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Random and Small-scale Quantum Ergodicity

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Abstract

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This thesis contains results in mathematical quantum ergodicity in a probabilistic or a complex analytic setting. For the former, we show that a random orthonormal basis of spherical harmonics is almost surely quantum ergodic, in which the randomness is induced by the generalized Wigner ensemble. For the latter, we show that small-scale quantum ergodicity holds on a compact Kähler manifold equipped with a prequantum line bundle, or the Grauert tube of a compact, negatively curved, real analytic manifold. Furthermore, the nodal sets of the eigensections or the complexified eigenfunctions are also equidistributed on small scales.

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CHAPTER 1

Introduction

Quantum ergodicity (QE) concerns the relationship between (the asymptotic behavior of) eigenfunctions of Schrödinger-type operators (1.1) and dynamics of the underlying Hamiltonian system (1.5). This thesis collects three sets of results on the subject. Theorem 3.1.1 is a phase space realization of the probabilistic QE theorem (Theorem 3.2.2) of Bourgade–Yau [2] for eigenvectors of random matrices. Theorem 4.1.4 and Theorem 5.1.4 are small-scale QE theorems for eigensections of a prequantum line bundle of a compact Kähler manifold and for complexified eigenfunctions on a Grauert tube. Thanks to complex analytic techniques, we are able to translate these small-scale equidistribution results into small-scale distributions of zero sets in Theorem 4.1.2 and Theorem 5.1.1. Such distribution results are unknown for zeros of real eigenfunctions in the Riemannian setting.

1.1. Quantum Mechanics

Before quantum mechanics, the hydrogen atom was treated roughly as a 2-body planetary system, with the electron orbiting around the nucleus as prescribed by the classical laws of motion. There was, however, a serious problem: because the electron is accelerating as it undergoes circular motion, it must lose energy (in the form of electromagnetic radiation). The classical model would predict that the electron spirals into the nucleus, contradicting the stability of the hydrogen atom.

The 'old quantum theory' in the early 20th century was a first attempt to correct the classical picture. It was postulated that special periodic orbits are quantized in accordance with the Bohr–Summerfield quantization condition to produce certain allowed states. While this theory works for the Hydrogen atom, it does not extend in any obvious way to more complicated atoms, like the Helium atom consisting of a nucleus and two electrons.

In 1926, Schrödinger introduced the eponymous operator

$$\hat{H}_h := -h^2\Delta + V, \tag{1.1}$$

where h is Planck's constant, $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator on $\mathbb{R}^3$, and $V = V(x)$ denotes multiplication by a potential[1]. An electron at a fixed energy $E(h)$ is no longer represented as a point particle, but as an L^2-normalized vector[2] $\psi \in L^2(\mathbb{R}^3)$ that satisfies the eigenequation

$$\hat{H}_h\psi(x) = E(h)\psi(x), \qquad \|\psi\|_{L^2(\mathbb{R}^3)} = 1. \tag{1.2}$$

Classical Hamiltonian mechanics is thereby replaced by functional analysis, namely an eigenvalue problem for the Schrödinger operator. The time evolution, governed by the time-dependent Schrödinger's equation, of an energy state (1.2) is given by

$$\psi(x) \mapsto e^{\frac{itE(h)}{h}}\psi(x).$$

[1]In the case of the hydrogen atom, $V(x) = |x|^{-1}$.

[2]We call ψ a wave function. As seen in (1.3), its modulus squared $|\psi|^2$ is interpreted as the probability density of detecting a particle of a given energy at a given region in space.

Schrödinger proposed that the only physically relevant quantities are matrix elements $\langle A\psi, \psi\rangle_{L^2(\mathbb{R}^3)}$ of bounded, self-adjoint operators A acting on $L^2(\mathbb{R}^3)$. For instance, when $A = \mathbf{1}_B$ is multiplication by the characteristic function of a nice [3] $B \subset \mathbb{R}^3$, then the corresponding matrix element

$$\langle \mathbf{1}_B\psi, \psi\rangle_{L^2(\mathbb{R}^3)} = \int_B |\psi(x)|^2\, dx \tag{1.3}$$

represents the probability of finding an electron at energy $E(h)$ in the region $B \subset \mathbb{R}^3$. Matrix elements are invariant under time evolution:

$$\langle Ae^{\frac{itE(h)}{h}}\psi, e^{\frac{itE(h)}{h}}\psi\rangle_{L^2(\mathbb{R}^3)} = |e^{\frac{itE(h)}{h}}|^2\langle A\psi, \psi\rangle_{L^2(\mathbb{R}^3)} = \langle A\psi, \psi\rangle_{L^2(\mathbb{R}^3)},$$

which explains how an orbiting electron can be moving and 'stationary' (in the sense that it does not spiral into the nucleus) at the same time.

1.2. Hamiltonian Mechanics

While Schrödinger's 'new quantum theory' is elegant and successful in explaining not only the hydrogen atom, but also much more complicated systems, it replaces intuitions from classical mechanics by abstract functional analysis. Recall that classically, Hamilton's equations are used to describe the motion of a point particle in phase space $T^*\mathbb{R}^3$, i.e., the cotangent bundle of the configuration space $\mathbb{R}^3$. The time evolution of a particle

[3] The boundary of B must have measure zero.

at position x with momentum ξ obeys the system of differential equations

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial \xi}(x(t), \xi(t)), \\ \xi'(t) = -\dfrac{\partial H}{\partial x}(x(t), \xi(t)), \end{cases} \tag{1.4}$$

where the *Hamiltonian*

$$H\colon T^*\mathbb{R}^3 \to \mathbb{R}, \qquad H(x,\xi) = |\xi|^2 + V(x)$$

is the sum of the kinetic and potential energy of the system.

Classical Hamiltonian dynamics refers to the dynamical system generated by the Hamiltonian flow[4] (1.4) on the energy surfaces[5]

$$\Sigma_E = \{(x,\xi) \in T^*\mathbb{R}^3 : H(x,\xi) = E\}.$$

The dynamics can be highly chaotic, completely integrable, or somewhere in between depending on the potential V.

1.3. Quantum ergodicity

The quantum-classical correspondence suggests that in the semiclassical limit $h \to 0$, the asymptotic behavior of eigenfunctions (1.2) should reflect the dynamics of the classical

[4]More generally, when working on the cotangent space T^*M of a Riemannian manifold (M^n, g), the *Hamiltonian vector field* X_H of a Hamiltonian H is defined by

$$\omega(\,\cdot\,, X_H) = dH,$$

where ω is the natural symplectic form given locally by $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$ and dH is the exterior derivative of H. The *Hamilton flow* of H is then

$$\exp(tX_H)\colon T^*M \to T^*M. \tag{1.5}$$

It can be shown that the flow preserves energy surfaces.

[5]It is an exercise in symplectic geometry that Σ_E is invariant under the flow of H.

system (1.5). Mathematical quantum ergodicity is the rigorous study of the effects of classical chaos on high frequency eigenfunctions. The fundamental theorem in this subject is Theorem 2.3.1. Roughly, it asserts that when the Hamilton flow is ergodic on the energy surface Σ_E, then eigenfunctions are asymptotically equidistributed in phase space $S^*\Sigma_E$.

CHAPTER 2

Notation and Background

This chapter discusses some basic tools and results in the subject; more technical background will be recalled as needed in subsequent chapters. We will henceforth assume that the potential V vanishes in (1.1) and study Laplace eigenfunctions in relation to the dynamics of (1.5), which reduces to the (homogeneous) geodesic flow in this setting.

2.1. The Laplacian on a Riemannian Manifold

Let (M, g) be an n-dimensional Riemannian manifold. The Laplacian $\Delta_g = \Delta$ with respect to the Riemannian metric $g = (g_{jk})$ is the second-order differential operator given locally by the formula

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(g^{ij} \sqrt{|\det g|} \frac{\partial}{\partial x_k} \right).$$

When M is compact and without boundary, eigenvalues of $-\Delta$ form a discrete set in the nonnegative real axis with accumulation only at the origin. We list the eigenvalues $0 = \lambda_0^2 < \lambda_1^2 \leq \cdots \uparrow \infty$ in increasing order (repeated with multiplicity). The corresponding L^2-normalized eigenfunctions

$$(\Delta + \lambda_j^2)\varphi_j = 0, \quad \|\varphi_j\|_{L^2(M)} = 1, \quad \langle \varphi_j, \varphi_k \rangle_{L^2(M)} = \delta_{jk}, \tag{2.1}$$

form an orthonormal basis of $L^2(M)$. When convenient, we suppress subscripts and write

$$(\Delta + \lambda^2)\varphi_\lambda = 0 \tag{2.2}$$

for a general L^2-normalized eigenfunction of $-\Delta$ with eigenvalue λ^2. Often, the semiclassical parameter $h = h_j$ is introduced via a change of variable $h = \lambda^{-1}$. Then, (2.2) takes the form

$$(h^2\Delta + 1)\varphi_h = 0.$$

The Hamiltonian corresponding[1] to the Laplacian is the the metric norm-squared of a covector. We take its square root and work instead with the Hamiltonian

$$H(x,\xi) = |\xi|_{g_x} = \left(\sum_{j,k=1}^{n} g^{ij}(x)\xi_j\xi_k \right)^{\frac{1}{2}}, \tag{2.3}$$

which corresponds to the operator[2] $\sqrt{-\Delta}$. Since eigenfunctions of $-\Delta$ coincides with those of $\sqrt{-\Delta}$, it is immaterial whether we work with the flow generated by $-\Delta$ or by its square root. The advantage of (2.3) is that its Hamilton flow (1.5) is the homogeneous geodesic flow $G^t(x,\xi)$, which can (without loss of generality) be restricted to the energy surface $\Sigma_1 = \{H = 1\}$:

$$G^t \colon S^*M \to S^*M, \quad S^*M = \{(x,\xi) \in T^*M : |\xi|_{g_x} = 1\} = \Sigma_1. \tag{2.4}$$

[1]The principal symbol (2.6) of $-\Delta$ is given by $\sigma_{-\Delta}(x,\xi) = |\xi|^2_{g_x}$.
[2]Note that $\sqrt{-\Delta}$ is a pseudodifferential operator. (In fact, by a theorem of Seeley [55], complex powers of an elliptic pseudodifferential operator on a compact manifold are pseudodifferential operators.) It can be defined spectrally by the eigenfunction expansion $\sqrt{-\Delta} = \sum_{j=1}^{\infty} \lambda_j \varphi_j \otimes \overline{\varphi}_j$, where (λ_j^2, φ_j) are the spectral data for $-\Delta$.

The relationship between ergodicity of G^t and the asymptotic behavior of φ_j is summarized in Theorem 2.3.1.

2.2. Standard Quantization

In general, quantization is a recipe for converting classical observables to quantum observables, i.e., for converting 'nice' functions on phase space, called symbols, to (pseudodifferential) operators acting on some Hilbert space. Excellent references for quantization and the calculus of pseudodifferential operators include [**27**, **19**, **21**, **75**].

Definition **2.2.1** (Classical symbol). *A* classical symbol $a \in S^m$ of order $m \in \mathbb{R}$ *is a smooth function $a(x,\xi) \in C^\infty(T^*M - \{0\})$ in phase space (away from the zero section) with the properties*[3]

(i) $|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}$ *for all multi-indices α, β, where $\langle \xi \rangle := (1 + |\xi|^2_{g_x})^{\frac{1}{2}}$ is a smoothed out version of the metric norm of a covector;*

(ii) $a(x,\xi) \sim \sum_{j=0}^\infty a_j(x,\xi)$, *where $a_j(x,t\xi) = t^j a_j(x,\xi)$ for $|\xi| \geq 1$.*

The asymptotic summation notation in (ii) means for every $J \in \mathbb{N}$, the difference $a - \sum_{j=0}^J a_j$ satisfies (i) with $m = J+1$.

Definition **2.2.2** (Pseudodifferential operator). *Given $a \in S^m$, define the (Schwartz kernel of the) operator $\mathrm{Op}(a)\colon L^2(M) \to L^2(M)$ by the oscillatory integral*

$$\mathrm{Op}(a) = \frac{1}{(2\pi)^n} \int_{T_y^*M} \chi(x,y) a(x,\xi) e^{i\langle \exp_y^{-1}(x), \xi \rangle} d\xi, \tag{2.5}$$

where

[3] Smooth functions satisfying only the derivative estimate (i) are called *Kohn–Nirenberg symbols*. Here, we also require (ii), which is the condition that a is asymptotically polyhomogeneous, so it makes sense to view it as a function on the cosphere bundle S^*M.

- *χ is a cutoff localizing near* $\{(x,y) \in M \times M : d(x,y) < \mathrm{inj}(M)\}$, *in which* $\mathrm{inj}(M)$ *denotes the injectivity radius;*
- $\exp\colon TM \to M$ *is the Riemannian exponential map;*
- *Angle brackets denote the pairing of the vector* $\exp_y^{-1}(x)$ *with the covector* ξ*.*

Then, $\mathrm{Op}(a) \in \Psi^m$ *is a* pseudodifferential operator of order m *with (full) symbol* $a \in S^m$*.*

The map $\mathrm{Op}\colon S^m \to \Psi^m$ is a (choice of) quantization. It is unique under change of quantization and change of local coordinates modulo Ψ^{m-1}. In the reverse direction is the principal symbol map

$$\sigma\colon \Psi^m \to S^m, \qquad A \mapsto \sigma_A(x,\xi) \tag{2.6}$$

taking a pseudodifferential operator to its principal symbol. Again, uniqueness holds only modulo lower order terms in S^{m-1}. There are generalizations of the definitions above to incorporate the semiclassical parameter h; these will be presented in Section 5.5 when small-scale quantum ergodicity (Theorem 5.5.1) is discussed.

More generally, we can quantize symplectic map as unitary Fourier integral operators (see [**62**, **28**, **75**]). These are operators acting on $L^2(M)$ whose Schwartz kernels are oscillatory integrals similar to (2.5), except that the exponential part (i.e., the phase function) can have more complicated expressions. Of particular importance is the quantization of the geodesic flow (2.4) as the half-wave group

$$U(t)(x,y) := e^{it\sqrt{-\Delta}}(x,y) = \int_{T_y^*M} \chi(x,y) e^{it|\xi|_{g_y}} e^{i\langle \exp_y^{-1}(x), \xi\rangle} d\xi.$$

(Again, χ here is a cutoff near the diagonal so that the exponential map is defined.) Functional calculus implies that $U(t)$ is defined spectrally as

$$U(t)(x,y) = \sum_{j=1}^{\infty} e^{it\lambda_j} \varphi_j(x) \otimes \varphi_j(y), \tag{2.7}$$

where (λ_j^2, φ_j) are the spectral data for $-\Delta$.

2.3. Quantum Ergodicity for Laplace Eigenfunctions

Quantum observables are matrix elements

$$\langle A\varphi_j, \varphi_j \rangle_{L^2(M)} \tag{2.8}$$

of bounded, self-adjoint operators A on $L^2(M)$ relative to an energy eigenstate φ_j in (2.1). We are interested in the case where $A \in \Psi^0$ is a zeroth order pseudodifferential operator (Definition 2.2.2), so that its matrix elements process useful asymptotics as $j \to \infty$.

We give a flavor of how knowledge about (2.8) can be used to relate classical dynamics to the behavior of eigenfunctions[4]. Notice that Laplace eigenfunctions φ_j are also eigenfunctions of the half-wave group $U(t)$:

$$U(t)\varphi_j = e^{it\lambda_j}\varphi_j, \qquad U(t) := e^{it\sqrt{-\Delta}}. \tag{2.9}$$

Therefore, at the level of matrix elements there is the identity

$$\langle A\varphi_j, \varphi_j \rangle_{L^2(M)} = \langle U(t)AU(-t)\varphi_j, \varphi_j \rangle_{L^2(M)} \quad \text{for all } A \in \Psi^0.$$

[4]See [75, Chapter 15] for a detailed proof.

Since the left-hand side is independent of t, the right-hand side can be replaced by its time average:

$$\langle A\varphi_j, \varphi_j\rangle_{L^2(M)} = \langle \langle A\rangle_T \varphi_j, \varphi_j\rangle_{L^2(M)} \quad \text{where} \quad \langle A\rangle_T := \frac{1}{T}\int_0^T U(t)AU(-t)\,dt.$$

If the geodesic flow is ergodic on S^*M, then the ergodic theorem can be used to convert this time average into a spatial average[5]. This leads to the celebrated quantum ergodicity theorem of Shnirelman [**59**], Zelditch [**66**], and Colin de Verdière [**12**].

THEOREM **2.3.1** (QE theorem [**59, 66, 12**]; see also [**75**, Theorem 15.5])**.** *Let (M, g) be a compact Riemannian manifold without boundary. Suppose the geodesic flow (2.4) is ergodic. Then, there exists a density one subsequence of frequencies λ_{j_k} such that*

$$\langle A\varphi_{j_k}, \varphi_{j_k}\rangle_{L^2(M)} \to \frac{1}{\mu_L(S^*M)}\int_{S^*M} \sigma_A(x,\xi)\,d\mu_L \quad \textit{for every } A \in \Psi^0. \tag{2.10}$$

Here,

- *Density one means* $\lim_{\lambda\to\infty} \frac{\#\{\lambda_{j_k}\le\lambda\}}{\#\{\lambda_j\le\lambda:\lambda_j^2 \text{ is an eigenvalue of } -\Delta\}} = 1$*;*
- $\mu_L = \frac{dx\wedge d\xi}{d|\xi|}\Big|_{|\xi|=1}$ *is the Liouville surface measure on S^*M;*
- σ_A *is the principal symbol (2.6) of A.*

[5]More precisely, the time-averaged principal symbol $\sigma_{\langle A\rangle_T}$ is converted to its spatial average $⨍_{\Sigma_E} \sigma_A\, d\mu_L$, assuming that the geodesic flow is ergodic on the energy surface Σ_E with respect to the measure μ_L.

The QE theorem can be rephrased in the following way: Define *Wigner distributions*[6] $d\Phi_j$ by

$$\int_{S^*M} a(x,\xi)\, d\Phi_j := \langle \mathrm{Op}(a)\varphi_j, \varphi_j\rangle_{L^2(M)}, \qquad a \in C^\infty(S^*M).$$

then QE is equivalent to the weak* convergence

$$d\Phi_{j_k}(x,\xi) \rightharpoonup \frac{1}{\mu_L(S^*M)}\, d\mu_L(x,\xi)$$

along a density one subsequence[7][8]. Since the Liouville measure (which is the natural measure on S^*M coming from the metric g) is supported everywhere on S^*M, (2.10) is interpreted as a statement about the eigenfunctions becoming 'diffuse' or 'equidistributed' in phase space.

A corollary[9] of (2.10) is that quantum ergodic eigenfunctions equidistribute in configuration space:

$$|\varphi_{j_k}(x)|^2\, dV_g(x) \rightharpoonup \frac{1}{\mathrm{Vol}(M)}\, dV_g(x),$$

---

[6]Note that the Wigner distributions depend on the choice of quantization. There exist positive quantizations (such as Friedrichs or anti-Wick quantizations) with the property that $a \geq 0$ implies its quantization is a positive operator.

[7]Quantum unique ergodicity (QUE) is the statement that the entire sequence $d\Phi_j$ converges to the Liouville measure $d\mu_L$ without needing to possibly discard a subsequence of density zero. A famous conjecture of Rudnick–Sarnak [**52**] postulates that $d\mu_L$ is the unique quantum limit for negatively curved manifolds. So far, QUE has only been proved for Hecke–Maas forms on arithmetic hyperbolic surfaces by Lindenstrauss [**42**] in the compact case and Soundararajan [**60**] in the noncompact case.

[8]The set of weak* limits of $d\Phi_j$ is independent of the choice of quantization. Every weak* limit (also called the *microlocal defect measures* associated to the sequence $\{\varphi_j\}$) is a positive probability measure that is invariant under the geodesic flow.

[9]This follows from using test functions $a(x,\xi) = a(x)$ that do not depend on the fiber variable and the Portmanteau theorem.

where dV_g is the Riemannian volume measure on M.

2.4. Small-scale QE

In the works of Hezari–Rivière [25] and Han [24], QE is shown to hold for a sequence of operators $A = A_{j_k}$ whose symbols are allowed to depend on the frequency parameter λ_{j_k}. In particular, eigenfunctions are shown to be equidistributed at length scale that is logarithmic in the frequency parameter.

THEOREM **2.4.1** (Small-scale QE, [**24, 25**]). *Let (M, g) be a compact, negatively curved, n-dimensional manifold without boundary. Let*

$$0 < \alpha < \frac{1}{3n} \quad \textit{and} \quad r(\lambda) = (\log \lambda)^{-\alpha}.$$

Then, there exists a density one subsequence such that

$$c\mathrm{Vol}(B(x, r_{j_k})) \leq \int_{B(x,r_{j_k})} |\varphi_{j_k}|^2 \, dV_g \leq C\mathrm{Vol}(B(x, r_{j_k})) \quad \textit{uniformly for all } x \in M, \tag{2.11}$$

where $c, C > 0$ depend only on (M, g).

The theorem above (and its semiclassical version Theorem 5.5.2) are small-scale versions of Theorem 2.3.1. Curiously, even though the latter gives the asymptotics (2.10), the small-scale version gives only the volume comparison (2.11). This is an artifact of the technique of the proof, which involves two extractions of subsequences. In the complex setting described in Section 2.5 and Section 2.7, volume comparison can be used to derive asymptotic distributions of zero sets.

2.5. Berezin–Toeplitz Quantization

One drawback of the standard quantization (Section 2.2) is that the procedure does not readily extend to more general phase spaces (i.e., symplectic manifolds that are not cotangent bundles). In this section we present a Berezin–Toeplitz quantization scheme that works for general compact Kähler manifolds; excellent references include [**69**, **45**, **34**].

