell)$ are all distinct, the arcs $L_i = \{z_i + t \mid 0 \leq t \leq t_i\}$ are distinct (some of these arcs may be points, since it could be that $z_i \in \Gamma$). We claim there exist boxes B_i, for $1 \leq i \leq n-1$, which satisfy the following:

(1) for each i, B_i contains L_i, and hence also z_i, in its interior

(2) $B_i \cap B_j = \emptyset$ if $i \neq j$

(3) $B_i \cap \Gamma$ is a clopen containing y_i.

Given these disjoint B_i, we may define $\eta\colon \Sigma_T X \to \mathbb{R}$ such that supp$\eta \subset \cup_{i=1}^{n-1} B_i$ and $\eta(z_i) = t_i$. Define $S_\eta \in \text{Homeo}^+\Sigma_T X$ by $S_\eta(z) = z + \eta(z)$, so $S_\eta(z_i) = y_i$ for $0 \leq i \leq n-1$. Let $g = S_\eta f S_\eta^{-1}$ and note that, since S_η is isotopic to the identity, g is isotopic to f. Furthermore, g satisfies $g(y_i) = y_{i+1}$ for $0 \leq i \leq n-2$. For the next step, we thicken up the points $y_i \in \Gamma$ to clopen sets F_i that form a section $\cup_{i=0}^{n-1} F_i$ with $[1_F]$ preserved by $f_* = g_*$. To do this, we construct a finite sequence of small boxes D_i, for $0 \leq i \leq n-1$, which each satisfy the following properties:

(1) $D_i \cap \Gamma = E_i$ is clopen

(2) $y_i \in E_i$

(3) $g(D_i) \cap D_i = \emptyset$

(4) $D_{i+1} \subset g(D_i)$ for $0 \leq i \leq n-2$.

This can be done inductively: to begin, we can choose a box D_0 such that $D_0 \cap \Gamma = E_0$ is clopen, E_0 contains y_0, and $g(D_0) \cap D_0 = \emptyset$. Now given D_i, choose a box D_{i+1} satisfying the first three properties such that $D_{i+1} \subset g(D_i)$, using continuity, and the fact that the y_i's are all distinct. Now let $D_F = (g^{-1})^{n-1}(D_{n-1}) = g^{1-n}(D_{n-1}) \subset D_0$. Then the sets $g^i(D_F)$ are disjoint for all $0 \leq i \leq n-1$, and hence so are the clopen sets $F_i = g^i(D_F) \cap \Gamma$. We claim

$$g_*([1_{F_i}]) = [1_{F_{i+1}}], \quad 0 \leq i \leq n-2. \tag{4.1.4}$$

This can be seen either by examining the proof of Proposition 2.24 in [**16**, Section 4], or by adjusting g by a small isotopy so that g actually takes F_i to F_{i+1}. Now consider the element of $\mathcal{G}_T$

$$\omega = \sum_{i=0}^{n-1} (g^i)^*([1_{F_0}]). \tag{4.1.5}$$

Since $[f]$ is order n, $[g]$ is order n, so the order of g^* divides n, and $g^*(\omega) = \omega$. On the other hand, by (4.1.4), and using the fact that the F_i's are disjoint, we have

$$\omega = \sum_{i=0}^{n-1} [1_{F_i}] = [1_F], \quad F = \bigcup_{i=1}^{n-1} F_i.$$

Thus g^*, and hence f^*, fixes $[1_F]$, so by Corollary 4.8, $[f] \in \operatorname{Image} \Psi_F$. □

4.2. Uniquely ergodic subshifts

Recall that a system (X, T) is called uniquely ergodic if it has only one T-invariant probability measure μ (see Section 2.1.3). In this case there is a unique state τ_μ and $p(T)$ corresponds to a ray. It follows that for any $[f] \in \mathcal{M}(T)$ there exists $\lambda_f \in \mathbb{R}^*_{>0}$ such that $f_* \tau_\mu = \lambda_f \tau_\mu$ and we may identify the map L_T in (2.4.9) with

$$\mathcal{M}(T) \xrightarrow{R_\mu} \mathbb{R}^*_{>0} [f] \longmapsto \tau_\mu(f^*([1])) \tag{4.2.1}$$

where $R^*_{>0}$ denotes the group of positive reals under multiplication.

Lemma 4.10. *Suppose (X, T) is a minimal Cantor system with a unique invariant probability measure μ. If $[f] \in \mathcal{M}(T)$ and (φ, C, D) is a flow code representing $[f]$, then $R_\mu([f]) = \mu(C)/\mu(D)$.*

Proof. Since $\varphi(C) = D$, we have

$$\mu(C) = \tau_\mu([1_C]) = \tau_\mu(f^*([1_D])) = \lambda_f \tau_\mu([1_D]) = \lambda_f \mu(D).$$

□

Following Durand, Ormes, and Petite [**31**], we say a minimal Cantor system (X, T) is *self-induced* if there exists a non-empty proper clopen set $C \subset X$ such that (X, T) and (C, T_C) are conjugate. The following result of [**31**] and Mossé's results on recognizability classify expansive self-induced minimal Cantor systems.

Theorem 4.11. [**31**, Theorem 14] *The system (X, T) is a self-induced expansive minimal Cantor system if and only if (X, T) is conjugate to a substitution subshift.*

Proposition 4.12. *Suppose (X, T) is a minimal uniquely ergodic Cantor system. If there exists $f \in \mathcal{M}(T)$ such that $R_\mu([f]) \neq 1$, then (X, T) is flow equivalent to a self-induced system.*

Proof. If $f^*(1) = \lambda \neq 1$, we can assume (by passing to f^{-1} instead, if necessary) that $\lambda > 1$. Choosing a flow code (φ, C, D) representing f, by Lemma 4.10 we have $f^*(1) = \mu(C)/\mu(D) > 1$, so $\mu(C) > \mu(D)$. Since (X, T) is uniquely ergodic, we claim there exists $C' \subset C$ and conjugacy $\tilde{\varphi}\colon (D, T_D) \to (C', T_{C'})$. A proof of this can be found in the proof of Proposition 7 in [**31**] (the argument there uses the self-inducing hypothesis only to ensure that the inequality between measures holds, something that is guaranteed by our hypotheses, as just explained). The composition $\tilde{\varphi} \circ \varphi$ gives the self-induction map $C \to C'$. Since (X, T) is flow equivalent to (C, T_C), the result follows. □

Corollary 4.13. *Let* (X, σ) *be a uniquely ergodic minimal subshift which is not flow equivalent to a substitution. If* $\operatorname{Inf} \mathcal{G}_\sigma = 0$*, then* $\mathcal{M}(\sigma)$ *is isomorphic to* $\operatorname{Aut}(X)/\langle\sigma\rangle$*.*

Proof. Since we are assuming (X, σ) is not flow equivalent to a substitution, Proposition 4.12 implies $\mathcal{M}(\sigma) = \ker R_\mu$. The condition $\operatorname{Inf} \mathcal{G}_\sigma = 0$ then implies $\mathcal{M}(\sigma) = \ker \pi_\sigma$, so by Theorem 4.5 we have $\mathcal{M}(\sigma) \subset \operatorname{Image} \Psi$. □

Example 4.14. Let (X_β, σ_β) be a minimal Sturmian subshift associated to an irrational $0 < \beta < 1$ (see [**35**, Ch 6.1], [**62**]). Then $\operatorname{Inf} \mathcal{G}_{\sigma_\beta} = 0$ (this can deduced from [**71**, Theorem 5.3]), and $\operatorname{Aut}(\sigma_\beta) \cong \mathbb{Z} = \langle\sigma_\beta\rangle$ (see [**67**, Example 4.1]). A classical result (see for example [**8**, Theorem 1]) is that (X_β, σ_β) is conjugate to a substitution system if and only if β is a *Sturm number*: that is, β is a quadratic irrational whose algebraic conjugate lies outside the interval $[0, 1]$. Thus we have the following dichotomy:

(1) Suppose β is not a Sturm number. Then Corollary 4.13 and the above discussion imply $\mathcal{M}(\sigma_\beta)$ is trivial.

(2) Suppose β is a Sturm number, so (X_β, σ_β) is conjugate to a substitution subshift. Then by Theorem 4.17, $\mathcal{M}(\sigma_\beta)$ is isomorphic to $\mathcal{F} \rtimes \mathbb{Z}$ where $\mathcal{F}$ is a finite group. However $\mathcal{F}$ must be trivial: by Theorem 4.9, an element of $\mathcal{F}$ is induced by an automorphism of a return system. But this return system can have at most one pair of asymptotic points, and hence has no finite order automorphisms (by [**29**]). Thus in this case, we have $\mathcal{M}(\sigma_\beta) \cong \mathbb{Z}$.

Remark 4.15. In Example 4.14, there are non-trivial elements of the mapping class group, namely those associated to substitutions, which fix a leaf. The existence of such elements demonstrates a key difference between $\mathcal{M}(\sigma)$ and $\mathrm{Aut}(X)$, since any automorphism of (X, σ) which is not a power of σ must act freely on the σ-orbits of X.

Remark 4.16. Flow equivalent systems must have isomorphic mapping class groups. However systems with isomorphic mapping class groups need not be flow equivalent. For example, by Fokkink's Theorem [**7**], for irrationals β, γ, the Sturmian systems (X_β, σ_β), $(X_\gamma, \sigma_\gamma)$ are flow equivalent if and only if β and γ are in the same orbit of the action of $SL(2, \mathbb{Z})$. It follows then from Example 4.14 that there are uncountably many Sturmian systems, each of whose mapping class group is trivial, which are pairwise not flow equivalent.