Definition **2.5.1** (Kähler manifolds)**.**

- *A* Hermitian manifold M *is a complex manifold endowed with a Riemannian metric* g *that is compatible with the complex structure* J*, i.e.,*

$$g(JX, JY) = g(X, Y) \quad \textit{for all } X, Y \in TM.$$

- *A Hermitian manifold* M *is said to be* Kähler *if the* $(1,1)$*-form*

$$\omega(X, Y) := g(JX, Y) \quad \textit{for all } X, Y \in TM$$

is closed. We call ω *the* Kähler form*.*

- *A* prequantum line bundle $(L, h) \to (M, \omega)$ *over a Kähler manifold is a holomorphic line bundle whose curvature form*[10] $c_1(h)$ *coincides with the Kähler form* ω*. We assume without loss of generality that* L *is very ample*[11]*.*

[10] If e_L is a non-vanishing local holomorphic frame for L over an open set $U \subset M$, then

$$c_1(h) = -\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log\|e_L\|_h,$$

where $\|e_L\|_h := h(e_L, e_L)^{1/2}$ denotes the h-norm of e_L.

[11] By Kodaira's embedding theorem, a line bundle L satisfying the curvature requirement is ample, i.e., there exists $N_0 \in \mathbb{N}$ such that global holomorphic sections of $L^{N_0} := L^{\otimes N_0}$ can be used to embed M into complex projective space of appropriate dimension. Note that this embedding is not an isometry. Replacing L by L^{N_0}, we may assume that L is very ample, i.e., the global sections of L define an embedding into projective space.

Let $(L,h) \to (M,\omega)$ be a prequantum line bundle over a compact Kähler manifold of complex dimension $\dim_{\mathbb{C}} M = m$. The (Liouville) volume form

$$dV_\omega := \frac{\omega^m}{m!}$$

on M induces an inner product on the space $\Gamma(M, L^N)$ of global smooth sections of tensor powers (L^N, h^N):

$$\begin{cases} \langle s_1, s_2 \rangle = \displaystyle\int_M h^N(s_1(z), s_2(z))\, dV_\omega & \text{for } s_1, s_2 \in \Gamma(M, L^N), \\ \|s\|^2_{h^N} = \langle s_1, s_2 \rangle & \text{for } s \in \Gamma(M, L^N). \end{cases} \tag{2.12}$$

Let $L^2(M, L^N)$ denote the completion of $\Gamma(M, L^N)$ with respect to (2.12). The space $H^0(M, L^N)$ of global holomorphic sections is a closed subspace of $L^2(M, L^N)$ of dimension[12]

$$d_N := \dim H^0(M, L^N) \sim N^m \quad \text{as } N \to \infty. \tag{2.13}$$

A key object is the orthogonal (Szegő) projection

$$\Pi_N \colon L^2(M, L^N) \to H^0(M, L^N). \tag{2.14}$$

Definition **2.5.2** (Toeplitz operator)**.** *The* Toeplitz operator (*of level* N) *associated to the smooth function* $a \in C^\infty(M)$ *is given by*[13]

$$\Pi_N a \Pi_N \colon H^0(M, L^N) \to H^0(M, L^N).$$

[12] $H^0(M, L^N)$ is necessarily finite dimensional because of the compactness of M.

[13] Since the domain of the operator is taken to be the space of holomorphic sections, the right-most factor of Π_N is redundant.

Here, Π_N is the Szegő projection as in (2.14), and a denotes multiplication by the function.

The Berezin–Toeplitz quantization scheme is as follows: Given a classical observable[14] $a \in C^\infty(M)$, we associate to it (a sequence of) Toeplitz operators $\Pi_N a \Pi_N$ acting on the Hilbert spaces $H^0(M, L^N)$ of global holomorphic sections of tensor powers of a very ample line bundle $(L, h) \to (M, \omega)$. In other words, the analogue of the quantization (2.5) in the line bundle setting is the map

$$C^\infty(M) \to \prod_{N=1}^{\infty} \operatorname{End}(H^0(M, L^N)), \qquad a \mapsto (\Pi_N a \Pi_N)_{N=1}^{\infty}.$$

The quantum observables are, as in the Riemannian case, matrix elements

$$\langle \Pi_N a \Pi_N s, s \rangle, \qquad s \in H^0(M, L^N).$$

2.6. QE and Equidistribution of Zeros for Eigensections

Quantum ergodicity can also be studied in the line bundle setting. Recall in the Riemannian case, the geodesic flow (2.4) is quantized as a Fourier integral operator (FIO), namely the half-wave group, whose eigenfunctions (2.9) coincide with those of the Laplacian. Here, we can similarly quantize a symplectic map $\chi\colon (M, \omega) \to (M, \omega)$ as (a sequence of) unitary FIOs

$$U_{\chi,N}\colon H^0(M, L^N) \to H^0(M, L^N), \tag{2.15}$$

[14]Note that the Kähler form ω is symplectic, so it is natural to regard (M, ω) itself (and not its cotangent bundle) as the phase space.

and study the relationship between dynamics of χ and eigensections $s_j^N \in H^0(M, L^N)$ of the quantum map

$$U_{\chi,N} s_j^N = e^{i\theta_{N,j}} s_j^N, \qquad \|s_j^N\|_{h^N} = 1, \qquad 1 \le j \le d_N,$$

where $e^{i\theta_{N,j}}$ are eigenphases and d_N is the dimension as in (2.13). The operator (2.15) is discussed in Section 4.2.2.

The line bundle analogue of the QE theorem is the due to Zelditch.

Theorem **2.6.1** (Zelditch [69]; see also [57] for random eigensections)**.** *Let $(L, h) \to (M, \omega)$ be a prequantum line bundle over a compact Kähler manifold without boundary. Let $\chi \colon M \to M$ be an ergodic symplectic map with $\chi^*\omega = \omega$. Let $U_{\chi,N} \colon H^0(M, L^N) \to H^0(M, L^N)$ be its quantization. Then, there exists a density one subsequence $J_N \subset \{1, \ldots, d_N\}$ of indices, i.e.,*

$$\lim_{N\to\infty} \frac{\# J_N}{d_N} = 1,$$

for which the corresponding eigensections $s_j^N \in H^0(M, L^N)$ satisfy

$$\int_M f(z)|s_j^N(z)|^2\, dV_\omega \xrightarrow[N\to\infty]{j\in J_N} \int_M f(z)\, dV_\omega \quad \text{for every } f \in C^0(M). \tag{2.16}$$

Equivalently, in the notation of weak convergence of measures, we have*

$$|s_j^N(z)|^2\, dV_\omega \xrightarrow[N\to\infty]{j\in J_N} dV_\omega.$$

Using complex analytic techniques, equidistribution of L^2-mass (2.16) can be used to prove equidistribution of the zero sets[15] of eigensections $s^N \in H^0(M, L^N)$. In a local frame e_L^N for L^N, we can write $s^N = f^{(N)} e_L^N$ with $f^{(N)}$ a holomorphic function. Let $g(z) := \|e_L(z)\|_h^2 = e^{-\varphi(z)}$ where φ is the Kähler potential, then $\|e_L^N(z)\|_{h^N}^2 = g(z)^N$ and $\|s^N\|_{h^N}^2 = |f^{(N)}|^2 g^N$. The Poincaré–Lelong formula states that the current $\left[Z_{s^N}\right]$ of integration over the zero divisor of s^N is given by

$$(2.17) \quad \left[Z_{s^N}\right] = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|f^{(N)}| = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log\|s^N\|_{h^N} + N\omega, \qquad s^N \in H^0(M, L^N).$$

Theorem **2.6.2** (Shiffman–Zelditch [**57**]). *Under the same assumptions and notation as in Theorem 2.6.1, we have*

$$\int_M f(z)\left[\frac{1}{N} Z_{s_j^N}\right] \wedge \omega^{m-1} \xrightarrow[N\to\infty]{j\in J_N} \int_M f(z)\, dV_\omega \quad \textit{for every } f \in C^\infty(M).$$

In the notation of weak convergence of currents, we have

$$\frac{1}{N}\left[Z_{s_j^N}\right] \xrightharpoonup[N\to\infty]{j\in J_N} \omega.$$

2.7. Complexification of Laplace Eigenfunctions to Grauert Tubes

The theory of Grauert tubes acts as a bridge between the Riemannian and the Kähler setting. In this section we introduce only the basic setup; the geometry and analysis of Grauert tubes will be recalled in Section 5.2 (see also [**22**, **23**, **38**, **39**, **36**]). The Grauert tube M_{τ_0} (of radius τ_0) of a *real analytic* manifold (M, g) can be identified with the co-ball

[15] Such precise distribution theorem for real zero sets of Laplacian eigenfunctions on a Riemannian manifold are unavailable.

bundle $B^*_{\tau_0}M$ (of radius τ_0), on which a complex structure $J = J_g$ compatible with the Riemannian metric can be defined (see Section 5.2.1). Let ω be the canonical symplectic form on T^*M, then the triple $(B^*_{\tau_0}M, \omega, J)$ is a Kähler manifold (Definition 2.5.1) with boundary.

Laplace eigenfunctions (2.1) on M can be complexified to the Grauert tube $B^*_{\tau_0}M$. This is done by analytically continuing the eigenfunction expansion (2.7) of the half-wave kernel $U(t) = e^{it\sqrt{-\Delta}}$ in the time and spatial variable

$$U(i\tau)(\zeta, y) = e^{-\tau\sqrt{-\Delta}}(\zeta, y) = \sum_{j=1}^{\infty} e^{-\tau\lambda_j}\varphi_j^{\mathbb{C}}(\zeta) \otimes \varphi_j(y), \quad \tau \in \mathbb{R}_{\geq 0}, \quad \zeta \in (B^*_{\tau_0}M, \omega, J).$$

It follows that the complexification $\varphi_j^{\mathbb{C}}$ of an eigenfunction φ_j to $B^*_{\tau_0}M$ is given by

$$\varphi_j^{\mathbb{C}}(\zeta) = e^{\tau\lambda_j}(e^{-\tau\sqrt{-\Delta}}\varphi_j)(\zeta), \quad \tau \in \mathbb{R}_{\geq 0}, \quad \zeta \in (B^*_{\tau_0}M, \omega, J). \tag{2.18}$$

It can be shown [**71**, Lemma 1.5] that the L^2-masses of $\varphi_j^{\mathbb{C}}$, appropriately normalized, equidistribute in the Grauert tube.

The advantage of working with complexified eigenfunctions on the Grauert tube is that, similar to the line bundle case, complex analysis lends powerful tools for studying the zero sets[16]

$$[Z_j] = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|\varphi_j^{\mathbb{C}}(z)|.$$

Equidistribution of $[Z_j]$ is due to Zelditch.

[16] As in the line bundle case, we view the zero sets as currents of integration (2.17).

Theorem **2.7.1** (Zelditch [71]). *Let (M,g) be a real analytic, n-dimensional manifold without boundary. Suppose the geodesic flow (2.4) is ergodic. Define the complexification $\varphi_j^{\mathbb{C}}$ of Laplace eigenfunctions to the Grauert tube $M_{\tau_0} \simeq B^*_{\tau_0}M$ by (2.18). Then, there exists a density one subsequence such that*

$$\int_{B^*_{\tau_0}M} f\left[\frac{1}{\lambda_{j_k}} Z_{j_k}\right] \wedge \frac{\omega^{n-1}}{(n-1)!} \to \frac{\sqrt{-1}}{\pi}\int_{B^*_{\tau_0}M} f\partial\bar{\partial}|\xi|_{g_x} \wedge \frac{\omega^{n-1}}{(n-1)!} \quad \textit{for all } f \in C(B^*_{\tau_0}M).$$

In the notation of weak convergence of currents, we have

$$\frac{1}{\lambda_{j_k}}[Z_{j_k}] \rightharpoonup \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}|\xi|_{g_x}.$$

2.8. Random Orthonormal Bases of Eigenfunctions

Much of the difficulty in the analysis of eigenfunctions is the lack of explicit formulae, except in very special settings. It often helps[17] to work probabilistically. One approach is to decompose the state space $\mathcal{H} = \bigoplus_{N=1}^{\infty}\mathcal{H}_N$ into a direct sum of eigenspaces. For instance, in the Riemannian setting with $M = S^2$, the state space $L^2(S^2)$ can be decomposed into the linear spans of degree N spherical harmonics (see Section 3.1 and [68, 74]). In the line bundle setting, there is a natural decomposition into a direct sum of the spaces $H^0(M, L^N)$ (see Section 4.2.1 and [57]). By fixing a background orthonormal basis $\{\varphi_1, \dots, \varphi_{d_N}\}$ of $\mathcal{H}_N$, any other orthonormal basis $\{\psi_1, \dots, \psi_{d_N}\}$ can be uniquely expressed as the linear combination

$$\psi_j = \sum_{k=1}^{d_N} u_{jk}\varphi_k, \qquad 1 \le j \le d_N.$$

[17]Law of large numbers and concentration of measure are particularly useful.

In this way, a random basis of $\mathcal{H}_N$ is identified with a random $d_N \times d_N$ unitary matrix $(u_{jk}) \in \mathrm{U}(d_N)$, and a random basis of $\mathcal{H} = \bigoplus_{N=1}^{\infty} \mathcal{H}_N$ is identified with an element of the product probability space $\prod_{N=1}^{\infty} \mathrm{U}(d_N)$.

CHAPTER 3

Quantum Ergodicity of Wigner Induced Random Spherical Harmonics

In this chapter we discuss the notion of a 'random orthonormal basis of spherical harmonics' of $L^2(S^2)$ using generalized Wigner ensembles and show that such a random basis is almost surely quantum ergodic. Similar quantum ergodicity results (with varying degrees of generality) are obtained in [**67**, **68**, **74**, **47**, **8**] for random Laplacian eigenfunctions defined using Haar measures on unitary groups. Our main contribution, Theorem 3.1.1, comes from the use of a more general measure than previously studied. We are able to work with this more general class of measures because Wigner eigenvectors are asymptotically Gaussian, a result proved in [**33**, **61**] (with additional assumptions on the moments) and [**2**]. Our quantum ergodicity statement also provides a semiclassical realization of the probabilistic 'local quantum unique ergodicity' of [**2**].

3.1. Main Results

The equidistribution condition (2.10) need not hold when the geodesic flow is not ergodic. On the sphere, for instance, the geodesic flow is completely integrable and direct computations show that the standard spherical harmonics localize not only on phase space, but also on the base manifold S^2. This fact notwithstanding, it is shown in [**67**] that a random orthonormal basis (defined using Haar measures on unitary groups) of spherical harmonics is almost surely quantum ergodic, a result that is extended to

Laplacian eigenfunctions on compact Riemannian manifolds in [68, 74, 47, 8]. In this chapter, we continue the investigation on the sphere and prove quantum ergodicity for a wider class of 'random' spherical harmonics.

Consider the orthogonal decomposition of $L^2(S^2)$ into a direct sum of subspaces $\mathcal{H}_N = \operatorname{span}\{Y_N^k \mid -N \le k \le N\}$ spanned by the standard degree N spherical harmonics. Here, by 'standard,' we mean spherical harmonics Y_N^k that are the joint eigenfunctions of the Laplacian $\Delta = \Delta_{S^2}$ and the z-component of the angular momentum operator $L_z = \frac{1}{i}\frac{d}{d\varphi}$, that is,

$$\begin{cases} \Delta Y_N^k = -N(N+1)Y_N^k, \\ \dfrac{1}{i}\dfrac{\partial}{\partial\varphi}Y_N^k = kY_N^k. \end{cases}$$

Let $d_N = \dim \mathcal{H}_N = 2N+1$ be the dimension of $\mathcal{H}_N$.

Let $H_N \in \operatorname{Herm}(d_N)$ be a generalized Wigner matrix. (See Section 3.2 for background on random matrix theory.) For $-N \le k \le N$, let $u_{N,k} = (u_{N,k}(\alpha))_{\alpha=-N}^N$ be the eigenvectors of H_N. Our object of study is the Wigner induced random basis $\{\psi_{N,k}\}_{k=-N}^N$ for $\mathcal{H}_N$ obtained by 'transplanting the Wigner eigenvectors onto the sphere' in the obvious way:

$$\psi_{N,k} := \sum_{\alpha=-N}^{N} u_{N,k}(\alpha) Y_N^\alpha, \quad -N \le k \le N. \tag{3.1}$$

An equivalent way of thinking about the random basis $\{\psi_{N,k}\}$ is to identify it with a unitary change-of-basis matrix $U_N = (u_{N,k}(\alpha))_{-N\le k,\alpha\le N}$ viewed as an element of the probability space $(\mathrm{U}(d_N), \mu_N)$. The probability measure μ_N on the unitary group $\mathrm{U}(d_N)$ is induced by a generalized Wigner matrix in the following way. Let π be the map from

Hermitian matrices to unitary matrices modulo the maximal torus $\mathrm{U}(1)^{d_N}$ defined by

$$\pi\colon\ \mathrm{Herm}(d_N) \to \mathrm{U}(d_N)/\,\mathrm{U}(1)^{d_N}, \quad H_N = U_N^* D(\boldsymbol{\lambda}) U_N \mapsto [U_N],$$

where U_N is a unitary matrix that diagonalizes H_N and $D(\boldsymbol{\lambda})$ is the resulting diagonal matrix. If we write μ_N^{W} for the measure on the Hermitian matrices that describes the generalized Wigner ensemble, then the induced measure μ_N on the unitary group is simply the pushforward of μ_N^{W} under the above map π, that is,

$$\mu_N := \pi_* \mu_N^{\mathrm{W}}. \tag{3.2}$$

The construction of a Wigner induced random basis (3.1) for the finite dimensional subspace $\mathcal{H}_N$ extends naturally to all of $L^2(S^2)$. Indeed, let U be the operator that acts block-diagonally on the decomposition $L^2(S^2) = \bigoplus_{N\geq 0} \mathcal{H}_N$ so that the restrictions $U|_{\mathcal{H}_N} = U_N \in \mathrm{U}(d_N)$ to the subspaces yield a sequence of independent unitary matrices of the appropriate dimensions. By the preceding paragraph, a Wigner induced random orthonormal basis $\boldsymbol{\Psi} = \{\psi_{N,k}\}_{-N\leq k\leq N, N\geq 0}$ for all of $L^2(S^2)$ may be identified with such an operator U viewed as an element of the product probability space $\prod_{n\geq 0}(\mathrm{U}(d_N), \mu_N)$. Henceforth, when the context is clear, we will refer to $\boldsymbol{\Psi}$ simply as a 'random basis' with the understanding that it is constructed randomly with respect to the product measure $\prod \mu_N$.

For technical reasons, certain indices k need to be excluded from our computations. Let $0 < \nu < \frac{3}{4}$ be a positive constant (guaranteed by Theorem 3.2.1), and let

$$I_N = [[-N, -N + N^{1/4}]] \cup [[-N + N^{1-\nu}, N - N^{1-\nu}]] \cup [[N - N^{1/4}, N]] \tag{3.3}$$

be the subset of indices $-N \leq k \leq N$ that are, in the random matrix theory language, 'in the bulk' and 'near the edges.' We can only work with indices belonging to I_N because the asymptotic normality result of Bourgade–Yau (Theorem 3.2.1), which we rely on, is established only for $k \in I_N$. (The set I_N displayed above is precisely the set $\mathbb{T}_N$ in the statement of Theorem 1.2 in the original paper [2], except that the our indexing convention is $k \in [-N, N]$, and the convention of [2] is $k \in [1, N]$.) It is expected that Theorem 3.2.1 holds for all indices k (see the remark immediately following Definition 5.1 in [2]). Luckily, the set I_N is sufficient for deriving a quantum ergodicity statement because we are still left with a density one subsequence after discarding indices in the intermediate regime, that is,

$$\frac{|\{k \in I_N\}|}{|\{k \in [-N, N]\}|} \to 1.$$

Given a pseudodifferential operator (see Definition 2.2.2) $A \in \Psi^0(M)$ of order zero and a random basis $\mathbf{\Psi}$, let $X_N = X_N^A(\{\psi_{N,k}\})\colon (\mathrm{U}(d_N), \mu_N) \to \mathbb{R}_{\geq 0}$ be random variables given by

$$X_N = X_N^A(\{\psi_{N,k}\}) = \frac{1}{d_N} \sum_{k \in I_N} |\langle A\psi_{N,k}, \psi_{N,k}\rangle - \omega(A)|^2, \tag{3.4}$$

where

$$\omega(A) := \frac{1}{\mu_L(S^*M)} \int_{S^*M} \sigma_A \, d\mu_L \tag{3.5}$$

denotes the average of the principal symbol of A. Even though the random variable (3.4) depends on the choice of a pseudo-differential operator and a random basis, for notational simplicity we will continue to write $X_N := X_N^A(\{\psi_{N,k}\})$. Our quantum ergodicity result is formulated in terms of X_N.

THEOREM **3.1.1** (QE of random spherical harmonics, Chang [9]). *Let $\mathbf{\Psi}$ be a Wigner induced random orthonormal basis of spherical harmonics for $L^2(S^2)$. Then $\mathbf{\Psi}$ is almost surely quantum ergodic with respect to the product probability measure $\prod \mu_N$ in the sense that*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{N=0}^{M} X_N = 0 \quad a.s.$$

for every $A \in \Psi^0(S^2)$.

REMARK **3.1.2.** *A standard extraction argument[1] implies the almost-sure existence of a density one subsequence of random spherical harmonics for which*

$$\langle A\psi_{N,k}, \psi_{N,k}\rangle \to \omega(A).$$

Note that the random variables X_N are independent by construction. Theorem 3.1.1 is therefore an easy consequence of the Kolmogorov convergence criterion and Strong Law of Large Numbers once we show that $\mathbb{E}X_N \to 0$ and $\mathbb{E}X_N^2$ is bounded. Indeed, the following holds.