4.3. The mapping class group of a substitution system

Throughout this section, let ξ denote a primitive aperiodic substitution on a finite alphabet $\{1, \cdots, d\}$, and (X, σ) is the minimal subshift associated to ξ. The first main goal of this section is to prove the following:

Theorem 4.17. *Suppose (X, σ) is a minimal subshift associated to a primitive substitution ξ. Then $\mathcal{M}(\sigma)$ fits into an exact sequence*

$$1 \to \mathcal{F} \to \mathcal{M}(\sigma) \xrightarrow{R_\mu} \mathbb{Z} \to 1 \tag{4.3.1}$$

where $\mathcal{F}$ is a finite group.

Since the group on the right in (4.3.1) is $\mathbb{Z}$, the sequence splits. Choosing a splitting map $s\colon \mathbb{Z} \to \mathcal{M}(\sigma)$ gives a presentation of $\mathcal{M}(\sigma)$ as a semidirect product $\mathcal{M}(\sigma) \cong \mathcal{F} \rtimes \mathbb{Z}$, and the associated structure map $\mathbb{Z} \to \mathrm{Aut}(\mathcal{F})$ can be determined by examining the action of $s(1)$ on the asymptotic leaves of $\Sigma_\sigma X$. It also follows that $\mathcal{M}(\sigma)$ is virtually $\mathbb{Z}$. Let us say a bit more about the group $\mathcal{F}$. The proof of Theorem 4.17 shows that $\mathcal{F}$ always embeds as a subgroup of the permutation group on the set of asymptotic classes, which is finite. Moreover, it follows from Theorem 4.9 that $\mathcal{F}$ consists of self-flow equivalences induced by automorphisms of return systems of (X, σ). The second main goal of this section is to show that, for a large class of substitutions which we now define, we can identify $\mathcal{F}$ with the group $\mathrm{Aut}(\sigma)/\langle\sigma\rangle$. Before defining this class, we first briefly introduce the necessary notation. Define the roof function $r_{PF}\colon X \to \mathbb{R}$ by $r_{PF}(x) = v_{x_0}$, where x_0 denotes the symbol of x at the 0^{th} coordinate. We denote the suspension $\Sigma_{T_\xi}^{r_{PF}} X$ by Ω_ξ; the space Ω_ξ is also known as the tiling space, or hull, associated to an aperiodic tiling of the real line which can be constructed from the data $\xi, v_{PF}^{(l)}$. There is a homeomorphism (see [**20**, Theorem 1.1])

$$h_\Omega\colon \Sigma_\sigma X \to \Omega_\xi h_\Omega\colon (x,t) \longmapsto (x, t \cdot r_{PF}(x)) \tag{4.3.2}$$

and the induced map $(h_\Omega)_*\colon \mathrm{Homeo}^+\Sigma_\sigma X \to \mathrm{Homeo}^+\Omega_\xi$ takes S_ξ, the map on $\Sigma_\sigma X$ induced by the substitution ξ, to a substitution map which we denote by ξ_Ω. The roof function r_{PF} is distinguished in that Ω_ξ makes ξ_Ω into a self-similarity, in the sense that

$$\xi_\Omega(x+t) = \xi_\Omega(x) + \lambda_\xi t$$

for all $x \in \Omega_\xi, t \in \mathbb{R}$ (see [**4**]).

For a constant $c > 0$, let r_c denote the constant roof function $r_c(x) = c$. For any $c > 0$, there is a homeomorphism

$$h_c\colon \Sigma_T X \to \Sigma_T^{r_c} X h_c\colon (x,t) \mapsto (x, ct). \tag{4.3.3}$$

We say a substitution ξ is of *type CR* if there exists a constant $c > 0$ (which may depend on ξ) and a conjugacy $h_{CR}\colon (\Omega_\xi, \mathbb{R}) \to (\Sigma_T^{r_c} X, \mathbb{R})$ satisfying the following:

$$\big(h_{CR} \circ h_\Omega\big)(l) = h_c(l) \text{ for every leaf } l \in \Sigma_\sigma X. \tag{4.3.4}$$

We denote by $\mathcal{S}_{CR}$ denote the collection of substitutions of type CR.

Any substitution whose left Perron-Frobenius eigenvector is a scalar multiple of $\mathbf{1}$ (where $\mathbf{1}$ is the vector of all 1s) is clearly of type CR (such substitutions are precisely those dual to a constant length substitution, i.e. there is a constant K such that for every letter $j \in \mathcal{A}$, $\sum_i M_{i,j} = K$). More generally, it follows from [**20**, Theorem 3.1] that if the incidence matrix M_ξ satisfies, for some $\alpha > 0$,

$$(\alpha \cdot v_{PF}^{(l)} - \mathbf{1}) M_\xi^m \xrightarrow{m\to\infty} 0$$

then ξ is of type CR. From this one can deduce (see [**20**, Corollary 3.2]) that substitutions of Pisot type (i.e. substitutions whose Perron-Frobenius eigenvalue is a Pisot number) are of type CR.

Theorem 4.18. *Let ξ be a primitive substitution and (X,σ) the minimal subshift associated to ξ. If ξ is of type CR then the sequence*

$$1 \to \mathrm{Aut}(\sigma)/\langle\sigma\rangle \xrightarrow{\Psi} \mathcal{M}(\sigma) \xrightarrow{R_\mu} \mathbb{Z} \to 1 \tag{4.3.5}$$

is exact.

Remark 4.19. Analogous to (4.3.1), the sequence (4.3.5) also splits, and hence in the CR case $\mathcal{M}(\sigma)$ is isomorphic to the semidirect product $\mathrm{Aut}(\sigma)/\langle\sigma\rangle \rtimes \mathbb{Z}$.

We now prepare for the proofs of Theorems 4.17 and 4.18. Let r be a roof function and suppose $(\Sigma_\sigma^r X, \mathbb{R})$ has a unique $\mathbb{R}$-invariant probability measure μ_r. For the system $(\Omega_\xi, \mathbb{R})$ we denote this measure by μ_ξ. Given $f \in \mathrm{Homeo}^+\Sigma_\sigma^r X$ there exists $\lambda_f \in \mathbb{R}$ such that for any $x \in \Sigma_\sigma^r X$

$$\frac{1}{t}\alpha_f(x,t) \xrightarrow{t\to\infty} \lambda_f.$$

This can be shown using the Ergodic Theorem (see [**56**, Lemma 4.1] for details). Recall for a space Y we denote by $\pi^1(Y)$ the group $[Y, S^1]$. A small variation of standard arguments using bump functions shows that the collection of functions in $C(\Sigma_T^r X, S^1)$ which are leaf-wise smooth (with respect to the leaf-wise derivative $df(z) = \lim_{t\to 0} \frac{f(z+t)-f(z)}{t}$) are uniformly dense. It follows that, for any class $[g] \in \pi^1(\Sigma_T^r X)$, we may choose a leaf-wise smooth representative $g_s\colon \Sigma_T^r X \to S^1$ of $[g]$ and define a homomorphism

$$C_{\mu_r}\colon \pi^1(\Sigma_T^r X) \longrightarrow \mathbb{R} C_{\mu_r}\colon [g] \longmapsto \int_{\Sigma_T^r X} \frac{1}{2\pi i} g_s^{-1} dg_s \; d\mu_r. \tag{4.3.6}$$

Upon identifying $\pi^1(\Sigma_T^r X)$ with $\check{H}^1(\Sigma_T^r X, \mathbb{Z})$, the map C_{μ_r} is the degree one part of the Ruelle-Sullivan map (see [**49**]).

The relevant cases for us we notate by

$$C_\Omega \coloneqq C_{\mu_\xi} \colon \pi^1(\Omega_\xi) \to \mathbb{R} C_{\Sigma_\sigma X} \coloneqq C_{\mu_1} \colon \pi^1(\Sigma_\sigma X) \to \mathbb{R}.$$

and denote the kernels by

$$\mathrm{Inf}_\Omega = \ker C_\Omega \mathrm{Inf}_{\Sigma_\sigma X} = \ker C_{\Sigma_\sigma X}.$$

Define $1_{\Sigma_\sigma X} \in C(\Sigma_\sigma X, S^1)$ by $1_{\Sigma_\sigma X}(x,t) = e^{2\pi i t}$. Recall $h_\Omega \colon \Sigma_\sigma X \to \Omega_\xi$ denotes the homeomorphism defined in (4.3.2).