THEOREM **3.1.3** (Moment bounds, Chang [9]). *We have $\mathbb{E}X_N = O(d_N^{-\varepsilon_0})$ and $\mathbb{E}X_N^2 = O(d_N^{-\varepsilon_0'})$ for some $\varepsilon_0, \varepsilon_0' > 0$ guaranteed by Theorem 3.2.1.*

This is a good place for some remarks. First, since we only work with random spherical harmonics in this chapter, we confine ourselves to describing the construction of random

[1]See [75, Chapter 15].

bases on S^2. A similar construction that involves partitioning the spectrum of the Laplacian appropriately can be used to make sense of random bases (defined using either Haar measures or Wigner induced measures on unitary groups) on any compact Riemannian manifold. Readers are referred to [68, 74, 47, 8] for the general construction. A natural next step is to extend our quantum ergodicity result to Wigner induced random bases of Laplacian eigenfunctions or approximate eigenfunctions on other manifolds.

Second, it is known that the eigenvectors of a Gaussian unitary ensemble is distributed by Haar measure on the unitary group. Since the generalized Wigner ensembles contain GUE as a special case, the measure with respect to which Wigner eigenvectors are distributed (i.e., the Wigner induced measure μ_N) is a vast generalization of Haar measure. It is unknown to the author if such measures can be given an explicit characterization. Nevertheless, universality results from random matrix theory are robust enough for showing that Wigner induced random bases enjoy the same quantum ergodicity property as 'GUE induced random bases' (i.e., random bases defined using Haar measure) on the sphere.

Finally, the methods presented in this chapter can be used to prove quantum ergodicity of Wigner induced random spherical harmonics on higher dimensional spheres S^p for any $p \geq 2$. It will be clear from the proof that ε_0 and ε_0' in the statement of Theorem 3.1.3 are independent of the dimension p because, in the notation of Theorem 3.2.1, we have $\varepsilon_0 = \varepsilon_0(Q_1)$ and $\varepsilon_0' = \varepsilon_0'(Q_2)$ where Q_1, Q_2 are polynomials of the form

$$Q_1(z_1, z_2, z_3, z_4) = z_1 z_2 \overline{z}_3 \overline{z}_4 \quad \text{and} \quad Q_2(z_1, \ldots, z_8) = z_1 z_2 z_3 z_4 \overline{z}_5 \overline{z}_6 \overline{z}_7 \overline{z}_8.$$

While $\varepsilon_0, \varepsilon_0'$ remain fixed for all $p \geq 2$, the dimension d_N of the space of degree N spherical harmonics grows like N^{p-1} on S^p. Substituting the asymptotics for d_N into the statement of Theorem 3.1.3 gives $\mathbb{E}X_N = O(N^{-\varepsilon_0(p-1)})$ and $\mathbb{E}X_N^2 = O(N^{-\varepsilon_0'(p-1)})$. Observe that, for all p sufficiently large, the Borel–Cantelli lemma becomes applicable and implies the stronger convergence statement that $X_N \to 0$ almost surely instead of the Cesàro means $\frac{1}{M}\sum_{N=0}^{M} X_N \to 0$.

The rest of the chapter is organized as follows. Section 3.2 provides a brief summary of random matrix theory that will be used in our proofs. The key result is Theorem 3.2.1, which states that Wigner eigenvectors (with the appropriate scaling) are asymptotically Gaussian random variables. Section 3.3 is devoted to proving Proposition 3.3.1, which is a special case of Theorem 3.1.3. The techniques developed for this special case extends easily to prove the main theorems in Section 3.4.

3.2. Background: The Wigner Ensemble and Bourgade–Yau Local QUE

We now summarize a universality result for Wigner eigenvectors proved in [2]. In keeping with the indexing convention for spherical harmonics, the indices in this section continue to range from $-N$ to N. Recall also that $d_N = 2N + 1$.

By a generalized Wigner matrix we mean a Hermitian matrix $H_N = (h_{jk})_{-N \leq j,k \leq N} \in \mathrm{Herm}(d_N)$ such that:

- The entries h_{jk} are independent random variables for $j \leq k$, each with mean zero and variance $\mathbb{E}h_{jk}^2 =: \sigma_{jk}^2$ satisfying the normalization condition $\sum_{j=-N}^{N} \sigma_{jk}^2 = 1$ for k fixed;

- There exists a constant $c_1 > 0$ independent of N such that $(c_1 N)^{-1} \leq \sigma_{jk}^2 \leq c_1 N$ for all $-N \leq j, k \leq N$;
- There exists a constant $c_2 > 0$ independent of N such that $\mathbb{E}(\boldsymbol{h}_{jk}^* \boldsymbol{h}_{jk}) \geq c_2 N^{-1}$ in the sense of inequality between 2×2 positive matrices, where $\boldsymbol{h}_{jk} := (\Re h_{jk}, \Im h_{jk})$;
- For any $q \in \mathbb{N}$, there exists a constant $C_q > 0$ such that for any N and any $-N \leq j, k \leq N$, we have $\mathbb{E}|\sqrt{d_N} h_{jk}|^q \leq C_q$.

Let $u_{N,k} = (u_{N,k}(\alpha))_{\alpha=-N}^N$ denote the eigenvectors of a generalized Wigner matrix $H_N \in \mathrm{Herm}(d_N)$. The eigenvectors, indexed by $k \in [-N, N]$, are ordered so that the corresponding eigenvalues form a nondecreasing sequence. Of course, an eigenvector is well-defined only up to a phase $e^{i\theta} \in \mathrm{U}(1)$. This phase ambiguity may be eliminated, for instance, by considering instead the equivalence class $[u_{N,k}]$.

Theorem **3.2.1** (Normality for eigenvectors, [2, Corollary 1.3]). *Let $\{H_N\}$ be a sequence of generalized Wigner matrices. Let I_N be the set of indices away from the intermediate regime as defined in (3.3) (note that I_N depends on a parameter ν). Then there exists $\nu > 0$ such that for any $k \in I_N$ and $J \subset \{-N, \dots, N\}$ with $|J| = m$, we have*

$$\sqrt{d_N}(u_{N,k}(\alpha))_{\alpha\in J} \to \left(\mathcal{N}_j^{(1)} + i\mathcal{N}_j^{(2)}\right)_{j=1}^m$$

in the sense of convergence in moments modulo phases, where $\mathcal{N}_j^{(1)}, \mathcal{N}_j^{(2)}$ are independent standard Gaussians. More precisely, for any polynomial Q in $2m$ variables, there exists $\varepsilon = \varepsilon(Q) > 0$ such that for sufficiently large N we have

$$\sup_{\substack{J\subset\{-N,\dots,N\} \\ |J|=m,\ k\in I_N}} \left|\mathbb{E} Q\left(\sqrt{2N}\left(e^{i\omega} u_{N,k}(\alpha), e^{-i\omega}\overline{u_{N,k}(\alpha)}\right)_{\alpha\in J}\right)\right.$$

$$- \mathbb{E}Q\left(\left(\mathcal{N}_j^{(1)} + i\mathcal{N}_j^{(2)}, \mathcal{N}_j^{(1)} - i\mathcal{N}_j^{(2)}\right)_{j=1}^{m}\right)\Big| \le d_N^{-\varepsilon}.$$

Here ω a phase independent of H_N and uniform on $(0, 2\pi)$.

In fact, a stronger statement is proved in [2, Theorem 1.2], namely the projection $\langle \boldsymbol{q}, u_{N,k}\rangle$ of an eigenvector to any unit vector $\boldsymbol{q} \in \mathbb{R}^{d_N}$ is asymptotically normal. As a corollary, generalized Wigner eigenvectors are 'locally quantum unique ergodic' in the following sense. Let $a_N \colon \{-N, \ldots, N\} \to [-1, 1]$ be a function with $\sum_{\alpha=-N}^{N} a_N(\alpha) = 0$ and let $|a_N| = \#\{-N \le \alpha \le N \mid a_N(\alpha) \ne 0\}$ be the size of its support.

Theorem **3.2.2** (Local QUE for eigenvectors, [2, Corollary 1.4])**.** *Let $\{H_N\}$ be a sequence of generalized Wigner matrices. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists a constant $C > 0$ so that for every sequence of functions $\{a_N\}$ as above and $k \in I_N$ we have*

$$\mathbb{P}\left(\left|\frac{d_N}{|a_N|}\langle a_N u_{N,k}, u_{N,k}\rangle\right| > \delta\right) \le C(d_N^{-\varepsilon} + |a_N|^{-1}), \tag{3.6}$$

where $\langle a_N u_{N,k}, u_{N,k}\rangle := \sum_{\alpha=-N}^{N} a_N(\alpha)|u_{N,k}(\alpha)|^2$.

Theorem 3.2.1 shows that Wigner eigenvectors are asymptotically flat even on small scales by choosing the test functions a_N to have small supports. Note that since the left-hand side of (3.6) depends only the eigenvectors but not the eigenvalues, the measure used in Theorem 3.2.2 is precisely the induced measure μ_N defined in (3.2).

We take this opportunity to remark that on a compact manifold (M, g), the analogue to the limiting formula (3.6) given by

$$\int_M f(x)|\varphi_k(x)|^2\,dx \to \int_M f(x)\,dx \quad \text{for every } f \in C(M) \tag{3.7}$$

is insufficient for concluding that $\{\varphi_k\}$ is quantum ergodic in the sense of Theorem 2.3.1. This is because delocalization on the base manifold M is a much weaker condition than diffuseness in the phase space S^*M. For instance, the Laplacian eigenfunctions $e^{i\langle \lambda, x\rangle}$ on a flat torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$ are delocalized in the sense of (3.7). But, if $\{\lambda_k\}$ is a sequence of lattice points for which the unit vectors $\lambda_k/|\lambda_k|$ tend to a limit vector $\xi \in \mathbb{R}^n$, then the asymptotic formula

$$\langle Ae^{i\langle \lambda_k, x\rangle}, e^{i\langle \lambda_k, x\rangle}\rangle \simeq \int_{\mathbb{R}^n/2\pi\mathbb{Z}^n} \sigma_A\left(x, \frac{\lambda_k}{|\lambda_k|}\right)dx \quad \text{for every } A \in \Psi^0(\mathbb{R}^n/2\pi\mathbb{Z}^n)$$

shows that the corresponding weak* limit is a delta mass on the invariant Lagrangian torus $T_\xi \subset S^*M$ for the geodesic flow. Since there always exists a sequence of $\lambda_k/|\lambda_k|$ converging to arbitrary $\xi \in \mathbb{R}^n$, the eigenfunctions $e^{i\langle \lambda, x\rangle}$ are far from diffuse in phase space. Of course, in the random matrix setting it is unclear even how to interpret the phase space when the base manifold is an index set $\{-N, \ldots, N\}$. We will need additional tools from semi-classical analysis to show that Theorem 3.1.1 holds.

3.3. Towards a Proof of Main Theorems: Rotationally Invariant Case

The purpose of this section is to prove Proposition 3.3.1 stated below. The difference between the proposition and Theorem 3.1.3 is the rotational invariance assumption we impose on A (and hence on the random variable X_N). This additional assumption allows us to isolate the key computational techniques and exhibit them in a simpler setting.

To clearly distinguish the special case we are currently considering from the general case, let us introduce some new notation. Let $B \in \Psi^0(S^2)$ denote pseudo-differential operators of degree zero that are invariant under z-axis rotations. To these rotationally invariant operators we associate random variables

$$Z_N = Z_N^B(\{\psi_{N,k}\}) = \frac{1}{d_N} \sum_{k \in I_N} |\langle B\psi_{N,k}, \psi_{N,k}\rangle - \omega(B)|^2, \tag{3.8}$$

where I_N is defined in (3.3) and $\omega(B)$ is defined in (3.5). Our goal is to show the following.

Proposition **3.3.1.** *In the above notation, we have* $\mathbb{E}Z_N = O(d_N^{-\varepsilon})$ *and* $\mathbb{E}Z_N^2 = O(d_N^{-\varepsilon'})$ *for some* $\varepsilon, \varepsilon' > 0$ *guaranteed by Theorem 3.2.1.*

Proof of Proposition 3.3.1. Note that the rotational invariance hypothesis implies that the matrix elements $\langle BY_N^\alpha, Y_N^\beta\rangle$ vanish whenever $\alpha \neq \beta$. Rewriting the random basis elements $\psi_{N,k}$ in terms of spherical harmonics Y_N^α using (3.1), the expression (3.8) becomes

$$\begin{aligned} Z_N &= \frac{1}{d_N} \sum_{k \in I_N} \left| \sum_{\alpha,\beta=-N}^{N} \langle BY_N^\alpha, Y_N^\beta\rangle u_{N,k}(\alpha) u_{N,k}(\beta) - \omega(B) \right|^2 \\ &= \frac{1}{d_N} \sum_{k \in I_N} \left| \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha\rangle |u_{N,k}(\alpha)|^2 - \omega(B) \right|^2 \\ &= S_1 + S_2, \end{aligned}$$

where

$$S_1 = \frac{1}{d_N} \sum_{k \in I_N} \sum_{\alpha,\beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2,$$

$$S_2 = -\frac{2\omega(B)}{d_N} \sum_{k \in I_N} \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha \rangle |u_{N,k}(\alpha)|^2 + \frac{1}{d_N} \sum_{k \in I_N} \omega(B)^2.$$

We use the Weingarten formula [64] to compute the expectation $\mathbb{E}Z_N = \mathbb{E}S_1 + \mathbb{E}S_2$. Let $(u_{N,k}(\alpha))_{-N \le k,\alpha \le N} \in \mathrm{U}(d_N)$ be a unitary matrix and for $1 \le j \le m$, let $k_j, k'_j, \alpha_j, \alpha'_j \in [-N, N]$ be indices. The Weingarten formula states that the integral

$$I_N(m) := \int_{\mathrm{U}(d_N)} u_{N,k_1}(\alpha_1) \cdots u_{N,k_m}(\alpha_m) \overline{u_{N,k'_1}(\alpha'_1)} \cdots \overline{u_{N,k'_m}(\alpha'_m)}\, dU_N$$

of a polynomial in the entries of $(u_{N,k}(\alpha))$ with respect to Haar measure dU_N has an asymptotic formula in terms of the Kronecker delta functions on the indices:

$$I_N(m) = d_N^{-m} \sum \delta_{k_1 k'_{j_1}} \delta_{\alpha_1 \alpha'_{j_1}} \cdots \delta_{k_\ell k'_{j_m}} \delta_{\alpha_\ell \alpha'_{j_m}} + O(d_N^{-m-1}), \tag{3.9}$$

where the sum is over all choices of $j_1, \ldots, j_m$ as a permutation of $1, \ldots, m$. Let Q be the polynomial in $2m$ variables defined by $Q\big((z_j, w_j)_{j=1}^m\big) := z_1 \cdots z_m \overline{w}_1 \cdots \overline{w}_m$. Then, in the notation of Theorem 3.2.1, direct computation with Gaussian random variables shows that

$$\left| \frac{1}{d_N^m} \mathbb{E} Q\left(\big(\mathcal{N}_j^{(1)} + i\mathcal{N}_J^{(2)}, \mathcal{N}_J^{(1)} - i\mathcal{N}_J^{(2)}\big)_{j=1}^m \right) - I_N(m) \right| = O(d_N^{-m-1}) \tag{3.10}$$

Putting together (3.9), (3.10), and Theorem 3.2.1 proves the following key lemma.

Lemma **3.3.2.** *Let $(u_{N,k}(\alpha)) \in \mathrm{U}(d_N)$ be a unitary matrix. For indices*

$$k_1, \dots, k_m, k'_1, \dots, k'_m \in I_N \quad \textit{and} \quad \alpha_1, \dots, \alpha_m, \alpha'_1, \dots, \alpha'_m \in [-N, N],$$

we have

$$\begin{aligned}(3.11) \qquad & \mathbb{E}\left(u_{N,k_1}(\alpha_1) \cdots u_{N,k_m}(\alpha_m)\overline{u_{N,k'_1}(\alpha'_1)} \cdots \overline{u_{N,k'_m}(\alpha'_m)}\right) \\ &= d_N^{-m} \sum \delta_{k_1 k'_{j_1}} \delta_{\alpha_1 \alpha'_{j_1}} \cdots \delta_{k_m k'_{j_m}} \delta_{\alpha_m \alpha'_{j_m}} + O(d_N^{-m-\varepsilon})\end{aligned}$$

for some $\varepsilon = \varepsilon(Q) > 0$ guaranteed by Theorem 3.2.1.

Returning to the quantity $\mathbb{E}Z_N = \mathbb{E}S_1 + \mathbb{E}S_2$, we find that (3.11) implies

$$\mathbb{E}\left(|u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2\right) = d_N^{-2}(1 + \delta_{\alpha\beta}) + O(d_N^{-2-\varepsilon_1}) \quad \text{for } k \in I_N,$$

which gives

$$\begin{aligned}(3.12) \qquad \mathbb{E}S_1 &= \frac{1}{d_N} \sum_{k \in I_N} \sum_{\alpha,\beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \mathbb{E}\left(|u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2\right) \\ &= \sum_{\alpha,\beta} \langle BY_N^\alpha, Y_N^\alpha \rangle \langle BY_N^\beta, Y_N^\beta \rangle \left(\frac{1}{d_N^2}(1 + \delta_{\alpha\beta}) + O(d_N^{-2-\varepsilon_1})\right) \\ &= \left(\frac{1}{d_N} \sum_\alpha \langle BY_N^\alpha, Y_N^\alpha \rangle\right)^2 + \frac{1}{d_N^2} \sum_\alpha \langle BY_N^\alpha, Y_N^\alpha \rangle^2 + O(d_N^{-\varepsilon_1}).\end{aligned}$$

The first sum in (3.12) can be rewritten using semi-classical analysis. Let $\Pi_N \colon L^2(S^2) \to \mathcal{H}_N$ denote the spectral projection onto the eigenspace of degree N spherical harmonics. Let $A \in \Psi^0(S^2)$ be any pseudo-differential operator of degree zero (not necessarily rotationally invariant), then Weyl's law states that

$$\frac{1}{d_N}\sum_{\alpha}\langle AY_N^{\alpha}, Y_N^{\alpha}\rangle = \frac{1}{d_N}\operatorname{tr}(\Pi_N A \Pi_N) = \omega(A) + O(d_N^{-1}). \tag{3.13}$$

For the second sum in (3.12), it suffices to note that the squares $\langle AY_N^{\alpha}, Y_N^{\alpha}\rangle^2$ of the matrix elements are uniformly bounded in N because the pseudo-differential operator $A \in \Psi^0(S^2)$ (again, not necessarily rotationally invariant) is a bounded operator from $L^2(S^2)$ to itself. Since we are summing over $-N \leq \alpha \leq N$ (i.e., summing d_N number of terms) and dividing by d_N^2, the second sum has only a lower order contribution:

$$\frac{1}{d_N^2}\sum_{\alpha}\langle BY_N^{\alpha}, Y_N^{\alpha}\rangle^2 = O(d_N^{-1}). \tag{3.14}$$

Combining (3.12), (3.13), and (3.14) yields

$$\mathbb{E}S_1 = \big(\omega(B) + O(d_N^{-1})\big)^2 + O(d_N^{-1}) + O(d_N^{-\varepsilon_1}) = \omega(B)^2 + O(d_N^{-\varepsilon_1}).$$

The asymptotics for $\mathbb{E}S_2$ is similarly computed. By (3.11), we have

$$\mathbb{E}|u_{N,k}(\alpha)|^2 = d_N^{-1} + O(d_N^{-1-\varepsilon_2}) \quad \text{for } k \in I_N,$$

whence

$$\begin{aligned}
\mathbb{E}S_2 &= -\frac{2\omega(B)}{d_N}\sum_{k\in I_N}\sum_{\alpha}\langle BY_N^{\alpha}, Y_N^{\alpha}\rangle \mathbb{E}|u_{N,k}(\alpha)|^2 + \frac{1}{d_N}\sum_{k\in I_N}\omega(B)^2 \\
&= -2\omega(B)\sum_{\alpha}\langle BY_N^{\alpha}, Y_N^{\alpha}\rangle\left(\frac{1}{d_N} + O(d_N^{-1-\varepsilon_2})\right) + \omega(B)^2 \\
&= -2\omega(B)^2 + \omega(B)^2 + O(d_N^{-\varepsilon_2}),
\end{aligned}$$

where the last equality follows from Weyl's law (3.13). Adding together the expressions for $\mathbb{E}S_1$ and $\mathbb{E}S_2$ shows that $\mathbb{E}Z_N = O(d_N^{-\min\{\varepsilon_1,\varepsilon_2\}}) = O(d_N^{-\varepsilon})$ as the factors of $\omega(B)^2$ cancel exactly. This proves the first part of Proposition 3.3.1.

The computations for the second moment $\mathbb{E}Z_N^2$ is more tedious, but no new techniques are required. Write a second copy of the random variable Z_N with the indices j, η, ξ in place of k, α, β, then direct computation shows

$$\mathbb{E}Z_N^2 = T_1 + T_2 + \cdots + T_5,$$

where

$$\begin{aligned}
T_1 &= \frac{1}{d_N^2} \sum_{k,j\in I_N} \sum_{\alpha,\beta,\eta,\xi} \langle BY_N^\alpha, Y_N^\alpha\rangle \langle BY_N^\beta, Y_N^\beta\rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2 \\
&\qquad\qquad \times \langle BY_N^\eta, Y_N^\eta\rangle \langle BY_N^\xi, Y_N^\xi\rangle |u_{N,j}(\eta)|^2 |u_{N,j}(\xi)|^2, \\
T_2 &= -\frac{4\omega(B)}{d_N^2} \sum_{k,j\in I_N} \sum_{\alpha,\beta,\eta} \langle BY_N^\alpha, Y_N^\alpha\rangle \langle BY_N^\beta, Y_N^\beta\rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2 \\
&\qquad\qquad \times \langle BY_N^\eta, Y_N^\eta\rangle |u_{N,j}(\eta)|^2, \\
T_3 &= \frac{2\omega(B)^2}{d_N^2} \sum_{k,j\in I_N} \sum_{\alpha,\beta} \langle BY_N^\alpha, Y_N^\alpha\rangle \langle BY_N^\beta, Y_N^\beta\rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2, \\
T_4 &= \frac{4\omega(B)^2}{d_N^2} \sum_{k,j\in I_N} \sum_{\alpha,\eta} \langle BY_N^\alpha, Y_N^\alpha\rangle |u_{N,k}(\alpha)|^2 \langle BY_N^\eta, Y_N^\eta\rangle |u_{N,j}(\eta)|^2, \\
T_5 &= -\frac{4\omega(B)^3}{d_N^2} \sum_{k,j\in I_N} \sum_{\alpha} \langle BY_N^\alpha, Y_N^\alpha\rangle |u_{N,k}(\alpha)|^2 + \frac{1}{d_N^2} \sum_{k,j\in I_N} \omega(B)^4.
\end{aligned}$$

We work out the asymptotics for $\mathbb{E}T_1$ in detail. Appealing once again to (3.11), we have

$$\mathbb{E}\big(|u_{N,k}(\alpha)|^2|u_{N,k}(\beta)|^2|u_{N,j}(\eta)|^2|u_{N,j}(\xi)|^2\big) = d_N^{-4}\big(C_1 + \delta_{kj}C_2\big) + O(d_N^{-4-\varepsilon_1'}), \tag{3.15}$$

where

$$\begin{aligned} C_1 &= C_1(\alpha,\beta,\eta,\xi) = (1+\delta_{\alpha\beta})(1+\delta_{\eta\xi}), \\ C_2 &= C_2(\alpha,\beta,\eta,\xi) = \delta_{\alpha\eta}(1+\delta_{\beta\xi}+2\delta_{\eta\xi}) + \delta_{\alpha\xi}(1+\delta_{\beta\eta}+2\delta_{\beta\xi}) \\ &\qquad\qquad + \delta_{\beta\eta}(1+2\delta_{\alpha\beta}) + \delta_{\beta\xi}(1+2\delta_{\eta\xi}) + 6\delta_{\alpha\beta}\delta_{\beta\xi}\delta_{\eta\xi}. \end{aligned}$$

These imply

$$\begin{aligned} \mathbb{E}T_1 &= \frac{1}{d_N^4}\sum_{\alpha,\beta,\eta,\xi} C_1(\alpha,\beta,\eta,\xi)\langle BY_N^\alpha, Y_N^\alpha\rangle\langle BY_N^\beta, Y_N^\beta\rangle\langle BY_N^\eta, Y_N^\eta\rangle\langle BY_N^\xi, Y_N^\xi\rangle \\ &+ \frac{1}{d_N^5}\sum_{\alpha,\beta,\eta,\xi} C_2(\alpha,\beta,\eta,\xi)\langle BY_N^\alpha, Y_N^\alpha\rangle\langle BY_N^\beta, Y_N^\beta\rangle\langle BY_N^\eta, Y_N^\eta\rangle\langle BY_N^\xi, Y_N^\xi\rangle + O(d_N^{-\varepsilon_1'}). \end{aligned} \tag{3.16}$$

Notice that the leading orders of C_1 and C_2 are different because there is a factor of δ_{kj} in front of C_2 but not C_1 in (3.15).

Consider the first line of the expression (3.16) (i.e., the part that involves only C_1). Recall that $C_1 = (1+\delta_{\alpha\beta})(1+\delta_{\eta\xi}) = 1+\delta_{\alpha\beta}+\delta_{\eta\xi}+\delta_{\alpha\beta}\delta_{\eta\xi}$ contains four terms. We claim that only the constant term has a top order contribution when computing the asymptotics of $\mathbb{E}T_1$; the other three terms containing Kronecker delta functions all have lower order contributions. Indeed, notice that

$$\frac{1}{d_N^4}\sum_{\alpha,\beta,\eta,\xi}\delta_{\alpha\beta}\langle BY_N^\alpha, Y_N^\alpha\rangle\langle BY_N^\beta, Y_N^\beta\rangle\langle BY_N^\eta, Y_N^\eta\rangle\langle BY_N^\xi, Y_N^\xi\rangle$$

is equal to

$$\frac{1}{d_N^4}\sum_{\alpha,\eta,\xi}\langle BY_N^\alpha, Y_N^\alpha\rangle^2\langle BY_N^\eta, Y_N^\eta\rangle\langle BY_N^\xi, Y_N^\xi\rangle = O(d_N^{-1}),$$

which is a lower order term because we are summing d_N^3 number of uniformly bounded products of matrix elements but dividing by d_N^4.

We now turn our attention to the second line of the expression (3.16) (i.e., the part that involves only C_2). Notice that each term of C_2 contains at least one Kronecker delta function on the indices α, β, η, ξ. At the same time, we are dividing the sum by d_N^5. Therefore, the entire second line is of order at most $O(d_N^{-2})$. These observations imply that the expected value of T_1 has the simple asymptotics

$$\begin{aligned}\mathbb{E}T_1 &= \frac{1}{d_N^4}\sum_{\alpha,\beta,\eta,\xi}\langle BY_N^\alpha, Y_N^\alpha\rangle\langle BY_N^\beta, Y_N^\beta\rangle\langle BY_N^\eta, Y_N^\eta\rangle\langle BY_N^\xi, Y_N^\xi\rangle + O(d_N^{-\varepsilon_1'})\\ &= \omega(B)^4 + O(d_N^{-\varepsilon_1'}).\end{aligned}$$

Similar arguments show that

$$\begin{aligned}\mathbb{E}T_2 = &-\frac{4\omega(B)}{d_N^2}\sum_{k,j\in I_N}\sum_{\alpha,\beta,\eta}\langle BY_N^\alpha, Y_N^\alpha\rangle\langle BY_N^\beta, Y_N^\beta\rangle\langle BY_N^\eta, Y_N^\eta\rangle\\ &\qquad\qquad\times\frac{1}{d_N^3}\big(1+\delta_{\alpha\beta}+\delta_{kj}(\delta_{\alpha\eta}+\delta_{\beta\eta}+2\delta_{\alpha\beta}\delta_{\beta\eta})\big) + O(d_N^{-\varepsilon_2'})\\ = &-4\omega(B)^4 + O(d_N^{-\varepsilon_2'}),\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}T_3 &= \frac{2\omega(B)^2}{d_N^2}\sum_{k,j\in I_N}\sum_{\alpha,\beta}\langle BY_N^\alpha, Y_N^\alpha\rangle\langle BY_N^\beta, Y_N^\beta\rangle\frac{1}{d_N^2}(1+\delta_{\alpha\beta}) + O(d_N^{-\varepsilon_3'})\\
&= 2\omega(B)^4 + O(d_N^{-\varepsilon_3'}),\\
\mathbb{E}T_4 &= \frac{4\omega(B)^2}{d_N^2}\sum_{k,j\in I_N}\sum_{\alpha,\eta}\langle BY_N^\alpha, Y_N^\alpha\rangle\langle BY_N^\eta, Y_N^\eta\rangle\frac{1}{d_N^2}(1+\delta_{kj}\delta_{\alpha\eta}) + O(d_N^{-\varepsilon_3'})\\
&= 4\omega(B)^4 + O(d_N^{-\varepsilon_3'}),\\
\mathbb{E}T_5 &= -\frac{4\omega(B)^3}{d_N^2}\sum_{k,j\in I_N}\sum_{\alpha}\langle BY_N^\alpha, Y_N^\alpha\rangle\frac{1}{d_N} + \frac{1}{d_N^2}\sum_{k,j\in I_N}\omega(B)^4 + O(d_N^{-\varepsilon_4'})\\
&= -4\omega(B)^4 + \omega(B)^4 + O(d_N^{-\varepsilon_4'}).
\end{aligned}
$$

As before, the factors of $\omega(B)^4$ cancel exactly, and we are left with

$$
\mathbb{E}Z_N^2 = \mathbb{E}T_1 + \cdots + \mathbb{E}T_5 = O(d_N^{-\min\{\varepsilon_1',\ldots,\varepsilon_4'\}}) = O(d_N^{-\varepsilon'}).
$$

This concludes the proof of Proposition 3.3.1. $\square$

3.4. Proof of Main Theorems

We now return to Theorem 3.1.1 and Theorem 3.1.3, which do not have invariance assumptions on the operator $A \in \Psi^0(S^2)$. This means that we can no longer assume *a priori* (as we did in the previous section) that the matrix elements $\langle AY_N^\alpha, Y_N^\beta\rangle$ vanish for $\alpha \neq \beta$. We will show, however, that by taking a Fourier series representation of the operator A and using orthogonality properties of the spherical harmonics, the general case reduces to the rotationally invariant case.

3.4.1. Reduction to Fourier coefficients

The goal of this section is to obtain a Fourier series representation for a general pseudo-differential operator. Let r_θ denote rotation about the z-axis by angle θ, that is, if we write a point $x = (\cos\tau \sin\varphi, \sin\tau\sin\varphi, \cos\varphi) \in S^2$ in spherical coordinates, then

$$r_\theta(x) := (\cos(\tau-\theta)\sin\varphi, \sin(\tau-\theta)\sin\varphi, \cos\varphi).$$

Given $A \in \Psi^0(S^2)$, form a new operator

$$A_\theta := r_\theta^* A r_{-\theta}^* \in \Psi^0(S^2),$$

where $(r_\theta^*\varphi)(x) := \varphi(r_\theta(x))$ for any smooth function $\varphi \in C^\infty(S^2)$. For $n \in \mathbb{Z}$, the Fourier coefficients $\hat{A}(n)$ of A_θ are defined by

$$\hat{A}(n) := \fint_{S^1} e^{-in\theta} A_\theta \, d\theta \in \Psi^0(S^2). \tag{3.17}$$

These new operators are related to the original operator A in the following way.

Lemma **3.4.1.** *The partial sums $\sum_{|n|\le N} \hat{A}(n)$ converge in the operator norm to A as $N \to \infty$.*

Proof of Lemma 3.4.1. Let D_θ denote the generator of z-axis rotation so that $r_\theta^* = e^{-i\theta D_\theta}$. Then, since D_θ and r_θ^* commute, we have

$$\frac{\partial}{\partial\theta} A_\theta = \left(\frac{\partial}{\partial\theta} r_\theta^*\right) A r_{-\theta}^* + r_\theta^* A \left(\frac{\partial}{\partial\theta} r_{-\theta}^*\right) = \frac{1}{i}(D_\theta A_\theta - A_\theta D_\theta) = \frac{1}{i}\,\mathrm{ad}_{D_\theta}(A_\theta) \in \Psi^0(M).$$

This implies that the map $\theta \mapsto A_\theta$ is differentiable, and by elementary properties of convolution with the Dirichlet kernel $D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}$ we get uniform convergence

$$\sum_{n=-N}^{N} \hat{A}(n) = \sum_{n=-N}^{N} \fint_{S^1} e^{-in\theta} A_\theta \, d\theta = \fint_{S^1} D_N(\theta) A_\theta \, d\theta \to A_0 = A. \qquad \square$$

Lemma **3.4.2.** *For $n \neq 0$, we have $\|\hat{A}(n)\| = O(n^{-\ell})$ for every $\ell \geq 1$.*

Proof of Lemma 3.4.2. Integrating (3.17) by parts gives

$$n\hat{A}(n) = \frac{i}{2\pi} e^{-in\theta} A_\theta \Big|_{\theta=0}^{2\pi} - \fint_{S^1} e^{-in\theta} \, \mathrm{ad}_{D_\theta}(A_\theta) \, d\theta = -\fint_{S^1} e^{-in\theta} \, \mathrm{ad}_{D_\theta}(A_\theta) \, d\theta.$$

It follows that integrating by parts ℓ times yields

$$(-n)^\ell \hat{A}(n) = \fint_{S^1} e^{-in\theta} (\mathrm{ad}_{D_\theta})^\ell (A_\theta) \, d\theta.$$

Since $(\mathrm{ad}_{D_\theta})^\ell(A_\theta) \in \Psi^0(S^2)$ for all $\ell \geq 1$, we conclude that $n^\ell \|\hat{A}(n)\| = O(1)$. $\square$

These lemmas allow us to replace A with finite sums of the form $\sum_{|n| \leq N} \hat{A}(n)$. We record several facts about the operators $\hat{A}(n)$. First, conjugating by rotation $A \mapsto r_\theta^* A r_{-\theta}^* = A_\theta$ changes the principal symbol of A by the canonical transformation on the cosphere bundle:

$$\sigma_{A_\theta}(x, \xi) = \sigma_A(r_\theta(x), (Dr_{-\theta}(x))^{-1} \xi).$$

It follows from definition (3.17) of $\hat{A}(n)$ that

$$\omega(\hat{A}(n)) := \int_{S^*M} \fint_{S^1} e^{-in\theta} \sigma_A(r_\theta(x), (Dr_{-\theta}(x))^{-1}\xi)\, d\theta d\mu_L = \begin{cases} \omega(A) & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases} \tag{3.18}$$

where the latter equality follows from interchanging the order of integration and using the fact that the Liouville measure μ_L is invariant under canonical transformations.

Second, from the definition of spherical harmonics, for each fixed n the matrix elements of $\hat{A}(n)$ are related to those of A by the identity

$$\langle \hat{A}(n) Y_N^\alpha, Y_N^\beta \rangle = \begin{cases} \langle A Y_N^\alpha, Y_N^{\alpha-n} \rangle & \text{if } \alpha = \beta + n \\ 0 & \text{if } \alpha \neq \beta + n \end{cases} \quad \text{simultaneously for all } N. \tag{3.19}$$

In other words, the infinite block-diagonal matrix with blocks $(\langle \hat{A}(n) Y_N^\alpha, Y_N^\beta \rangle)_{\alpha,\beta=-N}^N$ is obtained from the infinite block diagonal matrix with blocks $(\langle A Y_N^\alpha, Y_N^\beta \rangle)_{\alpha,\beta=-N}^N$ by replacing all the entries except those on the nth diagonal above (or below, depending on the sign of n) the main diagonal by zeros.

3.4.2. Computations with Fourier coefficients

Having defined Fourier coefficients $\hat{A}(n)$ and discussed their properties, we proceed to compute the expected value and second moment of the associated random variables

$$W_{n,N} := \frac{1}{d_N} \sum_{k \in I_N} |\langle \hat{A}(n) \psi_{N,k}, \psi_{N,k} \rangle - \omega(\hat{A}(n))|^2$$

$$= \begin{cases} \dfrac{1}{d_N} \displaystyle\sum_{k \in I_N} \left| \sum_{\alpha=-N+n}^{N} \langle A Y_N^\alpha, Y_N^\alpha \rangle u_{N,k}(\alpha) \overline{u_{N,k}(\alpha)} - \omega(A) \right|^2 & \text{if } n = 0, \\ \dfrac{1}{d_N} \displaystyle\sum_{k \in I_N} \left| \sum_{\alpha=-N+n}^{N} \langle A Y_N^\alpha, Y_N^{\alpha-n} \rangle u_{N,k}(\alpha) \overline{u_{N,k}(\alpha - n)} \right|^2 & \text{if } n \neq 0, \end{cases}$$

where the second equality is obtained by first writing $\psi_{N,k}$ in terms of Y_N^α using (3.1), and then applying (3.18) and (3.19). We make the crucial observation that the discussion following (3.19) implies the identity

$$X_N = \sum_{n \in \mathbb{Z}} W_{n,N} \quad \text{for each } N = 0, 1, 2, \ldots. \tag{3.20}$$

The asymptotics for $\mathbb{E} W_{n,N}$ and $\mathbb{E} W_{n,N}^2$ can be easily computed.

Lemma **3.4.3.** *For each fixed $n \in \mathbb{Z}$, we have $\mathbb{E} W_{n,N} = O(d_N^{-\varepsilon})$ and $\mathbb{E} W_{n,N}^2 = O(d_N^{-\varepsilon'})$ for some $\varepsilon, \varepsilon' > 0$ guaranteed by Theorem 3.2.1.*

Proof of Lemma 3.4.3. Thanks to (3.19), we recognize that $\hat{A}(0)$ is a rotationally invariant operator of the kind considered in Section 3.3. Thus, when $n = 0$ the statement of the lemma follows from Proposition 3.3.1.

When $n \neq 0$, expanding the square yields

$$W_{n,N} = \frac{1}{d_N} \sum_{k \in I_N} \sum_{\alpha, \beta} \langle A Y_N^\alpha, Y_N^{\alpha-n} \rangle \langle A Y_N^\beta, Y_N^{\beta-n} \rangle u_{N,k}(\alpha) u_{N,k}(\beta) \overline{u_{N,k}(\alpha - n) u_{N,k}(\beta - n)}.$$

Appealing once again to the asymptotic formula (3.11), we find

$$\mathbb{E} \left(u_{N,k}(\alpha) u_{N,k}(\beta) \overline{u_{N,k}(\alpha - n) u_{N,k}(\beta - n)} \right) = d_N^{-2} (\delta_{\alpha,\alpha-n} \delta_{\beta,\beta-n} + \delta_{\alpha,\beta-n} \delta_{\beta,\alpha-n}) + O(d_N^{-2-\varepsilon}).$$

Since $n \neq 0$ by hypothesis, by what is now a standard argument we conclude that all the terms in the expression of $\mathbb{E}W_{n,N}$ that contain Kronecker delta functions are of order at most $O(d_N^{-1})$, so $\mathbb{E}W_{n,N} = O(d_N^{-\varepsilon})$.

The second moment computation is equally straightforward. Indeed, we have

$$W_{n,N}^2 = \frac{1}{d_N^2} \sum_{k,j \in I_N} \sum_{\alpha,\beta,\eta,\xi} \langle AY_N^\alpha, Y_N^{\alpha-n}\rangle \langle AY_N^\beta, Y_N^{\beta-n}\rangle \langle AY_N^\eta, Y_N^{\eta-n}\rangle \langle AY_N^\xi, Y_N^{\xi-n}\rangle$$
$$\times\, u_{N,k}(\alpha) u_{N,k}(\beta) \overline{u_{N,k}(\alpha-n) u_{N,k}(\beta-n)} u_{N,j}(\eta) u_{N,j}(\xi) \overline{u_{N,j}(\eta-n) u_{N,j}(\xi-n)}.$$

It is easy to verify using (3.11) that the expected value of the product of eigenvector components is asymptotically zero because every term in the asymptotic formula contains a factor of $\delta_{\alpha,\alpha-n}$ for $n = 1, \ldots, 4$. $\square$

3.4.3. Approximation argument

We finish the computations for $\mathbb{E}X_N$ and $\mathbb{E}X_N^2$ by an approximation argument.

Proof of Theorem 3.1.3. Fix some small constant $\omega > 0$, then by (3.20) there exists $M > 0$ such that $\sum_{|n|>M} W_{n,N} < \omega$. Using Lemma 3.4.3 for the asymptotics of $\mathbb{E}W_{n,N}$ yields

$$\mathbb{E}X_N \le \mathbb{E}\Bigg(\sum_{|n| \le M} W_{n,N} + \omega \Bigg) = \sum_{|n|\le M} \mathbb{E}W_{n,N} + \omega = O(d_N^{-\varepsilon}) + \omega.$$

The asymptotics for the second moment is similarly computed using the elementary inequality $(a_1 + \cdots + a_m)^2 \leq m(a_1^2 + \cdots + a_m^2)$ and Lemma 3.4.3:

$$\begin{aligned}
\mathbb{E}X_N^2 &\leq \mathbb{E}\Big(\sum_{|n|\leq M} W_{n,N} + \omega \Big)^2 \\
&\leq (2M+1) \sum_{|n|\leq M} \mathbb{E}W_{n,N}^2 + 2\omega \sum_{|n|\leq M} \mathbb{E}W_{n,N} + \omega^2 \\
&= O(d_N^{-\varepsilon'}) + O(d_N^{-\varepsilon}) + \omega^2.
\end{aligned}$$

Since ω is arbitrary, Theorem 3.1.3 is proved with $\varepsilon_0 = \varepsilon$ and $\varepsilon_0' = \min\{\varepsilon, \varepsilon'\}$. □

Proof of Theorem 3.1.1. Let $\sigma_N^2 := \mathbb{E}X_N^2 - (\mathbb{E}X_N)^2$ be the variance of the random variable X_N. Theorem 3.1.1 shows that the sequence $\{X_N\}$ satisfies Kolmogorov's convergence criterion, that is, $\sum_{N=1}^{\infty} \sigma_N^2/N^2 < \infty$. We may therefore invoke the Strong Law of Large Numbers to conclude that the partial sums $\frac{1}{M}\sum_{N=0}^{M} X_N$ converge to its expected value almost surely. But $\mathbb{E}X_N = O(d_N^{-\varepsilon})$, which implies that the expected values of the partial sums converge to zero, finishing the proof of Theorem 3.1.1. □

CHAPTER 4

Log-scale Equidistribution of Zeros of Quantum Ergodic Eigensections

This chapter concerns the small-scale equidistribution of masses and of zeros of holomorphic eigensections in the line bundle setting introduced in Section 2.5–Section 2.6. The main theorems, Theorem 4.1.2 and Theorem 4.1.4, are small-scale versions of the ones presented in Section 2.6.