Lemma 4.20. *The isomorphism $(h_\Omega^*)^{-1} \colon \pi^1(\Sigma_\sigma X) \to \pi^1(\Omega_\xi)$ induced by the homeomorphism $h_\Omega^{-1} \colon \Omega_\xi \to \Sigma_\sigma X$ satisfies $(h_\Omega^*)^{-1}(\mathrm{Inf}_{\Sigma_\sigma X}) \subset \mathrm{Inf}_\Omega$.*

Proof. Let $a \in \mathrm{Inf}_{\Sigma_\sigma X}$. By [**16**, Sec. 1.9], a satisfies $na \leq [1_{\Sigma_\sigma X}]$ for all $n \in \mathbb{Z}$, or equivalently, $[1_{\Sigma_\sigma X}] - na \geq 0$ for all n. Since $(h_\Omega^{-1})^*$ is order preserving, we have $(h_\Omega^{-1})^*([1_{\Sigma_\sigma X}] - na) \geq 0$ for all n, and hence $(h_\Omega^{-1})^*([1_{\Sigma_\sigma X}]) - n(h_\Omega^{-1})^*(a) \geq 0$. Note that $(h_\Omega^{-1})_*([1_{\Sigma_\sigma X}]) \geq 0$. Then $(h_\Omega^{-1})^*([1_{\Sigma_\sigma X}]) \geq n(h_\Omega^{-1})^*(a)$ for all $n \in \mathbb{Z}$, which implies $C_\Omega(h_\Omega^{-1})^*(a) = 0$, so $(h_\Omega^{-1})^*(a) \in \mathrm{Inf}_\Omega$. $\square$

Recall the isomorphism $\mathcal{G}_\sigma \xrightarrow{\cong} \pi^1(\Sigma_\sigma X)$ of Proposition 2.24 given by

$$[\gamma] \longmapsto [\eta_\gamma], \;\; \eta_\gamma(x,t) = e^{2\pi i t \gamma(x)}.$$

Since $\eta_\gamma^{-1} d\eta_\gamma(x,t) = 2\pi i \gamma(x)$, by (2.4.8) the isomorphism $\mathcal{G}_\sigma \xrightarrow{\cong} \pi^1(\Sigma_\sigma X)$ takes $\mathrm{Inf}(\mathcal{G}_\sigma)$ to $\mathrm{Inf}_{\Sigma_\sigma X}$. Define $1_\Omega \in C(\Omega_\xi, S^1)$ by $1_\Omega(x,t) = e^{2\pi i r_{PF}(x)^{-1} t}$, and note that $C_\Omega([1_\Omega]) = 1$.

It follows from the Ergodic Theorem that $\lambda_f = C_{\Sigma_\sigma^r X}(f^*([1_\Omega]))$ (see for example [**47**, Sec. 7]).

For any homeomorphism $g\colon \Sigma_T^{r_1} X \to \Sigma_T^{r_2} X$ there is an induced isomorphism of topological groups which we denote by

$$g_*\colon \mathrm{Homeo}^+\Sigma_T^{r_1} X \to \mathrm{Homeo}^+\Sigma_T^{r_2} X g_*(f) = gfg^{-1}.$$

Thus for $h_\Omega\colon \Sigma_\sigma X \to \Omega_\xi$ we have

$$(h_\Omega)*\colon \mathrm{Homeo}^+\Sigma_\sigma X \to \mathrm{Homeo}^+\Omega_\xi (h_\Omega)_*(f) = h_\Omega f h_\Omega^{-1}.$$

Lemma 4.21. *If* $[f] \in \ker R_\mu$*, then* $\lambda_{(h_\Omega)_*(f)} = 1$*.*

Proof. Since $[f] \in \ker R_\mu$, for any $a \in \mathcal{G}_\sigma$ we have $f^*(a) - a \in \mathrm{Inf}\,(\mathcal{G}_T)$, and hence $f^* h_\Omega^*(1_\Omega) - h_\Omega^*(1_\Omega) \in \mathrm{Inf}_{\Sigma_\sigma X}$. Write $f^* h_\Omega^*(1_\Omega) = h_\Omega^*(1_\Omega) + b$ for some $b \in \mathrm{Inf}_{\Sigma_\sigma X}$. Then

$$C_\Omega\Big(\big((h_\Omega)_*(f)\big)^*(1_\Omega)\Big) = C_\Omega\big((h_\Omega^{-1})^* f^* h_\Omega^*(1_\Omega)\big)$$

$$= C_\Omega\big((h_\Omega^{-1})^*(h_\Omega^*(1_\Omega) + b)\big) = C_\Omega\big(1_\Omega + (h_\Omega^{-1})^*(b)\big) = C_\Omega(1_\Omega) = 1$$

where we have used that $C_\Omega((h_\Omega^{-1})^*(b)) = 0$ by Lemma 4.20. Thus

$$C_\Omega\Big(\big((h_\Omega)_*(f)\big)^*(1_\Omega)\Big) = 1,$$

and $\lambda_{(h_\Omega)_*(f)} = 1$. □

Finally, we note that for a primitive substitution ξ, $\Sigma_\sigma X$ always has a finite and non-zero number of asymptotic leaves (see either [**73**, V.21] or [**5**]).

The following lemma plays an important role.

Lemma 4.22. *Suppose (X,σ) is a minimal substitution subshift and $[f] \in \mathcal{M}(\sigma)$. If $[f] \in \ker R_\mu$ and f fixes an asymptotic leaf, then f is isotopic to the identity.*

Proof. Suppose $[f] \in \ker R_\mu$ and fixes an asymptotic leaf ℓ. For notational reasons, let us use the notation

$$(h_\Omega)_*(f) = f' \in \mathrm{Homeo}^+\Omega_\xi.$$

Since the number of asymptotic leaves is finite, there exists $j \in \mathbb{N}$ such that ξ_Ω^j fixes every asymptotic leaf. Consider the sequence of homeomorphisms

$$g'_m = (\xi_\Omega^{-j})^m f' (\xi_\Omega^j)^m = \xi_\Omega^{-jm} f' \xi_\Omega^{jm}.$$

By [**56**, Lemma 5.4], this sequence is equicontinuous in $\mathrm{Homeo}^+\Omega_\xi$, and by Arzela-Ascoli converges along some subsequence m_k to some $f'_\infty \in \mathrm{Homeo}^+\Omega_\xi$. Moreover, the homeomorphism f'_∞ satisfies $\alpha_{f'_\infty}(x,t) = \lambda_{f'} t$ for all $x \in \Omega_\xi, t \in \mathbb{R}$ (see Theorem 5.3 and its proof in [**56**]). Since $[f] \in \ker R_\mu$, Lemma 4.21 implies $\lambda_{f'} = 1$ and we have

$$\alpha_{f'_\infty}(x,t) = t \text{ for all } x \in \Omega_\xi, t \in \mathbb{R}. \tag{4.3.7}$$

Since $h_\Omega(\ell)$ is an asymptotic leaf in Ω_ξ, ξ^j fixes $h_\Omega(\ell)$. Then, recalling that f fixes ℓ, we have that g'_m, and thus f'_∞, fixes $h_\Omega(\ell)$. Since the leaf $h_\Omega(\ell)$ is dense in Ω_ξ, (4.3.7) implies f'_∞ is isotopic to the identity. Finally, we show f is isotopic to the identity. Let $g_k = (h_\Omega)_*^{-1}(g'_{m_k})$ and define $f_\infty = (h_\Omega)_*^{-1}((f')_\infty)$. Then $g_k \to f_\infty$ in $\mathrm{Homeo}^+\Sigma_\sigma X$ and f_∞ is isotopic to the identity. By [**2**, Prop. 1.2], the path component of the identity in

$\text{Homeo}^+\Sigma_\sigma X$ is open, so for some k we have g_k is isotopic to the identity. But this implies $S_\xi^{-jm_k} f S_\xi^{jm_k}$ is isotopic to the identity, and hence f is as well. □

We now prove Theorem 4.17.

Proof of Theorem 4.17. By Lemma 4.22, the restriction of the asymptotic leaf representation (defined in (2.2.2)) to $\ker R_\mu$

$$\pi_{as}|_{\ker R_\mu} \colon \ker R_\mu \to P(\text{as}(\sigma))$$

is injective. Since $\text{as}(T)$ is finite, $P(\text{as}(\sigma))$ is finite, showing that $\mathcal{F}$ is finite. It remains to show that $\text{Image}\, R_\mu \subset \mathbb{R}^*$ is isomorphic to $\mathbb{Z}$. Before doing this, we record the following folklore lemma. Recall $v_{pf}^{(l)} = (v_1, v_2, \ldots, v_d)$ denotes a normalized (i.e. $\sum_i v_i = 1$) left Perron-Frobenius eigenvector for the incidence matrix M_ξ, and let $s_{pf} \in \mathbb{Z}[\lambda_\xi]$ be such that $s_{pf} \cdot v_i \in \mathbb{Z}[\lambda_\xi]$ for all $1 \le i \le d$.

Lemma 4.23. *Let ξ be an aperiodic primitive substitution and let μ denote the unique invariant probability measure for (X, T_ξ). Then*

(4.3.8)
$$\text{Image}\, \tau_\mu = \{\frac{1}{\lambda_\xi^m} \sum_{i=0}^{d} n_i v_i \mid m \ge 0, n_i \in \mathbb{Z}\}.$$

Continuing the proof of Theorem 4.17, let $\mathfrak{S}_\xi = \{(\lambda_\xi)^m \cdot s_{pf}^n \mid m, n \ge 0\} \subset \mathbb{Z}[\lambda_\xi]$, and consider the localization $\mathfrak{S}_\xi^{-1}\mathbb{Z}[\lambda_\xi]$. Note that $\text{Image}\, \tau_\mu \subset \mathfrak{S}_\xi^{-1}\mathbb{Z}[\lambda_\xi]$ by Lemma 4.23, so that $\text{Image}\, R_\mu$ is contained in the group of units of $\mathfrak{S}_\xi^{-1}\mathbb{Z}[\lambda_\xi]$. By the S-unit Theorem ([**63**, III.3.5]), the group of units in $\mathfrak{S}_\xi^{-1}\mathbb{Z}[\lambda_\xi]$ is finitely-generated. It follows that $\text{Image}\, R_\mu$ is a finitely-generated free abelian group. All that remains then is to show that $\text{Image}\, R_\mu$ is rank one. To do this, we show the following: for any $\alpha \in \text{Image}(\text{R}_\mu)$ there exists $p, q \in \mathbb{N}$

such that $\alpha^p = \lambda_\xi^q$. Given this, it follows that Image $R_\mu/\langle\lambda_\xi\rangle$ is torsion and hence rank zero, so Image R_μ is rank one as desired.