Let $(L, h) \to (M, \omega)$ be a prequantum line bundle over a compact Kähler manifold of complex dimension m without boundary(Definition 2.5.1). Under certain quantization conditions (discussed in Section 4.2.2 and [69]), a symplectic map

$$\chi\colon (M, \omega) \to (M, \omega), \qquad \chi^*\omega = \omega$$

on the base manifold can be quantized as a sequence $\{U_{\chi,N}\}_{N=1}^{\infty}$ of unitary Fourier integral Toeplitz operators

$$U_{\chi,N}\colon H^0(M, L^N) \to H^0(M, L^N)$$

acting on the spaces $H^0(M, L^N)$ of global holomorphic sections of L^N with the inner product (2.12) induced by h.

The eigensections $s_j^N \in H^0(M, L^N)$ of the operators $U_{\chi,N}$ are characterized by

$$U_{\chi,N} s_j^N = e^{i\theta_{N,j}} s_j^N, \qquad 1 \le j \le d_N,$$

where $e^{i\theta_{N,j}}$ are eigenphases and $d_N = \dim H^0(M, L^N)$. We write

$$Z_{s_j^N} = \{z \in M \colon s_j^N(z) = 0\} \quad \text{and} \quad \left[Z_{s_j^N}\right] = \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log \|s_j^N(z)\|_{h^N}^2 + N\omega$$

for the zero set of s_j^N and the current of integration[1] over the zero set of s_j^N, respectively. Assuming χ is ergodic, Zelditch [**69**] proved that the eigensections of the quantum maps $U_{\chi,N}$ are quantum ergodic. Moreover, Nonnenmacher–Voros [**50**] and Shiffman–Zelditch [**57**] (see also Rudnick [**51**] for the modular surface setting) proved that the zeros of 'almost all' quantum ergodic eigensections are asymptotically equidistributed with respect to the Kähler volume form: There exists a subsequence $\Gamma \subset \{(N, j) \colon N \ge 1,\ j = 1, \dots, d_N\}$ of density one for which

$$\lim_{\substack{(N,j)\in\Gamma \\ N\to\infty}} \int_M f(z) \left[\frac{1}{N} Z_{s_j^N}\right] \wedge \omega^{m-1} = \int_M f \frac{\omega^m}{m!} \qquad \text{for all } f \in C(M). \tag{4.1}$$

4.1. Main Results

Let $m = \dim_{\mathbb{C}} M$. Fix a logarithmic scale ε_N depending on parameter γ:

$$\varepsilon_N := |\log N|^{-\gamma} \quad \text{for some constant } 0 < \gamma < \frac{1}{6m} \text{ independent of } N. \tag{4.2}$$

The main purpose of this paper is to show (with additional assumptions on χ, described below) that the equidistribution result (4.1) holds with the domain of integration M

[1] Recall the Poincaré–Lelong formula (2.17).

replaced by any ball $B(p, \varepsilon_N)$ centered at $p \in M$ with radius $\varepsilon_N = |\log N|^{-\gamma}$ for any $\gamma < (6m)^{-1}$. This is what is meant by "equidistribution of zeros at the logarithmic scale."

To obtain this log-scale improvement, we use two dynamical properties of χ:

- For $T \in \mathbb{Z}$, let χ^T denote the T-fold iterate of χ (or of its inverse χ^{-1}, depending on the sign of T). By the chain-rule χ satisfies the exponential growth estimate

$$\|\chi^T\|_{C^2} = \mathcal{O}(e^{|T|\delta_0}) \quad \text{for some fixed constant } \delta_0 > 0 \text{ independent of } T. \tag{4.3}$$

  In particular, if χ lifts to a contact transformation $\tilde{\chi}$ on the unit co-disk bundle $X \to M$ (see Section 4.2.2), then $\|F \circ \tilde{\chi}^\ell\|_{C^2}^2 = \mathcal{O}_F(e^{2|T|\delta_0})$ for any $F \in C^\infty(X)$.

- We assume that χ has sufficiently fast decay of correlations[2]. Namely, that there exist constants $0 < \beta < 1$, $c_1 > 0$, and $c_2 = c_2(\beta) > 1$ such that[3]

$$\left| \int_M (g \circ \chi^T) f \, dV - \int_M f \, dV \int_M g \, dV \right| \leq c_1 (1 + |T|)^{-c_2} \|f\|_{C^{0,\beta}} \|g\|_{C^{0,\beta}} \tag{4.4}$$

  for all $f, g \in C^{0,\beta}(M)$. Here and throughout,

$$dV = \frac{\omega^m}{m!} \tag{4.5}$$

  is the normalized volume form.

The explicit error estimate in Egorov's theorem for Toeplitz operators (Proposition 4.3.1, proved in Section 4.6) relies on assumption (4.3). Assumption (4.4) is used in the proof of logarithmic decay of quantum variances (Theorem 4.1.6) in Section 4.3.

[2]Even though an exponential decay rate (i.e., with $(1 + |T|)^{-c_2}$ replaced by $e^{-c_2|T|}$) is often assumed in the literature, much less is necessary for the proof; this was also noted in Schubert [53].
[3]Note that this condition implies χ is mixing and hence ergodic.

4.1.1. Log-scale equidistribution of zeros

The log-scale equidistribution of zeros states that zeros in balls of radii ε_N are uniformly distributed with respect to the volume form (4.5). It is simplest to state the result by dilating such shrinking balls by ε_N^{-1} back to a fixed reference ball of radius 1. In a local Kähler normal coordinate chart (U, z) with $z = 0$ at p, define local dilation maps

$$D_\varepsilon^p \colon B(p,1) \to B(p,\varepsilon), \qquad D_\varepsilon z = \varepsilon z. \tag{4.6}$$

Here we abuse notation by writing $B(p,1)$ when we mean the image of the metric unit ball centered at p in the local coordinate chart based at p. The inverse dilation is defined by

$$(D_\varepsilon^p)^{-1} \colon B(p,\varepsilon) \to B(p,1).$$

Remark **4.1.1.** *Recall Kähler normal coordinates $z_1, \cdots, z_m$ centered at point z_0 are holomorphic coordinates in which z_0 has coordinates $0 \in \mathbb{C}^m$, and*

$$\omega(z) = i \sum_{j=1}^m dz_j \wedge d\bar{z}_j + O(|z|^2).$$

We may also choose a local reference frame e_L of the line bundle in a neighborhood of z_0 such that the induced Kähler potential φ takes the form

$$\varphi(z) = |z|^2 + O(|z|^3).$$

See [20] for background.

Let D_ε^{p*} be the corresponding pullback operator on forms. For simplicity of notation we denote the pullback $(D_\varepsilon^p)^{*-1}$ of the inverse dilation by $D^p_{\varepsilon *}$ so that

$$D^p_{\varepsilon *}\colon \mathcal{D}^{m-1,m-1}(B(p,1)) \to \mathcal{D}^{m-1,m-1}(B(p,\varepsilon)),$$

where $\mathcal{D}^{m-1,m-1}$ is the space of compactly supported smooth $(m-1,m-1)$ test forms. In particular, for $\eta \in \mathcal{D}^{m-1,m-1}(B(p,1))$, we have

$$\int_{B(p,\varepsilon)} D^p_{\varepsilon *}\eta \wedge \frac{1}{N}\big[Z_{s_j^N}\big] = \int_{B(p,1)} \left(\eta \wedge \frac{1}{N} D_\varepsilon^{p*}\big[Z_{s_j^N}\big]\right).$$

Theorem **4.1.2** (Equidistribution of zeros, Chang–Zelditch [11])**.** *Let $(L,h) \to (M,\omega)$ be a prequantum line bundle. Let χ satisfy (4.3) and (4.4). Let $\{s_1^N, \dots, s_{d_N}^N\}$ be an orthonormal basis of eigensections of $U_{\chi,N}$ acting on $H^0(M, L^N)$. Then, for every $0 < \gamma < (6m)^{-1}$ and $\varepsilon_N = |\log N|^{-\gamma}$, there exists a full density subsequence $\Gamma \subset \{(N,j)\colon j = 1, \dots, d_N\}$ such that for every $p \in M$,*

$$\frac{1}{N\varepsilon_N^2} D^{p*}_{\varepsilon_N}\big[Z_{s_j^N}\big] \xrightharpoonup{\Gamma \ni (N,j)\to\infty} \omega_0^p \quad \text{in the weak sense of currents on } B(p,1),$$

where $\omega_0^p = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log|z|^2$ is the flat Kähler form in Kähler normal coordinates at p.

Remark **4.1.3.** *The weak convergence statement in Theorem 4.1.2 means*

$$\int_{B(p,1)} \left(\eta \wedge \frac{1}{N\varepsilon_N^2} D^{p*}_{\varepsilon_N}\big[Z_{s_j^N}\big]\right) = \int_{B(p,1)} \eta \wedge \omega_0^p + o(1)$$

for every test form $\eta \in \mathcal{D}^{m-1,m-1}(B(p,1))$.

The key ingredients of the proof are the log-scale mass comparison result (Theorem 4.1.4), the Poincaré–Lelong formula (2.17) and compactness results on logarithms of scaled sections.

4.1.2. Log-scale equidistribution of mass

The equidistribution result of Theorem 4.1.2 is based on log-scale volume comparison theorems similar to those of Hezari–Rivière [**25**, Lemma 3.1] and Han [**24**, Corollary 1.9].

Theorem **4.1.4** (Equidistribution of masses, Chang–Zelditch [**11**]). *Under the assumptions of Theorem 4.1.2. Then, given any $0 < \gamma' < (6m)^{-1}$ and $\varepsilon'_N = |\log N|^{-\gamma'}$ as defined by (4.2), there exist a full density subsequence Γ and constants C_1, C_2 uniform in $p \in M$ and independent of N such that*

$$C_1 \frac{\mathrm{Vol}(B(p, \varepsilon'_N))}{\mathrm{Vol}(M)} \leq \int_{B(p,\varepsilon'_N)} \|s_j^N\|^2_{h^N} \, dV \leq C_2 \frac{\mathrm{Vol}(B(p, \varepsilon'_N))}{\mathrm{Vol}(M)} \quad \text{as } \Gamma \ni (N, j) \to \infty.$$

Here, dV is the normalized volume form (4.5).

There is no need to put primes on γ or ε_N in the statement above, but we do so to foreshadow that in the proof of Theorem 4.1.2, the result of Theorem 4.1.4 is applied with $\gamma < \gamma'$ and $\varepsilon'_N < \varepsilon_N$. The comparison (as opposed to asymptotic) result on log-scale mass equidistribution is sufficient for deriving equidistribution of zeros at a slightly larger logarithmic scale. In fact, only the lower bound is used, and the bound itself is much stronger than necessary for the proof.

Theorem 4.1.4 is based on a quantitative quantum variance estimate (Theorem 4.1.6) in the holomorphic setting. Before stating the estimate, we record here another one of its corollaries, which is analogous to [**24**, Corollary 1.8].

PROPOSITION **4.1.5** (Asymptotics for fixed center, Chang–Zelditch [**11**]). *Assume the hypotheses of Theorem 4.1.2. Fix $z_0 \in M$. Then, given any $0 < \gamma < (4m)^{-1}$ and ε_N as defined by (4.2), there exists a subsequence $\Gamma_{z_0} \subset \{(N, j)\}$ of density one such that*

$$\int_{B(z_0,\varepsilon_N)} \|s_j^N\|_{h^N}^2 \, dV = \frac{\mathrm{Vol}(B(z_0, \varepsilon_N))}{\mathrm{Vol}(M)} + o(|\log N|^{-2m\gamma}).$$

Here, dV is the normalized volume form (4.5).

Recall $\dim_{\mathbb{C}} M = m$, so $\frac{\mathrm{Vol}(B(z_0,\varepsilon_N))}{\mathrm{Vol}(M)} = C(M, g)\varepsilon_N^{2m} = C(M, g)|\log N|^{-2m\gamma}$. The differences between Proposition 4.1.5 and Theorem 4.1.4 are that the former is an asymptotic result for a fixed base point, whereas the latter is a comparison result that holds for all points in M. Moreover, in the former case the range of values that γ can take is improved. Proposition 4.1.5 is not used in proving Theorem 4.1.2 or Theorem 4.1.4.

4.1.3. Log-scale quantum ergodicity

By the quantum variance associated to f we mean the quantity

$$\mathcal{V}_N(f) := \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \int_M f(z) \|s_j^N\|_{h^N}^2 \, dV - \fint_M f \, dV \right|^2 \quad \text{for } f \in C^\infty(M). \tag{4.7}$$

Here, dV is the normalized volume (4.5). Thanks to Egorov's theorem for Toeplitz operators (Proposition 4.3.1, proved in Section 4.6) and the decay of correlations assumption (4.4), we show the quantum variance has a logarithmic decay rate when $f \in C^\infty(M)$:

THEOREM **4.1.6** (Decay of quantum variances, Chang–Zelditch [11]). *Assume the hypotheses of Theorem 4.1.2. Then, there exists a constant $\kappa_0 > 0$ independent of N such that for every $0 < \beta < 1$ and for every $f \in C^2(M)$,*

$$\mathcal{V}_N(f) = \mathcal{O}\left(\frac{\|f\|^2_{C^{0,\beta}}}{\log N}\right) + \mathcal{O}\left(\frac{\|f\|^2_{C^2}|\log N|^2}{N^{\frac{1}{2}}}\right) + \mathcal{O}\left(\frac{\|f\|^2_{C^{0,\beta}}}{N\log N}\right),$$

where $\|\cdot\|_{C^{0,\beta}}$ is the β-Hölder norm.

We specialize to the following logarithmically dilated symbols. In Kähler normal coordinates, let $f_{z_0} \in C_0^\infty(B(z_0,2),\mathbb{R})$ be a smooth cut-off function that is equal to 1 on $B(z_0,1)$, vanishes outside of $B(z_0,2)$ and satisfies $0 \leq f_{z_0} \leq 1$. For "small-scale quantum ergodicity," we work with locally dilated symbols (recall the notation (4.6)) of the form

$$f_{z_0,\varepsilon}(z) := D^{z_0}_{\varepsilon *} f_{z_0}(z) = f\left(\frac{z}{\varepsilon}\right) \in C_0^\infty(B(z_0,2\varepsilon),\mathbb{R}), \quad \text{where } z_0 \in M \text{ and } \varepsilon > 0.$$

Then set $\varepsilon = \varepsilon_N$. It follows from Theorem 4.1.6 that, to leading order in N, the quantum variance associated to such symbols have the estimate

$$\mathcal{V}_N(f_{z_0,\varepsilon_N}) = \mathcal{O}(\|f_{z_0}\|^2_{C^{0,\beta}}|\log N|^{2\gamma\beta-1}).$$

Since $0 < \beta < 1$ and $\gamma < \frac{1}{6m}$, we have $2\gamma\beta - 1 < 0$. Since the second term is smaller than the first, we obtain:

Corollary **4.1.7** (Decay of quantum variances 2, Chang–Zelditch [**11**]). *Let ε_N be as defined in* (4.2). *Under the same hypotheses as in Theorem 4.1.2, we have*

$$\mathcal{V}_N(f_{z_0,\varepsilon_N}) = \mathcal{O}(\|f_{z_0}\|^2_{C^{0,\beta}} |\log N|^{2\gamma\beta-1}),$$

where the error estimate is uniform in z_0.

An application of Corollary 4.1.7 and a covering argument together imply Theorem 4.1.4.

4.1.4. Further results

The results of this paper are the line bundle analogues of the small-scale quantum ergodicity results in the Riemannian setting proved in [**25, 24**]. Specializing to the torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$, Lester–Rudnick [**41**, Theorem 1.1] proved the stronger uniform mass distribution result

$$\lim_{k\to\infty} \sup_{B(y,r)\in\mathcal{B}_{j_k}} \left| \frac{1}{\mathrm{Vol}(B(y,r))} \int_{B(y,r)} |\varphi_{j_k}|^2 \, dx - 1 \right| = 0,$$

for a density one subsequence of Laplace eigenfunctions. The supremum is taken over the set $\mathcal{B}_{j_k}$ of balls $B(y,r) \subset \mathbb{T}^d$ of radii $r > \lambda_{j_k}^{-1/(2d-2)+o(1)}$.

For Hecke modular eigenforms, Lester-Matomäki-Radziwiłł [**40**, Theorem 1.5] proved that for a sequence $\{f_k\}$ of Hecke modular cusp forms of weight k, there exists a certain $\delta > 0$ such that

$$\sup_{\mathcal{R}\subset\mathcal{F}} \left| \int_{\mathcal{R}} y^k |f_k(z)|^2 \frac{dxdy}{y^2} - \frac{3}{\pi} \int_{\mathcal{R}} \frac{dxdy}{y^2} \right| \le C_\varepsilon (\log k)^{-\delta+\varepsilon},$$

where the supremum is taken over all rectangles $\mathcal{R}$ with sides parallel to the coordinate axes. This is a stronger result because it is valid for all Hecke eigenforms and because the supremum is taken over rectangles of any size rather than of size comparable to $\varepsilon_k = |\log k|^{-\gamma}$. The authors also proved [40, Theorem 1.1] the equidistribution of zeros (again without needing to possibly discard a density zero subsequence of f_k's):

$$\frac{\#\{z \in B(z_0, r) \colon f_k(z) = 0\}}{\# Z_{f_k}} = \frac{3}{\pi} \int_{B(z_0,r)} \frac{dxdy}{y^2} + \mathcal{O}\left(r (\log k)^{-\delta+\varepsilon}\right)$$

when $r \geq (\log k)^{-\delta/2+\varepsilon}$.

In the line bundle setting, Shiffman–Zelditch [57] proved equidistribution of zeros (not at the logarithmic scale) for random orthonormal bases of $H^0(M, L^N)$ as well as for eigensections of quantized ergodic symplectic maps. It is probable that Theorem 4.1.2 can also be generalized to random orthonormal bases using the construction discussed in Section 2.8.

4.1.5. Existence of quantizable ergodic symplectic maps

An obvious question is whether quantizable ergodic symplectic maps satisfying the decay of correlations condition (4.4) exist on a given Kähler manifold. (Any diffeomorphism satisfies the exponential growth estimate (4.3) automatically.) There seem to exist few studies of ergodic symplectic dynamics in dimensions > 2. After consulting with several experts in the field, we give a brief summary of the examples that we are aware of.

The simplest and most-studied examples are hyperbolic symplectic toral automorphisms induced by an element of $\mathrm{Sp}(2n, \mathbb{Z})$ and small perturbations of such automorphisms (see [69, 30] for their Toeplitz quantizations). More generally, any hyperbolic or Anosov symplectic diffeomorphism satisfies the assumptions. There is a quantization condition, but as explained in [17], it is always satisfied if one tensors with a flat line bundle and modifies the contact form.

Most studies of smooth ergodic maps concern volume preserving diffeomorphisms. Studies of ergodic symplectic diffeomorphisms on manifolds other than tori are rare except in dimension two, in which case ergodic (indeed, Bernoulli) symplectic diffeomorphisms are known to exist (see [29] and [1, Theorem 1.26]) on surfaces if any genus. As mentioned above, they are quantizable. We also mention that pseudo-Anosov diffeomorphisms are singular ergodic symplectic diffeomorphisms which are smooth away from a finite number of singular points. They act hyperbolically with respect to two transverse (singular) measured foliations. Since they are singular, our techniques do not apply directly but it is plausible that they can be modified by suitably cutting off singular points. These examples may turn out to be the most explicitly computable ones on surfaces other than tori and are very likely to satisfy all the conditions of this article.

In higher dimensions, Anosov diffeomorphisms have been studied on certain types of nilmanifolds in addition to tori (see [13]). Partially hyperbolic symplectic diffeomorphisms are studied in [48]. There are further partially hyperbolic examples obtained by perturbation. As explained to the authors by A. Wilkinson, a symplectic toral automorphism (or any partially hyperbolic symplectic diffeomorphism) can be perturbed to produce a symplectic diffeomorphism which is stably accessible (see [16]). Moreover, if the original map is "center bunched," then the perturbed map is stably ergodic (see [7]). These examples are additional to the usual Anosov diffeomorphisms of tori and their perturbations. We refer to these articles for the definitions and further discussion.

4.2. Background: Quantization of Symplectic Diffeomorphisms

This section explains the quantization $U_{\chi,N}$ of a symplectic diffeomorphism $\chi\colon (M,\omega) \to (M,\omega)$ as unitary FIOs. Additional background and notation are found in Section 2.5–Section 2.6.

4.2.1. Hardy space of CR holomorphic functions

Let (L^*, h^*) be the dual line bundle to $L \to M$. Thanks to the positivity of $c_1(h)$, the unit co-disk bundle $D^* \subset L^*$ relative to the dual metric h^* is a strictly pseudoconvex domain whose boundary

$$X := \partial D^* = \{v \in L^* \colon h^*(v,v) = 1\} \subset L^*$$

is a CR manifold. The Hardy space $H^2(X)$ is the space of square integrable CR functions on X, or equivalently the space of boundary values of holomorphic functions on the unit disk bundle with finite $L^2(X)$ norm.