Suppose $\alpha \in \mathrm{Image(R_\mu)}$ and let (φ, C, D) be a flow code with $R_\mu(\varphi) = \alpha$. By considering φ^{-1} instead if necessary, we can assume that $\alpha > 1$. Lemma 4.10 implies $\mu(D) < \mu(C)$, so there exists (see the proof of [**31**, Prop. 7]) a clopen $E \subset C$ with $\mu(E) = \mu(D)$ such that (C, T_C) is conjugate to (E, T_E). By Proposition 4.12, (C, T_C) is thus a self-induced system. Note that (C, T_C) is also uniquely ergodic, with unique T_C-invariant measure $\bar{\mu}$ given by $\bar{\mu}(A) = \mu(A)/\mu(C)$ for any $A \subset C$.

By [**31**, Theorem 14], (C, T_C) must be conjugate to an aperiodic, primitive substitution (Y, S), given by $\theta : \mathcal{B} \to \mathcal{B}^*$ for some alphabet $\mathcal{B}$. Let $g\colon C \to Y$ denote such a conjugacy. The proof of [**31**, Theorem 14] shows that that for any $b \in \mathcal{B}$, $|\theta(b)| = r_E(x)$, where $x \in E$ satisfies $g(x)_0 = b$. The value of the return time is well-defined, regardless of the choice of x. The induced transformation for the system $(\theta(Y), S_{\theta(Y)})$ is given by the lengths of substitution θ (see [**73**, Cor. 5.11])

$$S_{\theta(Y)}(z) = S^{|\theta(y_0)|}(y) \text{ where } z = \theta(y), y \in Y.$$

It follows that the return times of x to E are precisely the return times of $z = \theta(g(x))$ to $\theta(Y)$.

Computing the Birkhoff sums

$$\frac{1}{N}\sum_{i=0}^{N-1} r_U(T_U^i x) = \frac{1}{N}\sum_{i=0}^{N-1} r_{\theta(Y)}(S_{\theta(Y)}^i z),$$

we see that $\bar{\mu}(E) = \nu(\theta(Y))$, where ν is the unique invariant probability measure on (Y, S). Thus, the Perron-Frobenius eigenvalue associated to θ is given by $1/\bar{\mu}(E) = \alpha$.

By construction, (X,T) and (Y,S) are aperiodic primitive substitution systems which are flow equivalent. Since $\Sigma_T X$ is homeomorphic to Ω_ξ and $\Sigma_S Y$ is homeomorphic to Ω_θ, it follows that Ω_ξ and Ω_θ are homeomorphic. Then by [**6**, Theorem 2.1], there must exist $m, n \in \mathbb{N}$ such that ξ_Ω^m and θ_Ω^n are topologically conjugate. Since the entropy of ξ_Ω and θ_Ω are given by λ, α respectively, we have $\lambda^m = \alpha^n$. □

Remark 4.24. For a substitution ξ, the element $[S_\xi] \in \mathcal{M}(\sigma)$ may or may not map to a generator of $\mathbb{Z}$ under the R_μ. For example, for any substitution ξ and $p \in \mathbb{N}$, let ξ^p denote the substitution obtained by composing ξ with itself p times. Then ξ itself gives a flow code defining a map $S_\xi \in \text{Homeo}^+\Sigma_{\sigma_{\xi^p}} X_{\xi^P}$, and the element $[S_\xi] \in \mathcal{M}(\sigma_{\xi^p})$ satisfies $(R_\mu([S_\xi])^p = R_\mu([S_{\xi^p}])$.

We now prove Theorem 4.18.

Proof of Theorem 4.18. In light of Theorem 4.17, we need only show that $\ker R_\mu$ is the image of Ψ. Let $[f] \in \ker R_\mu$ and consider $f' = (h_\Omega)_*(f)$ in $\text{Homeo}^+\Omega_\xi$. As in the proof of Lemma 4.22, the sequence $g'_{m_k} = \xi_\Omega^{-jm_k} f' \xi_\Omega^{jm_k}$ converges in $\text{Homeo}^+\Omega_\xi$ to a homeomorphism f'_∞ for which $\alpha_{f'_\infty}(x,t) = t$ for all $x \in \Omega_\xi, t \in \mathbb{R}$. Since ξ is of type CR, there exists $c > 0$ and a conjugacy $h_{CR} \colon \Omega_\xi \to \Sigma_T^{r_c} X$ inducing an isomorphism of topological groups $(h_{CR})_* \colon \text{Homeo}^+\Omega_\xi \to \text{Homeo}^+\Sigma_T^{r_c} X$. Let $f_a = (h_{CR})_*(f'_\infty)$. Since h_{CR} is a conjugacy, $h_{CR}(z+s) = h_{CR}(z) + s$ for all $z \in \Omega_\xi, s \in \mathbb{R}$. Thus f_a is isotopic to a map preserving the cross section $X \times \{0\}$ in $\Sigma_\sigma^{r_c} X$, and an argument analogous to the one given in Proposition 4.3 shows there exists an automorphism $\varphi \in \text{Aut}(X)$ such that f_a is isotopic to the map $\tilde{\varphi} \in \mathrm{Homeo}^{+}\Sigma_{\sigma}^{r_c} X$ defined by

$$\tilde{\varphi}\colon (x, s) \longmapsto (\varphi(x), s).$$

Let ℓ be an asymptotic leaf in $\Sigma_\sigma X$. We claim $f^{-1}\Psi(\varphi)$ in $\mathrm{Homeo}^{+}\Sigma_T X$ fixes ℓ. To see this, note that since ξ_Ω^j fixes every asymptotic leaf in Ω_ξ, we have $g'_{m_k}(h_\Omega(\ell)) = h_\Omega(f(\ell))$ for all m_k, and hence $f'_\infty(h_\Omega(\ell)) = h_\Omega(f(\ell))$. Likewise, $f_a\big(h_{CR}h_\Omega(\ell)\big) = \big(h_{CR}h_\Omega\big)(f(\ell))$. By condition (4.3.4), this implies $f_a(h_c(\ell)) = h_c(f(\ell))$, and since f_a and $\tilde{\varphi}$ are isotopic, we have $\tilde{\varphi}(h_c(\ell)) = h_c(f(\ell))$. One can now check that $\tilde{\varphi}(h_c(\ell)) = h_c(\Psi(\varphi)(\ell))$, giving $f(\ell) = \Psi(\varphi)(\ell)$, so $f^{-1}\Psi(\varphi)$ fixes ℓ as desired. Now to finish the proof, since both f and $\Psi(\varphi)$ are in $\ker R_\mu$, $f^{-1}\Psi(\varphi) \in \ker R_\mu$. By the above, $f^{-1}\Psi(\varphi)$ also fixes an asymptotic leaf, so by Lemma 4.22, $f^{-1}\Psi(\varphi)$ is isotopic to the identity, and $[f] = [\Psi(\varphi)]$ in $\mathcal{M}(\sigma)$. □

4.4. The mapping class group of a subshift of linear complexity

We consider in this section the mapping class group associated to subshifts whose complexity function grows linearly. We note that this is a more general setting than the primitive substitution subshifts of the previous section; while substitution subshifts have linear growth of complexity, the collection of subshifts of linear complexity is a significantly larger class. On the other hand, we must now assume a vanishing condition on the infinitesimals. The main result is the following.

Theorem 4.25. *Let (X, σ) be a minimal subshift satisfying*

$$\liminf_n \frac{P_X(n)}{n} < \infty.$$

If $\mathrm{Inf}\,\mathcal{G}_\sigma = 0$, then $\mathcal{M}(\sigma)$ is virtually abelian.

Remark 4.26. Any uniquely ergodic Cantor minimal system (X, T) where $(\mathcal{G}_T, \mathcal{G}_T^+)$ is totally ordered satisfies $\operatorname{Inf} \mathcal{G}_T = 0$. In particular, Theorem 4.25 applies to the collection of symbolic interval exchange transformations whose coinvariants are totally ordered. In [**66**, Theorem 1] it is shown that if $(H, H^+, [u])$ is any simple and totally ordered dimension group which is free abelian of finite rank, then there exists an interval exchange transformation (Y_H, S_H) whose natural symbolic cover has coinvariants which are isomorphic to $(H, H^+, [u])$.

Before beginning the proof of Theorem 4.25, let us make some comments about our approach. To summarize, there are two key finiteness results for subshifts with linear complexity which we make use of: namely, finitely many asymptotic leaves (Lemma 4.28), and finitely many ergodic probability measures (Theorem 4.27). Any $[f] \in \mathcal{M}(\sigma)$ then induces some permutation on the finite set of asymptotic leaves and the finite set of rays in the state space corresponding to extremal traces (at the end of the section, we give an example of an automorphism of a minimal system permuting two ergodic measures, showing this action can be non-trivial). What we then seek is a lemma analogous to Lemma 4.22. However, the presence of non-trivial infinitesimals provides meaningful obstructions to a result like Lemma 4.22 in general, and we must settle for Lemma 4.29. While substitution systems can have non-trivial infinitesimal subgroups (for example the Thue-Morse system), it is the substitution map (in particular, the self-similar version on the space Ω_ξ) and the ironing out procedure (as employed in [**56**]), that lets Lemma 4.22 avoid dealing with infinitesimals. (We remark that, while it is still possibly to manually smooth out the cocycle by repeatedly refining the domain of the flow code, this approach does not work for us; in particular, we do not invoke Lemma 6.1 of [**56**], as the author has confirmed with us in a personal communication that the proof of the Lemma there is incomplete.)

We first summarize some preliminary results on the finiteness of ergodic measures.