We introduce a defining function ρ for X, which will be featured in the Boutet de Monvel–Sjöstrand parametrix. We write points in the co-disk bundle as $x = (z, \lambda e_L^*(z))$, where $\lambda \le 1$ and $e_L^*(z)$ is a normalized dual frame centered at $z \in M$. Define

$$\rho\colon D^* \to \mathbb{R}, \quad \rho(z, \lambda e_L^*(z)) = 1 - |\lambda|^2 e^{-\varphi(z)} \text{ where } \varphi \text{ is the Kähler potential.} \tag{4.8}$$

Then ρ is a defining function for X satisfying (i) ρ is defined in a neighborhood of X; (ii) $\rho > 0$ in D^*; (iii) $\rho = 0$ on X; (iv) $d\rho \neq 0$ near X. Define the contact form

$$\alpha = d^c \rho|_X.$$

Let r_θ be the natural circle action on X, that is, $r_\theta x = e^{i\theta}x$ for $x \in X$. Note that a section $s \in H^0(M, L)$ determines an equivariant function $\hat{s}$ on L^* by the rule

$$\hat{s}(z, \lambda) = (\lambda, s(z)), \qquad z \in M,\, \lambda \in L_z^*.$$

It is easy to verify restricting $\hat{s}$ to X yields $\hat{s}(r_\theta x) = e^{i\theta}\hat{s}(x)$. Conversely, a section $s^N \in H^0(M, L^N)$ determines an equivariant function $\hat{s}^N$ on L^* whose restriction to X satisfies $\hat{s}(N)(r_\theta x) = e^{iN\theta}\hat{s}^N(x)$. The map $s^N \mapsto \hat{s}^N$ is in fact a unitary equivalence between the space $H^0(M, L^N)$ of holomorphic sections and the weight spaces

$$H_N^2(X) := \left\{F \in H^2(X)\colon F(r_\theta x) = e^{iN\theta}F(x)\right\} \quad \text{with} \quad H^2(X) = \bigoplus_{N\geq 0} H_N^2(X).$$

The Szegő projector is the orthogonal projection

$$\Pi\colon L^2(X) \to H^2(X)$$

and its Fourier components are denoted by

$$\Pi_N\colon L^2(X) \to H_N^2(X).$$

4.2.2. Quantization of symplectic maps

We use the dynamical Toeplitz quantization method of [69]. A symplectic map $\chi\colon M \to M$ is quantizable if and only if it lifts to a connection-preserving contact transformation $\tilde{\chi}\colon X \to X$, that is, $\tilde{\chi}^*\alpha = \alpha$. Denote by

$$T_{\tilde{\chi}}\colon L^2(X) \to L^2(X), \qquad T_{\tilde{\chi}}F = F \circ \tilde{\chi}$$

the pre-composition by the lift $\tilde{\chi}$. Note that $\tilde{\chi}$ commutes with the natural circle action r_θ on X, and $\|\tilde{\chi}\|_{C^2(X)} = c \cdot \|\chi\|_{C^2(M)}$ for some constant c.

The quantization of a quantizable map χ is defined to be a unitary Fourier integral operator

$$U_\chi := \Pi \sigma T_{\tilde{\chi}} \Pi \colon H^2(X) \to H^2(X). \tag{4.9}$$

Here, σ is a zeroth order symbol that makes the operator U_χ defined by (4.10) unitary. Its existence is guaranteed by the construction in [69]). We emphasize again that T_χ denotes translation by the lifted map; such translation is not well-defined on the base because it does not preserve the line bundle.

Under the identification $H^2(X) = \bigoplus_{N \geq 0} H^2_N(X)$, U_χ decomposes into a sequence of unitary Fourier integral operators $U_{\chi,N}$ defined by

$$U_{\chi,N} := \Pi_N \sigma_N T_{\tilde{\chi}} \Pi_N \colon H^2_N(X) \to H^2_N(X). \tag{4.10}$$

Here, σ_N is a zeroth order symbol making $U_{\chi,N}$ unitary. The Fourier coefficients Π_N have an explicit parametrix given in (4.12).

4.2.3. Boutet de Monvel–Sjöstrand parametrix for the Szegő projector

In preparation for the proof of Egorov's theorem for Toeplitz operators (Proposition 4.3.1), we briefly recall the Boutet de Monvel–Sjöstrand parametrix for the Szegő kernel. Let $\Pi(x, y)$ denote the kernel of the Szegő projector Π in (4.9), that is,

$$\Pi F(x) = \int_X \Pi(x,y) F(y)\, dV(y) \quad \text{for all } F \in L^2(X).$$

It is proved in [5] that Π is a complex Fourier integral operator of positive type. Near the diagonal, there is a parametrix of the form

$$\Pi(x,y) \sim \int_0^\infty e^{it\psi(x,y)} s(x,y,t)\,dt,$$

where

$$s(x,y,t) \sim \sum_{n=0}^{\infty} t^{m-n} s_n(x,y)$$

belongs to the symbol class $S^m(X \times X \times \mathbb{R}_{\geq 0})$ and $\psi \in C^\infty(D^* \times D^*)$ is a complex phase of positive type. (Recall that D^* stands for the unit co-disk bundle, of which X is the boundary.)

The phase function ψ is obtained as the almost-analytic continuation of the defining function ρ in (4.8). Explicitly, for $x_j = (z_j, \lambda_j e_L^*(z_j)) \in D^*$, we have

$$\psi(x_1,x_2) = \frac{1}{i}\left(1 - \lambda_1 \bar{\lambda}_2 e^{-\frac{\varphi(z_1)}{2} - \frac{\varphi(z_2)}{2} + \varphi(z_1,\bar{z}_2)}\right),$$

where $\varphi(z_1,\bar{z}_2)$ is obtained from the Kähler potential φ by writing $\varphi(z_1) = \varphi(z_1,\bar{z}_1)$ on the diagonal of $M \times \overline{M}$ and extending to a neighborhood of the diagonal. When the metric is real analytic the extension is analytic; in the general C^∞ case it is almost-analytic. If we assume in addition that $x_j \in X$ lie on the co-circle bundle, then $\lambda_j = e^{i\tau_j}$ is uni-modular, whence $x_j = (z_j, \tau_j)$ and

$$\psi(x_1,x_2) = \psi(z_1,\tau_1,z_2,\tau_2) = \frac{1}{i}\left(1 - e^{-\frac{\varphi(z_1)}{2} - \frac{\varphi(z_2)}{2} + \varphi(z_1,\bar{z}_2)} e^{i(\tau_1-\tau_2)}\right) \quad \text{on } X \times X. \tag{4.11}$$

The kernels of the partial Szegő projectors Π_N in (4.10) are the Fourier coefficients of $\Pi(x,y)$:

$$\begin{aligned}\Pi_N(x,y) &= \int_0^\infty \int_{S^1} e^{-iN\theta} e^{it\psi(r_\theta x, y)} s(r_\theta x, y, t)\, d\theta dt \\ &= N \int_0^\infty \int_{S^1} e^{iN[-\theta + t\psi(r_\theta x, y)]} s(r_\theta x, y, Nt)\, d\theta dt,\end{aligned} \tag{4.12}$$

where the second line follows from a change of variable $t \mapsto Nt$.

4.2.4. Off-diagonal estimates and scaling asymptotics

We will be using two off-diagonal estimates for the lifted Szegő kernel on $X \times X$. Again, write $x_j = (z_j, \tau_j)$ for points in the co-circle bundle X. Let $d(z,w)$ be the distance with respect to the Kähler metric on M.

The first is an Agmon-type estimate giving global off-diagonal bounds:
(4.13)

$$|\Pi_N(x_1,x_2)| \leq A_1 N^m e^{-A_2 \sqrt{N} d(z_1,z_2)} \quad \text{for constants } A_1, A_2 \text{ independent of } N, x_1, x_2$$

due to Lindholm [43], Delin [14] and others. The second is a near diagonal Gaussian decay estimate: There exists $A_3 < 1$ independent of N, x_1, x_2 such that

$$|\Pi_N(x_1,x_2)| \leq \left(\frac{1}{\pi^m} + o(1)\right) N^m e^{-\frac{1-A_3}{2} N d(z_1,z_2)^2} + O(N^{-\infty}) \quad \text{whenever } d(z,w) \leq N^{-\frac{1}{3}}.$$

We refer to [57, 58, 46] for background and references.

We further use near off-diagonal scaling asymptotics from [58, 44]. At each $z \in M$ there is an osculating Bargmann–Fock or Heisenberg model associated to $(T_z M, J_z, h_z)$.

Let $(u, \theta_1, v, \theta_2)$ be linear coordinates on $T_zM \times S^1 \times T_zM \times S^1$. The model Heisenberg Szegő kernel on the tangent space is denoted by

$$\Pi^{T_zM}_{h_z,J_z}(u, \theta_1, v, \theta_2) : L^2(T_zM) \to \mathcal{H}(T_zM, J_z, h_z) = \mathcal{H}_J. \tag{4.14}$$

We recall that the semi-classical Szegő kernels of the Heisenberg group have the form

$$\Pi^{\mathbf{H}}_N(x_1, x_2) = \frac{1}{\pi^m} N^m e^{iN(\tau_1-\tau_2)} e^{N(z_1\cdot\bar{z}_2 - \frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2)}. \tag{4.15}$$

In [44] the notion of K-coordinates is introduced, refining the notion of Heisenberg coordinates in [58]. These are Kähler-type coordinates in which (4.14) equals (4.15) to leading order (up to rescaling):

$$\Pi^{T_zM}_{h_z,J_z}(u, \theta_1, v, \theta_2) = \pi^{-m} e^{i(\theta_1-\theta_2)} e^{u\cdot\bar{v} - \frac{1}{2}(|u|^2+|v|^2)} = \pi^{-m} e^{i(\theta_1-\theta_2)} e^{i\Im(u\cdot\bar{v}) - \frac{1}{2}|u-v|^2}$$

The lifted Szegő kernel is shown in [58] and in [44, Theorem 2.3] to have the following scaling asymptotics.

Theorem **4.2.1.** *Fix $P_0 \in M$ and choose a K-frame centered at P_0. Then, identifying coordinates $(z_1, \tau_1, z_2, \tau_2)$ on X^2 with coordinates $(u, \theta_1, v, \theta_2)$ on $(T_zM \times S^1)^2$, we have*

$$\begin{aligned} &N^{-m}\Pi_N\left(\frac{u}{\sqrt{N}}, \frac{\theta_1}{N}, \frac{v}{\sqrt{N}}, \frac{\theta_2}{N}\right) \\ &\qquad = \Pi^{T_zM}_{h_z,J_z}(u, \theta_1, v, \theta_2)\left(1 + \sum_{r=1}^{K} N^{-r/2} b_r(P_0, u, v) + N^{-(K+1)/2} R_K(P_0, u, v, N)\right), \end{aligned}$$

where $\Pi^{T_zM}_{h_z,J_z}$ *is the osculating Bargmann–Fock Szegő kernel for the tangent space* $T_zM \simeq \mathbb{C}^m$ *equipped with the complex structure* J_z *and Hermitian metric* h_z*. Here,*

- $b_r = \sum_{\alpha=0}^{2[r/2]} \sum_{j=0}^{[3r/2]} (\psi_2)^\alpha Q_{r,\alpha,3r-2j}$, *where* $Q_{r,\alpha,d}$ *is homogeneous of degree* d *and*

$$\psi_2(u,v) = u \cdot \bar{v} - \frac{1}{2}(|u|^2 + |v|^2);$$

  (in particular, b_r *has only even homogeneity if* r *is even, and only odd homogeneity if* r *is odd);*

- $\|R_K(P_0,u,v,N)\|_{\mathcal{C}^j(\{|u|\le\rho,\ |v|\le\rho\}} \le C_{K,j,\rho}$ *for* $j \ge 0$, $\rho > 0$ *and* $C_{K,j,\rho}$ *is independent of the point* P_0 *and choice of coordinates.*

4.3. Proof of Theorem 4.1.6: Logarithmic Decay of Variances

The variance estimate is similar to the ones given in [**57**, **53**, **54**, **25**, **24**]. A key ingredient is Egorov's theorem in the Kähler setting, whose proof is deferred to Section 4.6. Let $\pi\colon X \to M$ be the natural projection from the unit co-disk bundle to the base manifold. A function $f \in C^\infty(M)$ pulls back $F := \pi^* f$ to a function on X that is constant along the fibers $X \to M$. Recall also that $\tilde{\chi}\colon X \to X$ is the contact lift of a symplectic diffeomorphism $\chi\colon M \to M$ for which the exponential growth estimate (4.3) and the polynomial decay of correlations (4.4) apply.

Proposition **4.3.1** (Egorov's theorem with remainder)**.** *Let* M_F *denote multiplication by a smooth function* $F := \pi^* f \in C^\infty(M)$ *that is the lift of some* $f \in C^\infty(M)$*. Let* $T \in \mathbb{Z}$ *be an integer. Then*

$$U^T_{\chi,N}(\Pi_N M_F \Pi_N)(U^*_{\chi,N})^T = \Pi_N M_{F\circ\tilde{\chi}^T}\Pi_N + R^T_N,$$

where $F \circ \tilde{\chi}^T$ denotes the T-fold composition of F with $\tilde{\chi}$, and R_N^T is a Toeplitz operator with

$$\frac{1}{d_N} \operatorname{Tr}[(R_N^T)^* R_N^T] = \mathcal{O}\left(\frac{T^2}{N} \|F\|_{C^2}^2 e^{2\delta_0 |T|}\right).$$

In particular, at the level of matrix elements one has

$$\left\langle U_{\chi,N}^T \Pi_N M_F \Pi_N (U_{\chi,N}^*)^T s_j^N, s_j^N \right\rangle = \left\langle \Pi_N M_{F\circ\tilde{\chi}^T} \Pi_N s_j^N, s_j^N \right\rangle + \mathcal{O}\left(\frac{T^2}{N} \|F\|_{C^2}^2 e^{2\delta_0 |T|}\right).$$

Taking Proposition 4.3.1 for granted, we proceed to prove Theorem 4.1.6. We write each integral in the Cesàro sum (4.7) as a matrix element:

$$\fint_M f(z) \|s_j^N\|_{h^N}^2 \, dV = \langle \Pi_N M_F \Pi_N s_j^N, s_j^N \rangle. \tag{4.16}$$

It is convenient to introduce shorthands for the time-averages:

$$\begin{cases} [\Pi_N M_F \Pi_N]_T := \dfrac{1}{2T+1} \displaystyle\sum_{n=-T}^{T} U_{\chi,N}^n (\Pi_N M_F \Pi_N) U_{\chi,N}^{*n}, \\[2ex] [F]_T := \dfrac{1}{2T+1} \displaystyle\sum_{n=-T}^{T} F \circ \tilde{\chi}^n, \\[2ex] [M_f]_T := M_{[f]_T}. \end{cases} \tag{4.17}$$

Since s_j^N are eigensections of $U_{\chi,N}$, we may replace $\Pi_N M_F \Pi_N$ in (4.16) by its time average defined in (4.17):

$$\int_M f(z) \|s_j^N\|_{h^N}^2 \, dV = \left\langle [\Pi_N M_F \Pi_N]_T s_j^N, s_j^N \right\rangle. \tag{4.18}$$

Proposition 4.3.1, that is Egorov's theorem, gives

$$[\Pi_N M_F \Pi_N]_T = \Pi_N [M_F]_T \Pi_N + R_N^{(T)}, \tag{4.19}$$

with the remainder term satisfying the error estimate

$$\frac{1}{d_N} \operatorname{Tr}[(R_N^{(T)})^* R_N^{(T)}] = \mathcal{O}\Big(\frac{T^2 \|F\|_{C^2}^2 e^{2\delta_0|T|}}{N}\Big). \tag{4.20}$$

Here the exponential growth condition (4.3) on χ is used.

By substituting (4.19) into (4.18), the quantum variance (4.7) can be rewritten as

$$\begin{aligned}
\mathcal{V}_N(f) &= \frac{1}{d_N} \sum_{j=1}^{d_N} \Big| \big\langle [M_F]_T s_j^N, s_j^N \big\rangle + \langle R_N^{(T)} s_j^N, s_j^N \rangle - \fint_M f \, dV \Big|^2 \\
&\le \frac{2}{d_N} \sum_{j=1}^{d_N} \Big| \big\langle [M_F]_T s_j^N, s_j^N \big\rangle - \fint_M f \, dV \Big|^2 + \frac{2}{d_N} \sum_{j=1}^{d_N} \Big| \langle R_N^{(T)} s_j^N, s_j^N \rangle \Big|^2 .
\end{aligned}$$

Applying the Cauchy–Schwarz inequality to the first term and the error estimate (4.20) to the second term, we find

$$\begin{aligned}
\mathcal{V}_N(f) &\le \frac{2}{d_N} \sum_{j=1}^{d_N} \fint_M \Big| [f]_T \|s_j^N\|_{h^N}^2 - \fint_M f \, dV \Big|^2 dV + \mathcal{O}\Big(\frac{T^2 \|F\|_{C^2}^2 e^{2\delta_0|T|}}{N}\Big) \\
&\le \frac{2}{d_N} \sum_{j=1}^{d_N} \fint_M \Big| [f]_T - \fint_M f \, dV \Big|^2 \|s_j^N\|_{h^N}^2 \, dV + \mathcal{O}\Big(\frac{T^2 \|F\|_{C^2}^2 e^{2\delta_0|T|}}{N}\Big) \\
&= \frac{2}{d_N} \fint_M \Big| [f]_T - \fint_M f \, dV \Big|^2 \Pi_N(z,z) \, dV + \mathcal{O}\Big(\frac{T^2 \|F\|_{C^2}^2 e^{2\delta_0|T|}}{N}\Big). \qquad (4.21)
\end{aligned}$$

Recall (cf. [70, 57]) the pointwise expansion for the Bergman kernel along the diagonal:

$$\Pi_N(z,z) = a_0 N^m + a_1(z) N^{m-1} + a_2(z) N^{m-2} + \cdots ,$$

where the coefficients $a_j(z)$ are invariant polynomials in derivatives of the metric h, and where the leading order coefficient is a constant equal to $a_0 = c_1(L)^m/m!$. Combining the Bergman kernel expansion with (4.21) yields

$$\mathcal{V}_N(f) \leq \left(\frac{2c_1(L)^m}{m!} + \mathcal{O}\left(\frac{1}{N}\right)\right)\left(\int_M \left|[f]_T - \fint_M f\,dV\right|^2 dV\right) + \mathcal{O}\left(\frac{T^2\|F\|^2_{C^2}e^{2\delta_0|T|}}{N}\right).$$

Set

$$T = T(N) = \frac{1}{4\delta_0}|\log N|,$$

then, thanks to the decay of correlations assumption (4.4), we get (for all $0 < \beta < 1$)

$$\mathcal{V}_N(f) = \mathcal{O}\left(\frac{\|f\|^2_{C^{0,\beta}}}{\log N}\right) + \mathcal{O}\left(\frac{\|f\|^2_{C^2}|\log N|^2}{N^{\frac{1}{2}}}\right) + \mathcal{O}\left(\frac{\|f\|^2_{C^{0,\beta}}}{N\log N}\right).$$

(Note $\|F\|_{C^2} = \|f\|_{C^2}$ by definition of $F = \pi^* f$.) This completes the proof of Theorem 4.1.6.

4.4. Proof of Theorem 4.1.4: Log-scale Mass Equidistribution

4.4.1. Proof of Proposition 4.1.5

We begin by defining constants κ_1, κ_2 that will appear in the proof. Let κ_1 be any constant satisfying

$$0 < \kappa_1 < 1 - 4m\gamma. \tag{4.22}$$

It follows that

$$\kappa_1 \leq 1 - 4\gamma(m+\beta) \quad \text{for some } 0 < \beta < 1,$$

whence

$$|\log N|^{4\gamma\beta - 1} \leq |\log N|^{-4m\gamma - \kappa_1}. \tag{4.23}$$

We also let κ_2 be any constant satisfying

$$0 < \kappa_2 < \frac{\kappa_1}{2}. \tag{4.24}$$

Now fix $z_0 \in M$. Define symbols $\rho_N \in C_0^\infty(B(z_0, 1 + 2|\log N|^{-\frac{\kappa_2}{\beta+1}}, [0,1])$ by

$$\rho_N(z) := \begin{cases} 1 & \text{for } z \in B(z_0, 1 + |\log N|^{-\frac{\kappa_2}{\beta+1}}), \\ 0 & \text{for } z \notin B(z_0, 1 + 2|\log N|^{-\frac{\kappa_2}{\beta+1}}). \end{cases}$$

Note that the support of ρ_N depends on N. We perform a further rescaling

$$(D_{\varepsilon_N}^{-1})^* \rho_N(z) = \rho_N(\varepsilon_N^{-1} z). \tag{4.25}$$

The statement of Corollary 4.1.7 (which follows easily from Theorem 4.1.6 as discussed in Section 4.1.3) with f_{z_0,ε_N} replaced by $\rho_N(\varepsilon_N^{-1} z)$ becomes

$$\begin{aligned} \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \left\langle M_{(D_{\varepsilon_N}^{-1})^* \rho_N} s_j^N, s_j^N \right\rangle - \fint_M \rho_N(\varepsilon_N^{-1} z)\, dV \right|^2 &= \mathcal{O}(\|\rho_N\|_{C^{0,\beta}}^2 |\log N|^{4\gamma\beta - 1}) \\ &\leq \mathcal{O}(\|\rho_N\|_{C^{0,\beta}}^2 |\log N|^{-4m\gamma} |\log N|^{-\kappa_1}) \end{aligned} \tag{4.26}$$

for any κ_1 satisfying (4.22). In the last line we used (4.23).