Theorem 4.27 (Boshernitzan [**10**], Cyr-Kra [**24**]). *Suppose (X, σ) is a transitive subshift and there exists $k \in \mathbb{N}$ satisfying*

$$\liminf_n \frac{P_X(n)}{n} < k.$$

Then (X, σ) has at most $k-1$ ergodic non-atomic probability measures.

We note that Boshernitzan originally proved this for minimal subshifts, and Cyr and Kra generalized to transitive subshifts.

In a minimal Cantor system, asymptotic points are in one-to-one correspondence with asymptotic leaves in the suspension. This means that the following result, due to Cyr and Kra (for transitive) and Donoso et al. (for minimal) independently, implies for subshifts of linear complexity, their suspensions have finitely many asymptotic components.

Lemma 4.28 ([**24, 29**]). *If (X, σ) is a transitive subshift for which*

$$\liminf_n \frac{P_X(n)}{n} < \infty$$

then the number of asymptotic orbits in (X, σ), and hence the number of asymptotic leaves in $\Sigma_\sigma X$, is finite.

Recall $m(\sigma)$ denotes the space of σ-invariant finite Borel measures. By Theorem 4.27 we may assume (X, σ) has K ergodic probability measures which we denote by $\{\mu_i\}_{i=1}^K$.

As defined in Chapter 2.4.2, there is a representation of $\mathcal{M}(\sigma)$ on the space of positive homomorphisms

$$L_\sigma \colon \mathcal{M}(\sigma) \to \mathrm{Aut}(p(\sigma)) L_\sigma([f]) = f_*$$

Upon choosing an isomorphism between $\mathbb{R}^K$ and the linear space spanned by the collection $\{\mu_i\}_{i=1}^K$, the map L_σ gives rise to a matrix representation

$$\mathcal{L}_\sigma \colon \mathcal{M}(\sigma) \to GL_K(\mathbb{R}).$$

Moreover, for any $[f] \in \mathcal{M}(\sigma)$, $L_\sigma([f])$ must preserve the extremal elements of $p(\sigma)$ and hence preserves the set of positive half-lines spanned by the τ_{μ_i}'s. It follows that the map $\mathcal{L}_\sigma$ lands in the subgroup $GL_K^{perm}(\mathbb{R})$ of generalized permutation matrices, i.e. matrices which have exactly one non-zero entry in each row and column. Let $D \subset GL_K^{perm}(\mathbb{R})$ denote the subgroup of diagonal matrices. The subgroup D is normal, and $GL_K^{perm}(\mathbb{R})$ fits in to an extension

$$1 \to D \to GL_K^{perm}(\mathbb{R}) \to S_K \to 1$$

where S_K denotes the permutation group on K symbols. Recall $\pi_{\mathrm{as}(\sigma)} \colon \mathcal{M}(\sigma) \to P(\mathrm{as}(\sigma))$ denotes the map defined in (2.2.2) taking $\mathcal{M}(\sigma)$ to the permutation group on asymptotic leaves. Let $F(\sigma)$ denote the kernel of $\pi_{\mathrm{as}(\sigma)}$. When (X, σ) has finitely many asymptotic equivalence classes, $F(\sigma)$ is a finite index subgroup. The following lemma plays a key role.

Lemma 4.29. *Let (X,T) be a minimal Cantor system such that* $\operatorname{Inf}\mathcal{G}_T = 0$*. Suppose $f \in \mathrm{Homeo}^+\Sigma_T X$ fixes a leaf l, and satisfies $f_*\mu = \mu$ for all ergodic measures $\mu \in m(T)$. Then f is isotopic to the identity.*

Proof. Since $f_*\mu = \mu$ for all ergodic measures μ, it follows that $\tau_\mu(f^*[1]) = f^*([1])$ for all extremal states τ_μ, and hence $\tau(f^*([1])) = f^*([1])$ for all states $\tau \in \mathcal{S}(T)$. This implies $f^*([1]) - [1] \in \mathrm{Inf}\mathcal{G}_T$, and hence $f^*([1]) = [1]$ since we are assuming $\mathrm{Inf}\mathcal{G}_T = 0$. Then Theorem 4.5 implies $[f] \in \mathrm{Image}\,\Psi$, so $[f] = \Psi(\alpha)$ for some $\alpha \in \mathrm{Aut}(X)$. Since f fixes a leaf, α must be a power of T, so f is isotopic to the identity. □

Proof of Theorem 4.25. Let $N = \mathcal{L}_\sigma^{-1}(D) \cap F(\sigma)$. We first show that N is a finite index normal subgroup of $F(\sigma)$. Consider the map Q defined to be the composition map

$$F(\sigma) \xrightarrow{\mathcal{L}_\sigma} GL_K^{perm}(\mathbb{R}) \to S_K.$$

Since $D = \ker(GL_K^{perm}(\mathbb{R}) \to S_K)$ and $N \subset \mathcal{L}_\sigma^{-1}(D)$, we have $N \subset \ker Q$. By Lemma 4.29, $\mathcal{L}_\sigma|_{F(\sigma)}$ is injective, and it follows that if $a \in \ker Q$, then $\mathcal{L}_\sigma(a) \in D$. Thus $\ker Q = N$, and since Q maps to a finite group, N is finite index. Since N is finite index in $F(\sigma)$ and $F(\sigma)$ is finite index in $\mathcal{M}(\sigma)$, N is finite index in $\mathcal{M}(\sigma)$. Finally, since $\mathcal{L}_\sigma|_N$ is injective and its image lies in D, N must also be abelian. □

Example 4.30 (Thue-Morse)**.** Consider the Thue-Morse substitution given by

$$\xi\colon\ 1 \mapsto 122 \mapsto 21$$

By Coven et al. [**21**], $\mathrm{Aut}(\sigma)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by the involution ι which permutes the letters and $\mathbb{Z}$ is generated by σ. Since ξ is of type

CR (see Definition 4.3) by Theorem 4.18 we have

$$\mathcal{M}(\sigma) \cong \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}.$$

In this case the involution ι commutes with the substitution, so in fact

$$\mathcal{M}(\sigma) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.$$

We note that, while $\mathrm{Aut}(\sigma)$ and $\mathcal{M}(\sigma)$ are abstractly isomorphic in this example, the generators for the $\mathbb{Z}$ components are very different:

$$\mathrm{Aut}(\sigma) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} = \langle \iota \rangle \times \langle \sigma \rangle \mathcal{M}(\sigma) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} = \langle \Psi(\iota) \rangle \times \langle \tilde{\xi} \rangle.$$

We also remark that the involution $\Psi(\iota) \in \mathcal{M}(\sigma)$ acts trivially on $(\mathcal{G}_\sigma, \mathcal{G}_\sigma^+)$, as can be checked directly.

The following example shows that $\mathrm{Aut}(X)$, and hence $\mathcal{M}(\sigma)$, can permute the ergodic measures of (X, σ). The construction is based on techniques from [**27**].

Example 4.31. We construct a minimal subshift of linear complexity and an automorphism which permutes two ergodic measures. Define base words

$$w_0^1 = \underbrace{0 \cdots 0}_{N_1} 1, \quad w_1^1 = \underbrace{1 \cdots 1}_{N_1} 0$$

where $N_1 \geq 1$. Note that w_0^1 is the image of w_1^1 under the involution $0 \leftrightarrow 1$. We define the second level words

$$w_0^2 = \underbrace{w_0^0 \cdots w_0^0}_{N_2} w_1^0, \quad w_1^2 = \underbrace{w_1^1 \cdots w_1^1}_{N_2} w_0^1$$

where $N_2 > N_1$ and recursively define $(i+1)$st level words by

$$w_0^{i+1} = \underbrace{w_0^i \cdots w_0^i}_{N_{i+1}} w_1^i, \quad w_1^{i+1} = \underbrace{w_1^i \cdots w_1^i}_{N_{i+1}} w_0^i$$

where N_i is a positive sequence with $N_{i+1} > N_i$ satisfying a growth condition to be stipulated later. At each stage, note that w_0^i is image of w_1^i under the involution $0 \leftrightarrow 1$. Consider the infinite word $w = \lim_{i\to\infty} w_0^i$. By construction, w is uniformly recurrent, so the subshift $(\overline{\mathcal{O}(w)}, \sigma)$ is minimal. Since the involution of w_0^i is contained in w_0^{i+1}, the involution of w is in the orbit closure of w. Thus, the 0-block code which permutes 0 and 1 induces an automorphism $\pi \in \mathcal{O}(w)$. The frequency of 0s which appear in w_0^i is greater than $\dfrac{N_1}{N_1+1} \cdots \dfrac{N_i}{N_i+1}$. We can choose $\{N_i\}$ to grow sufficiently quickly (for example geometrically) to ensure that the frequency of 0s in each w_0^i is greater than $1/2$ and that the complexity is linear. A standard construction then gives an ergodic measure μ_0 such that $\mu_0[0] > 1/2$ (see [**27**]). Similarly, by considering w_1^i, we get a distinct ergodic measure μ_1 with $\mu_1([1]) > 1/2$. Since the automorphism π maps the cylinder set $[0]$ to $[1]$, it must send μ_0 to μ_1.

4.5. Open questions

(1) Given a minimal d-IET, the group of coinvariants is isomorphic to $(\mathbb{Z}^d, \mathbb{Z}^d_+)$, generated by indicator functions of the subintervals $\mathbb{1}_{\Delta_i}$ (see Example 2.7). Thus,

if the lengths are totally irrational (there are no rational relations except that they sum to 1), there are no non-trivial infintesimals. In this case, Theorem 4.25 and Corollary 4.13 apply. This is true for a full measure set of IETs. However, there are quite interesting examples of IETs with rationally dependent lengths, for example those arising from powers of IETs. Since the coding of an IET is very structured, can we get around the obstructions of infintesimals in these cases to compute the mapping class group?

(2) There is a full measure set of uniquely ergodic IETs. For IETs which have rationally independent lengths but have multiple ergodic measures, we know that Theorem 4.25 applies, and the mapping class group is virtually abelian. Can we prove a stronger result for these IETs? Are the mapping class group actually virtually $\mathbb{Z}^k$ for some $k \in \mathbb{N}$?

(3) Given an IET and its coding, the two systems share many dynamical and ergodic properties. For example, an IET is minimal if and only if its coding is minimal; there is a 1-1 correspondence of ergodic measures, and the group of coinvariants are isomorphic. In smooth dynamics, IETs arise as return systems of translation surfaces. By studying the moduli spaces of the translation surfaces, Veech [**79**] and Masur [**59**] proved that generically, IETs are uniquely ergodic. Is there a way to relate the mapping class group of codings of IETs to translation surfaces?

(4) A Toeplitz system is an almost everywhere extension of an odometer. Donoso et al. [**30**] proved that the automorphism group of a Toeplitz system embeds into the automorphism group of its odometer factor. This is a remarkable result, as it is usually not possible to project automorphisms. Since the automorphism

of an odometer is the odometer, this implies that the automorphism group of a Toeplitz system is abelian. Since we know the automorphism group structure of a Toeplitz system, coupled with the fact that the suspension of an odometer is a solenoid, can we compute the mapping class group of a Toeplitz system?

The automorphism group of Toeplitz systems embeds into its odometer factor [**30**]. The suspension of an odometer is a solenoid, which are very well understood. Can we compute the mapping class group of Toeplitz systems?

CHAPTER 5

The Picard group of the crossed product algebra

In this chapter, we define another group that can be associated to a dynamical system, the Picard group, first introduced by Brown, Green, and Rieffel in [**19**]. This definition generalizes the definition of Picard groups of algebras over commutative rings. The construction is more algebraic than dynamic, passing to the crossed product algebra, which is a C^*-algebra that encodes the dynamical information. These objects link dynamics, K-theory, and operator algebras. This chapter expands on the results in [**78**].

In Section 5.1, we give background and definitions of the Picard group, as well as some examples. In Section 5.2 we explore in more depth how we can produce modules over the crossed product algebra given a flow code. In Section 5.3 we prove the main result:

Theorem 5.1 ([**78**])**.** *Suppose* (X, T) *is a Cantor system. Then there is a homomorphism*

$$\Theta : \mathcal{M}(T) \to \mathrm{Pic}A.$$

If (X, T) *is minimal, then* Θ *is injective.*

We conclude the chapter with several open questions in Section 5.4.

5.1. Definitions and Background

Let (X, T) be a Cantor system, and let $C(X)$ denote the continuous functions on X. We can build a non-commutative C^*-algebra $C(X) \rtimes_T \mathbb{Z}$, where the action of $\mathbb{Z}$ on $C(X)$

corresponds to applying the transformation T. A C^*-*algebra* is a Banach algebra with an additional involution map $*$ which satisfies adjoint properties (see [**3**] for a complete reference).

To define $C(X) \rtimes_T \mathbb{Z}$, it suffices to describe how the generator $u \in \mathbb{Z}$ acts on $C(X)$ (we call the generator u because it acts as a unitary operator on $C(X)$):

$$ufu^* := T(f) \qquad \text{for all } f \in C(X).$$

We note here that the algebraic structure of $C(X) \rtimes_T \mathbb{Z}$ is quite complicated and intractable. However, by computing the K-theory of this C^*-algebra, we can recover previous algebraic invariants. In particular, $K_0(C(X) \rtimes_T \mathbb{Z})$ is isomorphic to the group of coinvariants $\mathcal{G}_T$ (see [**72**, Theorem 1.1]), which gives motivation on the construction of $\mathcal{G}_T$. For more details see [**72**] in this case and [**69**, Ch 7] for crossed products in general.

Given two C^*-algebras A and B, let Z be an $A-B$-bimodule (a left A-module and a right B-module). We say that Z is an $A-B$ *imprimitivity bimodule* if the following hold:

(1) Z is a full left Hilbert A-module and a full right Hilbert B-module; let ${}_A\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_B$ denote the A- and B-valued, respectively, inner products,

(2) for all $z_1, z_2 \in Z, a \in A, b \in B$,

$$\langle az_1, z_2 \rangle_B = \langle z_1, a^* z_2 \rangle_B \quad \text{and} \quad {}_A\langle z_1 b, z_2 \rangle = {}_A\langle z_1, z_2 b^* \rangle,$$

(3) for all $z_1, z_2, z_3 \in Z$,

$${}_A\langle z_1, z_2 \rangle \cdot z_3 = z_1 \cdot \langle z_2, z_3 \rangle_B.$$

Example 5.2. Any C^*-algebra A is an $A - A$ imprimitivity bimodule, where the bimodule structure is multiplication in A, and the inner products are ${}_A\langle a, b\rangle = ab^*$ and $\langle a, b\rangle_A = a^*b$.

Example 5.3 (Projections)**.** Let A be a C^*-algebra. An element $p \in A$ is a *projection* if $p = p^* = p^2$. Consider the subsets $Ap := \{ap : a \in A\}$ and $pAp := \{pap : a \in A\}$. We claim that Ap is an $A - pAp$ imprimitivity bimodule.

Since Ap and pAp are both subsets of A, we use the same inner products as the previous example, restricted to Ap:

$$ {}_A\langle ap, bp\rangle = app^*b^* = apb^* $$

$$ \langle ap, bp\rangle_{pAp} = pab * p $$

We note that the terminology projection comes from the fact that such Hilbert modules arise as orthogonal projections onto closed subspaces. This example is particularly important, as the indicator functions $\chi_C \in C(X)$ are projections, and a key component of how we construct imprimitivity bimodules from flow codes.

Given C^*-algebras A and B, if there exists an $A - B$ imprimitivity bimodule, then we say that A and B are *Morita equivalent*. It is straightforward to see that this defines an equivalence relation. As with conjugacy and flow equivalence, we are interested in non-trivial self-Morita equivalences, in this case, of the crossed product algebra of a Cantor system.

The equivalence classes of $A - A$ imprimitivity bimodules form a group under $\otimes_A$, which is the *Picard group*, denoted PicA. See [**74**] for a more complete reference on Morita equivalence and the Picard group.

The Picard group of A is closely related to automorphisms of A. Given an bounded linear function $\pi : A \to B$ between C^*-algebras A and B, π is a **-homomorphism* if π is multiplicative and respects the involution *: for all $x, y \in A$, π satisfies

$$\pi(xy) = \pi(x)\pi(y) \quad \text{and} \quad \pi(x^*) = \pi(x)^*.$$

A *C^*-isomorphism* is a bijective *-homomorphism. Let Aut$A$ denote the group of C^*-isomorphisms from A to itself. An *inner automorphism* φ of A an automorphism of the form

$$\varphi(x) = uxu^* \qquad \text{where } u \text{ is a unitary element.}$$

The inner automorphisms Inn$A$ form a normal subgroup of Aut$A$. The relationship bewteen these groups can be summarized with the following proposition:

Proposition 5.4 (Brown, Green, and Rieffel [**19**])**.** *Let A be a unital C^*-algebra. There is an exact sequence*

$$1 \to \mathrm{Inn}A \to \mathrm{Aut}A \to \mathrm{Pic}A.$$

Remark 5.5. The result in [**19**, Prop. 3.1] does not require the hypothesis of unitary C^*-algebra, but this simplified version suffices for our purposes.

The outer automorphism group is defined to be $\mathrm{Out}A := \mathrm{Aut}A/\mathrm{Inn}A$. The proposition says that there is a map from $\mathrm{Out}A \to \mathrm{Pic}A$, which need not be surjective, as we see in the next example.

Example 5.6 (Irrational rotations of the circle, [**52**])**.** Let θ be an irrational number in $[0, 1)$. Let $([0, 1), T_\theta)$ be the irrational rotation system, where $T_\theta(x) = x + \theta \bmod 1$.

Recall from 2.4 that a Sturmian is the symbolic coding of an irrational rotation. While a Sturmian is an almost-everywhere 1-1 extension of an irrational rotation and shares similar properties, there are some important dynamical distinctions. Most pertinent to this thesis, rotations are not Cantor systems and are not expansive. Theorem 5.1 does not apply in this case. However, the example is still quite illuminating, and may give insight into how to compute the Picard group of Sturmians.

The same dichotomy occurs for irrational rotations as in Sturmians: either θ is a quadratic (corresponding to the substitutive case), or not.

If θ is not a quadratic irrational (and thus does not come from a substitution), then the exact sequence from 5.4 can be extended to

$$1 \to \mathrm{Inn}A \to \mathrm{Aut}A \to \mathrm{Pic}A \to 1.$$

In this case, $\mathrm{Pic}A \cong \mathrm{Aut}A/\mathrm{Inn}A$.

If θ is a quadratic irrational, then $\mathrm{Pic}A$ is isomorphic to $\mathrm{Aut}A/\mathrm{Inn}A \rtimes \mathbb{Z}$. In this case, the map from Proposition 5.4 is not surjective.

5.2. Flow codes

Throughout this section, let (X, T) be a Cantor system. Let A be the crossed product algebra $C(X) \rtimes_T \mathbb{Z}$, which is generated by $C(X)$ and a unitary denoted by u. For a clopen $C \subset X$, χ_C is a projection in A.