Now apply Markov's inequality $\mathbb{P}(X \geq a) \leq a^{-1}\mathbb{E}X$. We view each term of the sum on the left-hand side of (4.26) as a random variable indexed by (N, j). The probability measure is the normalized counting measure on the indices $\{0 \leq j \leq d_N\}$. Finally take a to equal $|\log N|^{\varepsilon}$ (for some small $\varepsilon > 0$) times the right side of (4.26). It follows that for any constant κ_2 satisfying (4.24) there exists a full density subsequence $\Gamma'_{z_0} \subset \{(N, j)\}$ such that the corresponding eigensections satisfy

(4.27)

$$\left| \int_{B(z_0,2)} \rho_N(\varepsilon_N^{-1} z) \|s_j^N\|_{h^N}^2 - \frac{1}{\mathrm{Vol}(M)} \int_{B(z_0,2)} \rho_N(\varepsilon_N^{-1} z) \right| \leq C\|\rho_N\|_{C^{0,\beta}} |\log N|^{-2m\gamma} |\log N|^{-\kappa_2}$$

for $(N, j) \in \Gamma'_{z_0}$. In other words, almost all the terms in the averaged sum (4.27) each satisfies the slightly worse than the average upper bound $C\|\rho_N\|_{C^{0,\beta}} |\log N|^{-2m\gamma} |\log N|^{-\kappa_2}$.

We then have

$$\begin{aligned}
\int_{B(z_0,\varepsilon_N)} \|s_j^N\|_{h^N}^2 \, dV &\leq \int_{B(z_0,2)} \rho_N(\varepsilon_N^{-1} z) \|s_j^N\|_{h^N}^2 \, dV \\
&\leq \frac{1}{\mathrm{Vol}(M)} \int_{B(z_0,2)} \rho_N(\varepsilon_N^{-1} z) \, dV + C\|\rho_N\|_{C^{0,\beta}} |\log N|^{-2m\gamma} |\log N|^{-\kappa_2} \\
&\leq \frac{\mathrm{Vol}(B(z_0, \varepsilon_N))}{\mathrm{Vol}(M)} + C\left(|\log N|^{-2m\gamma - \frac{\kappa_2}{\beta+1}} + \|\rho_N\|_{C^{0,\beta}} |\log N|^{-2m\gamma - \kappa_2} \right).
\end{aligned}$$

The first inequality follows from the definition (4.25) of ρ_N. The second inequality follows from the estimate (4.27). The third inequality follows from the support condition of (4.25) and from the volume of spherical shells (the "thickness" of the shell being $2|\log N|^{-\frac{\kappa_2}{\beta+1}}$):

$$\int_{B(z_0,2)} \rho_N(\varepsilon_N^{-1} z) \, dV = \int_{B(z_0, 1+2|\log N|^{-\frac{\kappa_2}{\beta+1}}) \setminus B(z_0,1)} \rho_N(\varepsilon_N^{-1} z) \, dV + \int_{B(z_0,1)} \rho_N(\varepsilon_N^{-1} z) \, dV$$

$$\leq \varepsilon_N^{2m} \int_{B(z_0, 1+2|\log N|^{-\frac{\kappa_2}{\beta+1}}) \setminus B(z_0,1)} dV + \int_{B(z_0,\varepsilon_N)} dV$$
$$\leq C\varepsilon_N^{2m} |\log N|^{-\frac{\kappa_2}{\beta+1}} + \mathrm{Vol}(B(z_0, \varepsilon_N)),$$

where C depends only on (M, ω) and the choice of ρ.

Note that $\|\rho_N\|_{C^{0,\beta}} \leq C(|\log N|^{\frac{\kappa_2}{\beta+1}})^{-\beta}$, which gives

$$\int_{B(z_0,\varepsilon_N)} \|s_j^N\|^2_{h^N} \, dV \leq \frac{\mathrm{Vol}(B(z_0,\varepsilon_N))}{\mathrm{Vol}(M)} + C|\log N|^{-2m\gamma} \left(|\log N|^{-\frac{\beta\kappa_2}{\beta+1}} + |\log N|^{-\frac{\beta\kappa_2}{\beta+1}} \right)$$
$$= \frac{\mathrm{Vol}(B(z_0,\varepsilon_N))}{\mathrm{Vol}(M)} + o(|\log N|^{-2m\gamma}). \tag{4.28}$$

(From (4.22) and (4.24) of how κ_1, κ_2 are defined, we have $0 < \beta\kappa_2/(\beta+1) < 1$.)

A similar argument using appropriately chosen $\tilde{\rho}_N$ of the form $\tilde{\rho}_N(z) = \rho_N(3z)$ gives the opposite inequality

$$\int_{B(z_0,\varepsilon_N)} \|s_j^N\|^2_{h^N} \, dV \geq \frac{\mathrm{Vol}(B(z_0,\varepsilon_N))}{\mathrm{Vol}(M)} + o(|\log N|^{-2m\gamma}) \tag{4.29}$$

for a full density subsequence Γ''_{z_0} of eigensections. The intersection $\Gamma'_{z_0} \cap \Gamma''_{z_0} =: \Gamma_{z_0}$ indexes a full density subsequence of eigensections for which (4.28) and (4.29) hold simultaneously. This completes the proof of Proposition 4.1.5.

4.4.2. Proof of Theorem 4.1.4

Note that one must first fix a single base point $z_0 \in M$ for the asymptotic statement of Proposition 4.1.5 to hold. To move towards global statements that hold for all $z \in M$ simultaneously, we introduce the concept of a *log-good cover*, for which we have uniform estimates on each element (i.e., a Kähler ball) of the cover. The existence of a cover satisfying the following conditions is proved in [**24**].

Definition **4.4.1.** *Let* $\varepsilon_N = |\log N|^{-\gamma}$ *for any fixed* $0 < \gamma < (6m)^{-1}$ *as before. A* log-good cover $\mathcal{U}_N$ *is a cover of* M *by geodesic balls* $\{B(z_{N,\alpha}, \varepsilon_N)\}_{\alpha=1}^{R(\varepsilon_N)}$ *with the following properties:*

- *The number* $R(\varepsilon_N)$ *of balls in the cover is bounded above*

$$R(\varepsilon_N) \leq c_1 \varepsilon_N^{-2m} \qquad (\dim_{\mathbb{R}} M = 2m)$$

  by some constant (*independent of* N) *multiple of* ε_N^{-2m}.

- *An arbitrary ball* $B(p, \varepsilon_N) \subset M$ *is covered by at most* c_2 (*independent of* N) *number of balls from the cover.*
- *An arbitrary ball* $B(p, \varepsilon_N) \subset M$ *contains at least one of the shrunken balls* $B(z_{N,\alpha}, \frac{\varepsilon_N}{3})$.

We now proceed with the proof of Theorem 4.1.4, suppressing the prime notation on γ and ε_N. Let $0 < \gamma < (6m)^{-1}$ be given and set $\varepsilon_N = |\log N|^{-\gamma}$. For each N, fix a log-good cover $\mathcal{U}_N$ as defined above. As before, let $0 \leq f_{z_{N,\alpha}} \leq 1$ be a smooth cut-off function that is equal to 1 on $B(z_{N,\alpha}, 1)$, and vanishes outside $B(z_{N,\alpha}, 2)$. Let $f_{z_{N,\alpha},\varepsilon_N} = f_{z_{N,\alpha}}(\varepsilon_N z)$. (This is a slight abuse of notation, where we mean balls in Kähler normal coordinate charts centered at $z_{N,\alpha}$.) In what follows, $\kappa_3 > 0$ is a parameter independent of N, j, to be chosen later.

The extraction argument uses Markov's inequality $\mathbb{P}(X \geq a) \leq a^{-1}\mathbb{E}X$. To this end, for each $1 \leq j \leq d_N$ and $1 \leq \alpha \leq R(\varepsilon_N)$ set

$$X_{N,j,\alpha} := \left| \int_M f_{z_{N,\alpha},\varepsilon_N} \|s_j^N\|_{h^N}^2 \, dV - \fint_M f_{z_{N,\alpha},\varepsilon_N} \, dV \right|^2.$$

We view $X_{N,j,\alpha}$ as a random variable with respect to the normalized counting measure on the set of indices $1 \leq j \leq d_N$. Thanks to Corollary 4.1.7 and (4.23), its expected value is

$$\mathbb{E}X_{N,j,\alpha} = \mathcal{O}(|\log N|^{-(1-2\gamma\beta)}) = \mathcal{O}(|\log N|^{-(4m\gamma+\kappa_1)}) \quad \text{for any } \kappa_1 \text{ satisfying (4.22)}.$$

(The error is uniform in $z_{N,\alpha}$.) In particular, we may choose κ_1 to equal

$$0 < \kappa_1 := 1 - 4m(\gamma + \beta) < 1 \quad \text{for some } 0 < \beta < \frac{1 - 6m\gamma}{4m} < 1. \tag{4.30}$$

It follows from an application of Markov's inequality with $X = X_{N,j,\alpha}$; with the normalized counting measure on $\{1, \ldots, d_N\}$; and with $a = |\log N|^{-(4m\gamma - \kappa_3)}$, that the 'exceptional sets'

$$\Lambda_\alpha(N) := \left\{ j = 1, \ldots, d_N \colon \left| \int_M f_{z_{N,\alpha},\varepsilon_N} \|s_j^N\|_{h^N}^2 \, dV - \fint_M f_{z_{N,\alpha},\varepsilon_N} \, dV \right|^2 \geq |\log N|^{-4m\gamma - \kappa_3} \right\}$$

satisfy

$$\frac{\#\Lambda_\alpha(N)}{d_N} \leq C|\log N|^{4m\gamma - \kappa_3} |\log N|^{-(4m\gamma+\kappa_1)} = C|\log N|^{-(1-4m(\gamma+\beta)-\kappa_3)}.$$

Now define 'generic sets'

$$\Sigma_\alpha(N) := \{j\colon 1 \le j \le d_N\} \setminus \Lambda_\alpha(N) \quad \text{and} \quad \Sigma(N) := \bigcap_{\alpha\colon B(z_{N,\alpha},\varepsilon_N)\in\mathcal{U}_N} \Sigma_\alpha(N).$$

The number of elements in the cover $\mathcal{U}_N$ is of order $\varepsilon_N^{-2m} = |\log N|^{2m\gamma}$, whence

$$\begin{aligned}\frac{\#\Sigma(N)}{d_N} &\ge 1 - \sum_\alpha \frac{\#\Lambda_\alpha(N)}{d_N}\\ &\ge 1 - C|\log N|^{2m\gamma}|\log N|^{-(1-4m(\gamma+\beta)-\kappa_3)}\\ &= 1 - C|\log N|^{-(1-6m\gamma-4m\beta-\kappa_3)}\\ &\to 1 \quad \text{by choosing } \beta,\kappa_3>0 \text{ sufficiently small.} \end{aligned} \tag{4.31}$$

Indeed, by choice (4.30) of β, we have $1-6m\gamma-4m\beta>0$, so κ_3 can always be chosen to ensure (4.31) holds. This is analogous to the estimate in [**25**] preceding Lemma 3.1 or in [**24**, p.3263].

The construction of indexing sets $\Sigma(N)$ yields a full density subsequence

$$\Sigma := \bigcup_{N\ge 1} \Sigma(N)$$

such that, for every $B(z_\alpha,\varepsilon_N)\in\mathcal{U}_N$, we have

$$\begin{aligned}\int_{B(z_{N,\alpha},\varepsilon_N)} \|s_j^N\|_{h^N}^2\,dV &\le \int_{B(0,2)} f_{z_{N,\alpha},\varepsilon_N}\|s_j^N\|_{h^N}^2\,dV\\ &\le \frac{1}{\mathrm{Vol}(M)}\int_{B(0,2)} f_{z_{N,\alpha},\varepsilon_N}\,dV + C|\log N|^{-(2m\gamma+\kappa_3/2)}\\ &\le \frac{\mathrm{Vol}(B(z_{N,\alpha},2\varepsilon_N))}{\mathrm{Vol}(M)} + o(|\log N|^{-2m\gamma})\end{aligned}$$

$$\leq C\mathrm{Vol}(B(z_{N,\alpha}, \varepsilon_N))$$

simultaneously for all $\alpha = 1, \dots, R(\varepsilon_N)$ as $\Sigma \ni (N, j) \to \infty$. The constant C is independent of α.

Now let $p \in M$ be arbitrary. By construction, the ball $B(p, \varepsilon_N)$ is contained in at most c_2 number (independent of N) of elements of the log-good cover $\mathcal{U}_N$. Thus,

$$\int_{B(p,\varepsilon_N)} \|s_j^N\|_{h^N}^2 \, dV \leq \sum_{i=1}^{c_2} \frac{1}{\mathrm{Vol}(M)} \int_{B(0,2)} f_{z_{N,\alpha_i},\varepsilon_N} \, dV + o(|\log N|^{-2m\gamma}) \leq C\mathrm{Vol}(B(p, \varepsilon_N))$$

for every $p \in M$ as $\Sigma \ni (N, j) \to \infty$. The constant C is independent of p. This is the statement of the volume upper bound.

It remains to repeat the same construction by dilating the symbol $0 \leq g_{z_\alpha} \leq 1$ that is a smooth cut-off function supported in $B(z_\alpha, 1/3)$ and equals to 1 in $B(0, 1/6)$. There exists a full density subsequence Σ' such that

$$\begin{aligned}
\int_{B(z_{N,\alpha},\varepsilon_N/3)} \|s_j^N\|_{h^N}^2 \, dV &\geq \int_{B(z_\alpha,1/3)} g_{z_\alpha,\varepsilon_N} \|s_j^N\|_{h^N}^2 \, dV \\
&\geq \frac{1}{\mathrm{Vol}(M)} \int_{B(z_{N,\alpha},1/3)} g_{z_\alpha,\varepsilon_N/3} \, dV - C|\log N|^{-(2m\gamma+\kappa_3/2)} \\
&\geq \frac{\mathrm{Vol}(B(z_{N,\alpha}, \varepsilon_N/6))}{\mathrm{Vol}(M)} - o(|\log N|^{-2m\gamma}) \\
&\geq c\mathrm{Vol}(B(z_{N,\alpha}, \varepsilon_N))
\end{aligned}$$

simultaneously for all $\alpha = 1, \dots, R(\varepsilon_N)$ as $\Sigma \ni (N, j) \to \infty$. Now let $p \in M$ be arbitrary. Every ball $B(p, \varepsilon_N)$ contains at least one element $B(z_{N,\alpha}, \varepsilon_N/3) \in \mathcal{U}_N$ of the log-good cover, whence

$$\int_{B(p,\varepsilon_N)} \|s_j^N\|_{h^N}^2\, dV \geq c\mathrm{Vol}(B(p,\varepsilon_N))$$

for every $p \in M$ as $\Sigma \ni (N,j) \to \infty$. This is the statement of the volume lower bound.

The intersection $\Gamma = \Sigma \cap \Sigma'$ is again a full density subsequence. By construction, the eigensections indexed by Γ satisfy the two-sided bound: for all $p \in M$,

$$c\mathrm{Vol}(B(p,\varepsilon_N)) \leq \int_{B(p,\varepsilon_N)} \|s_j^N\|_{h^N}^2\, dV \leq C\mathrm{Vol}(B(p,\varepsilon_N)) \quad \text{as } \Gamma \ni (N,j) \to \infty.$$

This completes the proof of Theorem 4.1.4.

4.5. Proof of Theorem 4.1.2: Log-scale Equidistribution of Zeros

Let $0 < \gamma < (6m)^{-1}$ from the statement of Theorem 4.1.2 be given. We distinguish two logarithmic scales by fixing another parameter γ':

$$0 < \gamma < \gamma' < \frac{1}{6m} \quad \text{so that} \quad |\log N|^{-\gamma'} = \varepsilon_N' < \varepsilon_N = |\log N|^{-\gamma}.$$

Let Γ be the full density subsequence corresponding to scale ε' as guaranteed by Theorem 4.1.4. We show that the same Γ satisfies the statement of Theorem 4.1.2 at the scale $\varepsilon_N > \varepsilon_N'$.

In the notation of Section 2.6, relative to a local frame we write the eigensections locally as

$$s_j^N = f_j^{(N)} e_L^N, \quad f_j^{(N)} \text{ a local holomorphic function.}$$

The Poincaré–Lelong formula (2.17) reduces the growth rate of zeros to the growth rate of the local plurisubharmonic function $N^{-1}\log|f_j^{(N)}|^2$ or to the global quasi-plurisubharmonic function[4] $u_j^{(N)}(z) = N^{-1}\log\|s_j^N(z)\|^2_{h^N}$. Fix $p \in M$ and consider the dilated function

$$u_j^{(N)}(z) := \frac{1}{N}\log\|s_j^N(\varepsilon_N z)\|^2_{h^N} = D_{\varepsilon_N}^{p*}\left[\frac{1}{N}\log\|s_j^N(z)\|^2_{h^N}\right] \quad \text{on } B(p,1), \tag{4.32}$$

where $D^p_{\varepsilon_N}$ is the local dilation defined by (4.6) in Kähler normal coordinates centered at $p = 0$. Since $D^p_{\varepsilon_N}$ is a local holomorphic map, (4.32) remains quasi-plurisubharmonic. We state a key lemma:

LEMMA **4.5.1.** *Let Γ be the subsequence of density one for the finer scale ε'_N of Theorem 4.1.4. For $(N,j) \in \Gamma$, the logarithmically dilated potential (4.32) satisfies*

$$\|u_j^{(N)}\|_{L^1(B(p,1))} = o(\varepsilon_N^2),$$

where the remainder is at a coarser scale ε_N.

REMARK **4.5.2.** *We emphasize that we are assuming the eigensections indexed by Γ satisfy*

$$C_1\frac{\mathrm{Vol}(B(p,\varepsilon'_N))}{\mathrm{Vol}(M)} \le \int_{B(p,\varepsilon'_N)} \|s_j^N\|^2_{h^N}\, dV \le C_2\frac{\mathrm{Vol}(B(p,\varepsilon'_N))}{\mathrm{Vol}(M)} \tag{4.33}$$

and then inverse dilating $B(p,\varepsilon_N)$ to $B(p,1)$, so that any ball $B(q,\varepsilon'_N) \subset B(p,\varepsilon)$ gets inverse dilated to (slightly deformed) by $(D^p_{\varepsilon_N})^{-1}$ to (slightly deformed) balls of radius $\varepsilon_N^{-1}\varepsilon'_N \simeq |\log N|^{-\gamma'+\gamma}$ in $B(p,1)$.

[4]'Quasi' means p.s.h. up to a fixed continuous term, here the potential $\log g$ where $g(z) := \|e_L(z)\|^2_h$.

Let's assume Lemma 4.5.1 for now and proceed to finish the proof of Theorem 4.1.2. Using the Poincaré-Lelong formula and the fact that the holomorphic rescaling D^p_ε commutes with $\partial\bar\partial$, we obtain

$$\frac{1}{N} D^{p*}_{\varepsilon_N} [Z_{s_j^N}] = \frac{\sqrt{-1}}{2\pi N} \partial\bar\partial \log |f_j^{(N)}(\varepsilon_N z)|^2 = \frac{\sqrt{-1}}{2\pi N} \partial\bar\partial \log \|s_j^N(\varepsilon_N z)\|^2_{h^N} + D^{p*}_{\varepsilon_N}\omega.$$

For every test form $\eta \in \mathcal{D}^{m-1,m-1}(B(p,1))$ and $\Gamma \ni (N,j) \to \infty$, integration by parts and Lemma 4.5.1 give

$$\begin{aligned} \int_{B(p,1)} \left(\eta \wedge \frac{1}{N} D^{p*}_{\varepsilon_N}[Z_{s_j^N}]\right) &= \int_{B(p,1)} \eta \wedge D^{p*}_{\varepsilon_N}\omega + \int_{B(p,1)} \frac{\sqrt{-1}}{2\pi N} \log \|s_j^N(\varepsilon_N z)\|^2_{h^N} \partial\bar\partial\eta(z) \\ &= \int_{B(p,1)} \eta \wedge D^{p*}_{\varepsilon_N}\omega + o(\varepsilon_N^2). \end{aligned} \tag{4.34}$$

Locally at $p = 0$, the Kähler potential can be written as $\varphi(z) = |z|^2 + \mathcal{O}(|z|^4)$, so

$$D^{p*}_{\varepsilon_N}\omega = \frac{\sqrt{-1}}{2\pi} D^{p*}_{\varepsilon_N} \partial\bar\partial\varphi = \varepsilon_N^2 \frac{\sqrt{-1}}{2\pi} \partial\bar\partial |z|^2 + \mathcal{O}(\varepsilon_N^4) = \varepsilon_N^2 \omega_0^p + \mathcal{O}(\varepsilon_N^4), \tag{4.35}$$

with ω_0^p the flat Kähler form. Combining (4.34) and (4.35) (and dividing by ε_N^2) yields

$$\int_{B(p,1)} \left(\eta \wedge \frac{1}{N\varepsilon_N^2} D^{p*}_{\varepsilon_N}[Z_{s_j^N}]\right) = \int_{B(p,1)} \eta \wedge \omega_0^p + o(1) \quad \text{as } \Gamma \ni (N,j) \to \infty,$$

which is equivalent to the statement of Theorem 4.1.2.

Proof of Lemma 4.5.1. The argument is similar to the one in [57] except for the dilation of the plurisubharmonic functions. The log-scale quantum ergodicity successfully replaces unscaled quantum ergodicity in the key step of the argument due to the fact that the local dilation is holomorphic. But we need to use two logarithmic scales and for later applications we need the remainder estimate.

Let N_0 be sufficiently large so that for all $N \geq N_0$, e_L is a local frame for L over an open subset U containing $\overline{B(p,1)}$ and e_L^N is the corresponding frame for L^N. Since $g(z) = \|e_L(z)\|_h^2$, we have

$$\|e_L^N(z)\|_{h^N}^2 = g^N \quad \text{and} \quad \|s_j^N(\varepsilon_N z)\|_{h^N}^2 = |f_j^N(\varepsilon_N z)|^2 g^N(\varepsilon_N z).$$

We first show that $\|u_j^{(N)}\|_{L^1} \to 0$, and then indicate how the argument can be adapted to yield the $o(\varepsilon_N^2)$ improvement.