Following Example 5.3 let A_C denote the algebra $\chi_C A \chi_C$. Note that $A_C \cong C(C) \rtimes_{r_C} \mathbb{Z}$ (see [**41**, Prop. 3.9]). Then $A\chi_C$ is an $A - A_C$ bimodule, which we call P_C. Given another clopen D and an isomorphism $A_D \overset{\alpha}{\to} A_C$, let A_D^α be the vector space A_D, and equip A_D^α with the structure of an $A_D - A_C$ bimodule defined as follows: the left action is the natural multiplication, right action is given by $x \cdot a = x\alpha^{-1}(a)$, where $x \in A_D, a \in A_C$, and the inner products are defined by

$$_{A_D}\langle x, y\rangle = xy^*$$

$$\langle x, y\rangle_{A_C} = \alpha(x^* y).$$

Given a flow code (φ, C, D) there is an induced isomorphism $A_D \xrightarrow{\varphi^*} A_C$, and we define an $A - A$ bimodule

$$X_\varphi = P_D \otimes A_D^{\varphi^*} \otimes P_C^{-1}.$$

The flow code produces an interesting and nontrivial bimodule. However, the induced isomorphism of vector spaces results in the wrong module structure. The role of the projection modules P_C and P_D is the fix the module structure to create an $A - A$ bimodule.

We conclude this section by presenting a lemma which allows us to restrict flow codes, thereby making compositions of flow codes well-defined. This prepares us to define the group structure of the Picard group in the next section.

Lemma 5.7. *Let (φ, C, D) be a flow code, $E \subset C$ a discrete cross section, and let $F = \varphi(E)$. Then $(\varphi|_E, E, F)$ is a flow code.*

Proof. Since $\varphi\colon (C, T_C) \to (D, T_D)$ is a conjugacy, the return times to E and $\varphi(E)$ are the same. It follows that $\varphi T_E(x) = T_F\varphi(x)$ for any $x \in E$, which implies $(\varphi|_E, E, F)$ is a flow code. Since $E \subset C$, $T_E(x) = T_C^k(x)$ for some k. Then $\varphi T_E(x) = \varphi T_C^k(x) = T_D^k\varphi(x)$, so it suffices to show that $r_F(\varphi(x)) = \sum_{i=0}^{k-1} r_D T_D^i(\varphi(x))$. Since $T_C^k(x) \in E$, we have $T_D^k\varphi(x) = \varphi T_C^k(x) \in F$, and hence $r_F(\varphi(x)) \leq \sum_{i=1}^{k-1} r_D(T_D^i\varphi(x))$. For the other direction, if $T^j\varphi(x) \in F$, then $T^j\varphi(x) = T_D^\ell\varphi(x)$ for some ℓ, since $F \subset D$. Then $T^j\varphi(x) = T_D^\ell\varphi(x) = \varphi T_C^\ell(x)$, and hence $T_C^\ell(x) \in E$, so $\ell \geq k$. It follows that $r_F\varphi(x) \geq \sum_{i=0}^{k-1} r_D T_D^i(x)$. □

To show how we might compose modules arising from flow codes, we show that if (φ, C, D) is any flow code, then the following hold:

(1) If S_φ is isotopic to the identity, then X_φ is equivalent to ${}_AA_A$.

(2) If $E \subset C$ is a discrete cross section, the restriction $(\varphi|_E, E, \varphi(E))$ is a flow code, and X_φ is equivalent to $X_{\varphi|_E}$.

(3) If (ψ, B, C) is another flow code, then $X_\varphi \otimes X_\psi$ is equivalent to $X_{\varphi\psi}$.

Proof of (1). Suppose (φ, C, D) is a flow code for which S_φ is isotopic to the identity. By Proposition 4.1, there exists $\alpha \in C(C, \mathbb{Z})$ such that $\varphi(x) = T^{\alpha(x)}x$ for all $x \in C$. Let $C_i = \alpha^{-1}(i)$, so $\{C_i\}$ provides a (finite) partition of C, and let $v = \sum_{i\in\mathbb{Z}} u^i\chi_{C_i} \in A$.

Claim 5.8. *The sum $v := \sum_{i\in\mathbb{Z}} u^i\chi_{C_i}$ is a partial isometry with $v^*v = \chi_C, vv^* = \chi_D$, for which the isomorphism $\varphi^*\colon A_D \to A_C$ is given by $Ad_{v^*}(x) = v^*xv$.*

Proof. We can write the product

$$v^*v = \left(\sum_{i\in\mathbb{Z}} \chi_{C_i} u^{i*}\right)\left(\sum_{j\in\mathbb{Z}} u^j \chi_{C_j}\right)$$

as a sum of terms of the form

$$\chi_{C_i} u^{j-i} \chi_{C_j} = \chi_{C_i} \chi_{C_j} \circ T^{-(j-i)} u^{j-i} = \chi_{C_i} \chi_{T^{j-i}(C_j)} u^{j-i}.$$

The clopen sets $\{T^i(C_i)\}$ form a partition of D, so if $i \neq j$, $C_i \cap T^{j-i}(C_j) = \emptyset$. Thus, we are left with

$$\sum_{i\in\mathbb{Z}} \chi_{C_i} = \chi_C.$$

Similarly,

$$vv^* = \left(\sum_{j\in\mathbb{Z}} u^j \chi_{C_j}\right)\left(\sum_{i\in\mathbb{Z}} \chi_{C_i} u^{i*}\right) = \sum_{i\in\mathbb{Z}} \chi_{C_i} \circ T^{-i} = \chi_D.$$

□

Consider the linear map

$$\Phi\colon P_D \otimes A_D^{\varphi^*} \to P_C$$

given by $\Phi(a\chi_D \otimes \chi_D b \chi_D) = a\chi_D b \chi_D v$. For any $c \in A$, we have $\Phi(cv^*\chi_D \otimes \chi_D) = cv^*\chi_D v = c\chi_C$, so Φ is onto. By [**74**, Remark 3.27], to show Φ is an equivalence of $A - A_C$-bimodules, it is enough then to show that Φ preserves the inner products, which is a straightforward calculation.

Proof of (2). First note that, by Lemma 5.7, $\varphi|_E\colon E \to F$ is also a flow code. We want to show that there is an equivalence of bimodules

$$P_D \otimes A_D^{\varphi^*} \otimes P_C^{-1} \cong P_F \otimes A_F^{\varphi|_E^*} \otimes P_E^{-1}. \tag{5.2.1}$$

For any clopens $V \subset U$, we may consider P_V as an $A_U - A_V$-bimodule, and the map $a\chi_U \otimes b\chi_V \mapsto a\chi_U b\chi_V$ induces an equivalence of bimodules $P_U \otimes ({}_{A_U}P_{V\,A_V}) \cong P_V$. Applying this to $E \subset C, F \subset D$, it follows that $P_C^{-1} \otimes P_E$ is equivalent to ${}_{A_C}P_{E\,A_E}$, and $P_D^{-1} \otimes P_F$ is equivalent to ${}_{A_D}P_{F\,A_F}$. Thus to show (5.2.1) it is enough to show that $A_D^{\varphi^*} \otimes P_E$ and $P_F \otimes A_F^{\varphi|_E^*}$ are equivalent. Consider the map $\Phi\colon A_D^{\varphi^*} \otimes P_E \to P_F \otimes A_F^{\varphi|_E^*}$, where

$$\Phi\colon \chi_D a \chi_D \otimes \chi_C b \chi_E \mapsto \chi_D a \chi_F \otimes (\varphi|_E^*)^{-1}(\chi_E b \chi_E).$$

Let $A_D = \chi_D A \chi_D$. We can view A_D as an A_D-A_C bimodule, where the left action is the natural multiplication, and $x \cdot a = x\varphi^*(a)$, where $x \in A_D, a \in A_C$. The inner products are defined as follows:

$${}_{A_D}\langle x, y\rangle = xy^*, \langle x, y\rangle_{A_C} = \varphi^*(a^*b).$$

It suffices to check that Φ has dense range and preserves inner products. The map Φ has dense range because $(\varphi|_E^*)^{-1} : A_E \to A_F$ is an isomorphism of C*-algebras. It is a straightforward calculation to show that Φ preserves inner products.

$$\begin{aligned}
{}_{A_D}\langle \chi_D a \chi_D \otimes \chi_C b \chi_E, \chi_D c \chi_D \otimes \chi_C d \chi_E\rangle &= {}_{A_D}\langle \chi_D a \chi_D, \chi_D a \chi_D \cdot_{A_C} \langle \chi_C d \chi_E, \chi_C b \chi_E\rangle\rangle \\
&= {}_{A_D}\langle \chi_D a \chi_D, \chi_D c \chi_D (\varphi^*)^{-1}(\chi_C d \chi_E b^* \chi_C)\rangle \\
&= \chi_D a \chi_D (\varphi^*)^{-1}(\chi_C b \chi_E d^* \chi_C)\chi_D c^* \chi_D.
\end{aligned}$$

A similar computation shows that

$$_{A_D}\langle \Phi(\chi_D a \chi_D \otimes \chi_C b \chi_E), \Phi(\chi_D c \chi_D \otimes \chi_C d \chi_E\rangle) = \chi_D a \chi_F (\varphi|_E^*)^{-1}(\chi_E b \chi_E d^* \chi_E)\chi_F c^* \chi_D.$$

Both expressions are supported on F and clearly agree on $F = \varphi^{-1}(E)$. On E, analogous computations show that

$$\langle \chi_D a \chi_D \otimes \chi_C b \chi_E, \chi_D c \chi_D \otimes \chi_C d \chi_E\rangle_{A_E} = \chi_E b^* \chi_C \varphi^*(\chi_D a^* \chi_D c \chi_D)\chi_C d \chi_E.$$

and

$$\langle \Phi(\chi_D a \chi_D \otimes \chi_C b \chi_E), \Phi(\chi_D c \chi_D \otimes \chi_C d \chi_E\rangle_{A_E} = \chi_E b^* \chi_E \varphi^*(\chi_F a^* \chi_D c \chi_F)\chi_E d \chi_E.$$

Such elements are supported on E and agree on E.

Proof of (3). First observe that $A_D^{\varphi^*} \otimes A_C^{\psi^*} \cong A_D^{\psi^* \varphi^*}$; this can be checked directly (or see [**19**, Section 3]). Then we have

$$X_\varphi \otimes X_\psi = P_D \otimes A_D^{\varphi^*} \otimes P_C^{-1} \otimes P_C \otimes A_C^{\psi^*} \otimes P_B^{-1} \cong P_D \otimes A_D^{\varphi^*} \otimes A_C^{\psi^*} \otimes P_B^{-1}$$
$$\cong P_D \otimes A_D^{\psi^* \varphi^*} \otimes P_B^{-1} \cong P_D \otimes A_D^{(\varphi\psi)^*} \otimes P_B^{-1} = X_{\varphi\psi}.$$

5.3. From mapping class to Picard group

We are now equipped to define the map

$$\Theta_{\mathcal{M}} \colon \mathcal{M}(T) \to \mathrm{Pic} A$$

Before we give the definition, we note that this section is applicable to all Cantor systems. Flow equivalence can be defined between any topological systems. In this section, we show

how to produce an imprimitivity bimodule from a flow code. Parry and Sullivan's result in [**68**] states that any flow equivalence between Cantor systems is isotopic to a flow code. This completes the definition of the map $\Theta_{\mathcal{M}}$ for Cantor systems.

Given $[f] \in \mathcal{M}(T)$, by Theorem 2.20 there exists a flow code (φ, C, D) such that $[S_\varphi] = [f]$, and we set $\Theta_{\mathcal{M}}([f]) = [X_\varphi]^{-1}$ (the choice of the inverse is to make Θ a homomorphism instead of an antihomomorphism). We show that Θ is a well-defined homomorphism using $(1), (2)$ and (3) above. First, we record the following lemma from [**15**], which shows how to compose flow codes (up to isotopy).

Lemma 5.9. [**15**, Prop. A.3] *If $(\varphi, C, D), (\psi, E, F)$ are a pair of flow codes, there exists clopen subsets $K \subset C, L \subset E$ and a flow code $\eta\colon \varphi(K) \to L$ such that η is isotopic to the identity and $S_\psi \circ S_\varphi$ is isotopic to $S_{\psi|_L \circ \eta \circ \varphi|_{\varphi^{-1}(K)}}$.*

If $(\varphi, C, D), (\psi, E, F)$ are a pair of flow codes, by Lemma 5.9 there exists clopen subsets K, L and a flow code $\eta\colon \varphi(K) \to L$ such that η is isotopic to the identity and $S_\psi \circ S_\varphi$ is isotopic to $S_{\psi|_L \circ \eta \circ \varphi|_{\varphi^{-1}(K)}}$. Using $(1), (2)$, and (3) above we have that $X_{\psi|_L \circ \eta \circ \varphi|_{\varphi^{-1}(K)}}$ is equivalent to $X_\varphi \otimes X_\psi$. Note that by (3) and (1) we have $[X_{\varphi^{-1}}] = [X_\varphi]^{-1}$. Now to show that $\Theta_{\mathcal{M}}$ is well-defined, if $(\varphi_1, C_1, D_1), (\varphi_2, C_2, D_2)$ are flow codes for which $[S_{\varphi_1}] = [S_{\varphi_2}]$, then $S_{\varphi_2^{-1}} \circ S_{\varphi_1}$ is isotopic to $S_{\varphi_2}^{-1} \circ S_{\varphi_1}$ which is isotopic to the identity. Applying Lemma 5.9 gives η, K, L for which $S_{\varphi_2^{-1}|_L \circ \eta \circ \varphi_1|_{\varphi^{-1}(K)}}$ is isotopic to the identity. Then using $(1), (2)$, and (3) above, in Pic$A$ we have $1 = [X_{\varphi_2^{-1}} \otimes X_{\varphi_1}] = [X_{\varphi_2}]^{-1}[X_{\varphi_1}]$, so $[X_{\varphi_1}] = [X_{\varphi_2}]$. That $\Theta_{\mathcal{M}}$ is a homomorphism also follows similarly using $(1), (2)$ and (3).

Proposition 5.10. *If (X, T) is a minimal Cantor system then $\Theta_{\mathcal{M}}\colon \mathcal{M}(T) \to \mathrm{Pic}A$ is injective.*

The group Pic$A$ acts on $K_0(A)$; in general, this action may be defined using the linking algebra, as in the discussion preceding Theorem 2.5 in [**75**]. Letting $\mathrm{Aut}(K_0(A))$ denote the group of order-preserving automorphisms of $K_0(A)$ (not necessarily preserving [1]), there is then a homomorphism $\mathscr{P}\colon \mathrm{Pic}A \to \mathrm{Aut}(K_0(A))$. The groups $\mathcal{G}_T$ and $K_0(A)$ are isomorphic (see [**72**, Theorem 1.1]), and upon identifying $\mathcal{G}_T$ and $K_0(A)$, the composition map $\mathscr{P} \circ \Theta_{\mathcal{M}}$

$$\begin{array}{ccc} \mathcal{M}(T) & \xrightarrow{\Theta_{\mathcal{M}}} & \mathrm{Pic}A \\ & \searrow^{\mathscr{P}\circ\Theta_{\mathcal{M}}} & \downarrow \mathscr{P} \\ & & \mathrm{Aut}(K_0(A)) \end{array} \tag{5.3.1}$$

agrees with π_T.

Proof of Proposition 5.10. If $[f]$ is in the kernel of Θ_M then it is in the kernel of $\mathscr{P} \circ \Theta_M$. It follows from the above discussion that $[f]$ is then in the kernel of π_T, so by Theorem 4.5, $[f] = \Psi(\alpha)$ for some $\alpha \in \mathrm{Aut}(X)$. The composition $\mathrm{Aut}(X) \to \mathcal{M}(T) \to \mathrm{Pic}A$ coincides with the composition map $\mathrm{Aut}(X) \to \mathrm{Aut}A \to \mathrm{Pic}A$. If $\Theta_{\mathcal{M}}\Psi(\alpha)$ is trivial in Pic$A$, then α gives rise to an inner automorphism of A. Since α_* preserves $C(X)$ in A, there exists some continuous function $\delta\colon X \to \mathbb{Z}$ such that for $g \in C(X)$, $g \circ \alpha = \alpha_*(g) = g(T^{-\delta(x)}x)$ (this follows from [**72**, Lemma 5.1]; for details, we refer the reader to the discussion preceding Proposition 2.4 in [**42**]). But this implies α maps a T-orbit to itself (in fact every T-orbit), and hence must be in $\langle T \rangle$, giving $[f] = [\alpha] = 1$ in $\mathcal{M}(T)$. $\square$

5.4. Open questions

(1) Can we compute the Picard group for Sturmians, and more ambitiously, for interval exchange transformations (IETs)? Example 5.6 computes the Picard group of irrational rotations, which are not Cantor systems. The analysis in [**52**] relies on viewing the rotation as a matrix element in $\mathrm{SL}(2,\mathbb{Z})$ and observing how θ affects the inner and outer automorphisms of the crossed product algebra. While Theorem 5.1 does not apply, as flow equivalences cannot be written as flow codes in the irrational rotation case, the resulting Picard group does fit in the Diagram 5.3.1. It may be possible to adapt ideas from [**52**] to Sturmians and IETS.

(2) By 5.10, $\mathcal{M}(T)$ embeds into $\mathrm{Pic}A$ via $\Theta_{\mathcal{M}}$ for a minimal Cantor system. It is natural to ask how large the image of $\mathcal{T}$ under $\Theta_{\mathcal{M}}$ is relative to $\mathrm{Pic}A$.

Furthermore, by [**19**, Prop.3.1] the kernel of the map $\mathrm{Aut}A \to \mathrm{Pic}A$ is the subgroup $\mathrm{Inn}A$ of inner automorphisms of A, giving an injective map $\mathrm{Out}A \to \mathrm{Pic}A$. It follows from Theorem 4.5 that

$$\mathrm{Image}\,\Theta_{\mathcal{M}} \cap \mathrm{Out}A = \mathrm{Image}(\Psi\colon \mathrm{Aut}(X)/\langle T\rangle \to \mathrm{Out}A)$$

It would be interesting to know how large the image of $\mathcal{M}(\mathcal{T})$ is under the composition

$$\mathcal{M}(T) \xrightarrow{\Theta_{\mathcal{M}}} \mathrm{Pic}A \to \mathrm{Pic}A/\mathrm{Out}A$$

and how it relates to the work in [**64**], [**65**].

(3) What we say about the structure of the Picard group of an SFT, and how does it differ from minimal cases? Proposition 5.10 also holds in the case where

(X, T) is an irreducible SFT. This follows from (5.3.1), and injectivity of the map $\mathcal{M}(T) \to \mathrm{Aut}(K_0(A))$, which itself can be deduced using Corollary 3.3 in [**15**]. This parallels the phenomenon that minimality (all orbits are dense) and irreducibel SFTs (periodic orbits are dense) allow us to embed the automorphism group (modulo the subgroup generated by the shift) into the mapping class group.

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