Observe that any L^2-normalized section satisfies

$$\|s^N(z)\|_{h^N}^2 \leq \Pi_N(z,z) = \left(\frac{c_1(L)^m}{m!} + O\left(\frac{1}{N}\right)\right) N^m.$$

Hence $\|s^N(z)\|_{h^N} \leq CN^{m/2}$ for some $C < \infty$ and taking the logarithm gives

(i) The functions $u^{(N)}$ are uniformly bounded above on M;

(ii) $\limsup_{N\to\infty} u_N \leq 0$.

Now consider the plurisubharmonic function

$$v_j^{(N)}(z) := \frac{1}{N} \log|f_j^{(N)}(\varepsilon_N z)|^2 = u_j^{(N)}(z) - \log g(\varepsilon_N z) \in \mathrm{PSH}(B(p,1)).$$

It is clear that $v_j^{(N)}$ are uniformly upper bounded. A standard result on plurisubharmonic functions (see [26, Theorem 4.1.9]) then implies a subsequence $v_j^{(N_k)}$ either converges uniformly to $-\infty$ on $B(p,1)$ or else has a subsequence that is convergent in $L^1_{\mathrm{loc}}(B(p,1))$.

Let us rule out the first possibility. If it occurred, there would exist $K > 0$ such that

$$\frac{1}{N_k} \log \|s_j^{N_k}(\varepsilon_{N_k} z)\|^2_{h^{N_k}} \le -1 \iff \|s_j^{N_k}(\varepsilon_{N_k} z)\|^2_{h^{N_k}} \le e^{-N_k} \quad \text{on } B(p,1) \text{ for all } k \ge K.$$

Equivalently, the same exponential decay estimate holds on $B(p, \varepsilon_{N_k})$ for the undilated sections. But this contradicts the lower bound of (4.33).

Therefore the sequence $v_j^{(N)}$ is pre-compact in $L^1(B(p,1))$, and every sequence contains a subsequence, which we continue to denote by $\{v_j^{(N_k)}\}$, that converges in $L^1(B(p,1))$ to some $v \in L^1(B(p,1))$. By passing if necessary to a further subsequence, we may assume that $\{v_j^{(N_k)}\}$ converges pointwise almost everywhere in $B(p,1)$ to v, and hence by observation (ii),

$$v(z) = \limsup_{(N_k,j)\to\infty} \left(u_j^{(N_k)}(z) - \log g(\varepsilon_{N_k} z) \right) \le 0 \quad \text{a.e. on } B(p,1).$$

Let

$$v^*(z) := \limsup_{w \to z} v(w) \le 0$$

be the upper-semicontinuous regularization of v. Then v^* is plurisubharmonic on $B(p,1)$ and $v^* = v$ almost everywhere. We claim that $v^* = 0$. To this end, we use the second scale ε'_N. If $v^* \neq 0$, then

$$\|v_j^{(N_k)} + D^{p*}_{\varepsilon_{N_k}} \log g\|_{L^1(B(p,1))} = \|u_j^{(N_k)}\|_{L^1(B(p,1))} \ge \delta > 0.$$

Hence, for some $c > 0$, the open set $U_c = \{z \in B(p,1) \colon v^*(z) < -c\}$ is nonempty. For sufficiently large k, this set contains a ball $B(q, \varepsilon'_{N_k} \varepsilon^{-1}_{N_k})$. By Hartogs' Lemma, there exists a positive integer K such that $v_j^{(N_k)}(z) \leq -c/2$ for $z \in B(q, \varepsilon'_{N_k}\varepsilon_{N_k}^{-1})$ and $k \geq K$, that is

$$\|s_j^{N_k}(\varepsilon_N z)\|_{h^{N_k}}^2 \leq e^{-cN_k/2} \quad \text{on } B(q, \varepsilon'_{N_k}\varepsilon_{N_k}^{-1}) \text{ for all } k \geq K.$$

But this again contradicts the lower bound in Theorem 4.1.4 on $B(q, \varepsilon'_{N_k})$. We have therefore proved $\|u_j^{(N)}\|_{L^1(B(p,1))} = o(1)$.

We now exploit the exponential decay to prove the sharper result $\|u_j^{(N)}\|_{L^1(B(p,1))} = o(\varepsilon_N^2)$. Consider the renormalized sequence

$$\varepsilon_N^{-2} u_j^{(N)} = \frac{1}{N\varepsilon_N^2} D_{\varepsilon_N}^* \log \|s_j^N(z)\|_{h^N}^2.$$

Note that this is still an upper-bounded sequence of plurisubharmonic functions because of the exact cancellation between dilating by $D_{\varepsilon_N}^{p*}$ and dividing by ε_N^2. Indeed, $\log g = |z|^2 + \mathcal{O}(|z|^4)$ as $|z| \to p = 0$ in local coordinates, so $\varepsilon_N^{-2} D_{\varepsilon_N}^{p*} \log g$ remains bounded.

We now run through the previous argument again with this re-normalized sequence. If $\varepsilon_{N_k}^{-2} v_j^{N_k} \to -\infty$ uniformly on compact subsets of $B(p,1)$, then

$$\frac{1}{N_k \varepsilon_{N_k}^2} \|s_j^{N_k}(\varepsilon_{N_k} z)|_{h^{N_k}}^2 \leq -1 \iff \|s_j^{N_k}(\varepsilon_{N_k} z)\|_{h^{N_k}}^2 \leq e^{-\varepsilon_{N_k}^2 N_k} \quad \text{on } B(p,1),$$

a contradiction to (4.33) as before. The alternative (namely $\varepsilon_{N_k}^{-2} v_j^{N_k}$ being pre-compact) leads to the estimate

$$\|s_j^{N_k}(\varepsilon_N z)\|_{h^{N_k}}^2 \leq e^{-c\varepsilon_{N_k}^2 N_k/2} \quad \text{on } B(q, \varepsilon'_{N_k}\varepsilon_{N_k}^{-1}) \text{ for all } k \geq K,$$

again a contradiction. This completes the proof of Lemma 4.5.1. $\square$

4.6. Appendix: Egorov's Theorem

The purpose of this section is to prove a long time Egorov's theorem with remainder as stated in Proposition 4.3.1. It is convenient to work on the contact manifold (X, α) by lifting χ on M to the contact transformation $\tilde{\chi}$ on X and viewing sections $s_j^N \in H^0(M, L^N)$ as equivariant functions $\hat{s}_j^N \in L^2(X)$ as discussed in Section 4.2.1.

We recall the setting. Let χ be a quantizable symplectic map (whose quantization $U_{\chi,N}$ is defined in (4.10)) satisfying the exponential growth condition (4.3) and decay of correlations condition (4.4). Let M_F denote multiplication by $F \in C^\infty(X)$ and $F \circ \tilde{\chi}^T$ the composition of F with the T-fold iterate of $\tilde{\chi}$ (or $\tilde{\chi}^{-1}$, depending on the sign of T). Proposition 4.3.1, which is a statement on the base manifold M, is equivalent to the following statement on the co-circle bundle X.

Proposition **4.6.1.** *Let χ be a quantizable symplectic map on M satisfying conditions (4.3) and (4.4). Let $\tilde{\chi}$ denote its lift to (X, α) as a contact transformation. Let $F \in C^\infty(X)$ and $T \in \mathbb{N}$. Then*

$$U_{\chi,N}^T (\Pi_N M_F \Pi_N)(U_{\chi,N}^*)^T = \Pi_N M_{F\circ\tilde{\chi}^T} \Pi_N + R_N^{(T)},$$

where $R_N^{(T)}$ is a Toeplitz operator with

$$\frac{1}{d_N} \|R_N^{(T)}\|_{\mathrm{HS}}^2 = \frac{1}{d_N} \operatorname{Tr}[(R_N^{(T)})^* R_N^{(T)}] = \mathcal{O}_{\tilde{\chi},F,h}\left(\frac{T^2}{N} \|F\|_{C^2}^2 e^{2\delta_0 |T|}\right),$$

where the $\mathcal{O}$ symbol depends on the metric h and a fixed number of derivatives of $\tilde{\chi}, F$ depending on the dimension.

The proposition is the analogue for Toeplitz operators of the well-known estimate of the Egorov remainder, except that the remainder is stated in terms of the normalized Hilbert–Schmidt norm rather than the operator norm[5]. The Hilbert-Schmidt norm is simpler to estimate since it is defined by a trace, and the remainder estimate is simply the standard one in the stationary phase expansion of Hörmander [**26**]. Sharper remainder estimates have been proved for quantizations of Hamiltonian flows on $T^*\mathbb{R}^n$ in [**6**, Theorem 1.4, Theorem 1.8]. Subsequently, there are many articles proving related results for T^*M. But there do not seem to exist parallel results for Toeplitz operators in the Kähler setting, in particular for powers of a map rather than for Hamiltonian flows. In special cases such as symplectic toral automorphisms and their perturbations, Egorov's theorem with remainder have been proved (see [**53, 54**]) but the proofs use special properties of the metaplectic representation and do not generalize to our setting. Egorov's theorem without estimate of the time-dependence of the remainder may be obtained from the composition theorem for Toeplitz operators in [**3**].

Remark **4.6.2.** *The strategy of the proof is to use induction on T. At each stage, the remainder terms from the previous stage are left 'untouched', and are estimated using that unitary conjugations do not change Hilbert-Schmidt norms. Unlike most statements of the Egorov theorem, we only need the principal term and a remainder of order N^{-1}, and we do not try to give a formula for the lower order terms in the symbol. Thus, at the Tth stage we only conjugate by one power of $U_{\chi,N}$ a Toeplitz operator whose symbol is of the form $F \circ \tilde{\chi}^{T-1}$. This is why the resulting remainder after T steps involves the*

[5]The more difficult norm estimate of the remainder will be presented elsewhere.

C^2 norm of $F \circ \tilde{\chi}^T$ and otherwise only involves a fixed number of derivatives of the data $\tilde{\chi}, h, F$.

4.6.1. Reduction to $T = 1$ case

In this section we reduce the proof of Proposition 4.6.1 to the proof of the following lemma.

LEMMA **4.6.3.** *Under the same assumption as Proposition 4.6.1, we have*

$$U_{\chi,N}\Pi_N M_F \Pi_N U^*_{\chi,N} = \Pi_N M_{F\circ\tilde{\chi}}\Pi_N + R_N, \tag{4.36}$$

where R_N is a Toeplitz operator with

$$\frac{1}{d_N}\|R_N\|^2_{\mathrm{HS}} = \frac{1}{d_N}\operatorname{Tr}[R_N^* R_N] = \mathcal{O}_{\tilde{\chi},F,h}\left(\frac{1}{N}\|F\|^2_{C^2}e^{2\delta_0}\right).$$

We now indicate how Lemma 4.6.3 implies the statement of Egorov's theorem. The rest of the section is then devoted to proving Lemma 4.6.3.

PROOF OF PROPOSITION 4.6.1 GIVEN LEMMA 4.6.3. Given $T \in \mathbb{N}$ and two operators U and A, we introduce the shorthand

$$\mathrm{Ad}^T(U)(A) = U^T A (U^*)^T$$

for the T-fold conjugation of A by U. To keep track of the remainders we henceforth denote R_N in the statement of Lemma 4.6.3 by $R_N^{(1)}$. Invoking the assumption (4.3) that

$\|\tilde{\chi}^T\|_{C^2}^2 = \mathcal{O}(e^{2|T|\delta_0})$, Lemma 4.6.3 reads

$$\begin{cases} \mathrm{Ad}(U_{\chi,N})\Pi_N M_F \Pi_N = \Pi_N M_{F\circ\tilde{\chi}}\Pi_N + R_N, \\ \dfrac{1}{d_N}\,\mathrm{Tr}[R_N^* R_N] = \mathcal{O}\left(\dfrac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0}\right). \end{cases}$$

We now iterate the conjugation. Conjugating a second time by $U_{\chi,N}$ yields two terms:

$$\mathrm{Ad}^2(U_{\chi,N})\Pi_N M_F \Pi_N = \mathrm{Ad}(U_{\chi,N})\Pi_N M_{F\circ\tilde{\chi}}\Pi_N + \mathrm{Ad}(U_{\chi,N})R_N^{(1)}. \tag{4.37}$$

It follows from Lemma 4.6.3 (with M_F replaced by $M_{F\circ\tilde{\chi}}$) that the first term on the right-hand side of (4.37) equals

$$\begin{cases} \mathrm{Ad}(U_{\chi,N})\Pi_N M_{F\circ\tilde{\chi}}\Pi_N = \Pi_N M_{F\circ\tilde{\chi}^2}\Pi_N + \tilde{R}_N^{(2)}, \\ \dfrac{1}{d_N}\,\mathrm{Tr}[(\tilde{R}_N^{(2)})^*\tilde{R}_N^{(2)}] = \mathcal{O}\left(\dfrac{1}{N}\|F\circ\tilde{\chi}\|_{C^2}^2 e^{2\delta_0}\right) = \mathcal{O}\left(\dfrac{1}{N}\|F\|_{C^2}^2 e^{4\delta_0}\right). \end{cases} \tag{4.38}$$

In the error estimate we again made use of the exponential growth assumption (4.3).

The unitarity of $U_{\chi,N}$ implies that the second term $\mathrm{Ad}(U_{\chi,N})R_N^{(1)}$ in (4.38) satisfies

$$\mathrm{Tr}[(\mathrm{Ad}(U_{\chi,N})R_N^{(1)})^*\,\mathrm{Ad}(U_{\chi,N})R_N^{(1)}] = \mathrm{Tr}[(R_N^{(1)})^* R_N^{(1)}] = \mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0}\right). \tag{4.39}$$

Combining (4.37) and (4.38) gives

$$\mathrm{Ad}^2(U_{\chi,N})\Pi_N M_F \Pi_N = \Pi_N M_{F\circ\tilde{\chi}^2}\Pi_N + \tilde{R}_N^{(2)} + \mathrm{Ad}(U_{\chi,N})R_N^{(1)}. \tag{4.40}$$

Set

$$R_N^{(2)} := \tilde{R}_N^{(2)} + \mathrm{Ad}(U_{\chi,N})R_N^{(1)}, \tag{4.41}$$

then (4.38) and (4.39) imply

$$
\begin{aligned}
\frac{1}{d_N}\operatorname{Tr}[(R_N^{(2)})^* R_N^{(2)}] &\le \frac{2}{d_N}\operatorname{Tr}[(\tilde{R}_N^{(2)})^*\tilde{R}_N^{(2)} + (R_N^{(1)})^* R_N^{(1)}] \\
&= 2\left(\mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{4\delta_0}\right) + \mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0}\right)\right) \\
&= 3\mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{4\delta_0}\right). \qquad (4.42)
\end{aligned}
$$

The statement of Proposition 4.6.1 with $T = 2$ is proved thanks to (4.40), (4.41) and (4.42).

The calculation is similar when $\operatorname{Ad}(U_{\chi,N})$ is iterated T times. By a similar stationary phase computation presented in the subsequent section, it is easy to see that on the Tth iterate, we pick up the leading order term:

$$
\begin{cases}
\operatorname{Ad}(U_{\chi,N})\Pi_N M_{F\circ\tilde{\chi}^{T-1}}\Pi_N = \Pi_N M_{F\circ\tilde{\chi}^{T}}\Pi_N + \tilde{R}_N^{(T)}, \\
\dfrac{1}{d_N}\operatorname{Tr}[(\tilde{R}_N^{(T)})^*\tilde{R}_N^{(T)}] = \mathcal{O}\left(\dfrac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0|T|}\right).
\end{cases}
$$

We also have to conjugate the $(T-1)$ 'old' remainders from the $(T-1)$st iterate:

$$
\operatorname{Ad}(U_{\chi,N})\tilde{R}_N^{(T-1)} + \operatorname{Ad}^2(U_{\chi,N})\tilde{R}_N^{(T-2)} + \operatorname{Ad}^3(U_{\chi,N})\tilde{R}_N^{(T-3)} + \cdots + \operatorname{Ad}^{T-1}(U_{\chi,N})\tilde{R}_N^{(1)}.
$$

The Hilbert–Schmidt norm of $\tilde{R}_N^{(\ell)}$ does not change under conjugation by $U_{\chi,N}$. Therefore, the combined remainder term

$$
R_N^{(T)} := \tilde{R}_N^{(T)} + \operatorname{Ad}(U_{\chi,N})\tilde{R}_N^{(T-1)} + \operatorname{Ad}^2(U_{\chi,N})\tilde{R}_N^{(T-2)} + \cdots + \operatorname{Ad}^{T-1}(U_{\chi,N})\tilde{R}_N^{(1)}
$$

at the Tth stage of the iterate has the estimate

$$\frac{1}{d_N}\operatorname{Tr}[(R_N^{(T)})^*R_N^{(T)}] \leq \frac{T}{d_N}\sum_{\ell=1}^{T}\operatorname{Tr}[(R_N^{(\ell)})^*R_N^{(\ell)}] = T\sum_{\ell=1}^{T}\mathcal{O}\Big(\frac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0|\ell|}\Big).$$

Replacing each $e^{2\delta_0|\ell|}$ in the above sum by $e^{2\delta_0|T|}$ for $\ell = 1, 2, \ldots, T$ completes the proof of Proposition 4.6.1 assuming Lemma 4.6.3. $\square$

4.6.2. Proof of Lemma 4.6.3 via stationary phase computation

Let

$$\tilde{L}_N := U_{\chi,N}\Pi_N M_F \Pi_N U_{\chi,N}^* \quad \text{and} \quad L_N := \Pi_N M_{F\circ\tilde{\chi}}\Pi_N,$$

From the definition (4.10) of Toeplitz quantization, the conjugated operator has the form

$$\tilde{L}_N = \Pi_N \sigma_N T_{\tilde{\chi}} \Pi_N M_F \Pi_N T_{\tilde{\chi}^{-1}} \bar{\sigma}_N \Pi_N.$$

Next, insert the identity operator $\mathrm{Id} = T_{\tilde{\chi}^{-1}}T_{\tilde{\chi}}$ between the operators Π_N and M_F in the above expression. Note that $T_{\tilde{\chi}}FT_{\tilde{\chi}^{-1}} = F\circ\tilde{\chi}$. Hence, the expression becomes

$$\tilde{L}_N = \Pi_N \sigma_N \Pi_N^{\tilde{\chi}} M_{F\circ\tilde{\chi}} \Pi_N^{\tilde{\chi}} \bar{\sigma}_N \Pi_N. \tag{4.43}$$

where $\Pi_N^{\tilde{\chi}} := T_{\tilde{\chi}}\Pi_N T_{\tilde{\chi}^{-1}}$ is the operator with Schwartz kernel $\Pi_N^{\tilde{\chi}}(x, y) = \Pi_N(\chi(\tilde{x}), \chi(\tilde{y}))$.

In the notation (4.36),

$$R_N = \tilde{L}_N - L_N = \Pi_N\big(\sigma_N \Pi_N^{\tilde{\chi}} M_{F\circ\tilde{\chi}} \Pi_N^{\tilde{\chi}} \bar{\sigma}_N - M_{F\circ\tilde{\chi}}\big)\Pi_N.$$

Evidently,

$$\operatorname{Tr}[R_N^* R_N] = \operatorname{Tr}[\tilde{L}_N^* \tilde{L}_N] - 2\operatorname{Tr}[\tilde{L}_N L_N] + \operatorname{Tr}[L_N^* L_N]. \tag{4.44}$$

We evaluate each term asymptotically by stationary phase with remainder and add the terms. Lemma 4.6.3 follows from:

LEMMA **4.6.4.** *We have*

$$\frac{1}{d_N}\operatorname{Tr}[L_N^* L_N] = \int_M |F \circ \tilde{\chi}|^2 \, dV + \mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0}\right). \tag{4.45}$$

Moreover,

$$\frac{1}{d_N}\operatorname{Tr}[\tilde{L}_N^* \tilde{L}_N] = \frac{1}{d_N}\operatorname{Tr}[\tilde{L}_N^* L_N] + \mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0}\right) = \frac{1}{d_N}\operatorname{Tr}[L_N^* L_N] + \mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0}\right).$$

In particular, thanks to (4.44) *we have*

$$\frac{1}{d_N}\operatorname{Tr}[R_N^* R_N] = \mathcal{O}\left(\frac{1}{N}\|F\|_{C^2}^2 e^{2\delta_0}\right).$$

The first statement (4.45) is the well-known Szegő limit formula with remainder. Since $\tilde{\chi}$ is symplectic it may be removed from $F \circ \tilde{\chi}$ in the integral. The leading order term is calculated in [3] using the homogeneous calculus of Toeplitz operators. The semiclassical calculation and the remainder estimate may be calculated by the method below.

For the rest of the section, we calculate the most difficult of the three terms, namely $d_N^{-1}\operatorname{Tr}[\tilde{L}_N^* \tilde{L}_N]$, asymptotically to leading order by the method of stationary phase for oscillatory integrals with complex phases of positive type. The calculations of the other two terms are similar and therefore omitted.

All three traces in (4.44) have the same leading order term (4.45), and so the leading term cancels when taking the sum (4.44). The cancellation between the 'symbols' σ_N and the Hessian determinants in the calculation of the leading order terms (4.45) is guaranteed by unitarity of $U_{\chi,N}$ (see also [**69**] for explicit calculation of the symbol).

From (4.43), we have

$$\frac{1}{d_N}\operatorname{Tr}[\tilde{L}_N^*\tilde{L}_N] = \frac{1}{d_N}\operatorname{Tr}\left[\Pi_N\bar{\sigma}_N\Pi_N^{\tilde{\chi}}M_{\overline{F\circ\tilde{\chi}}}\Pi_N^{\tilde{\chi}}\sigma_N\Pi_N\sigma_N\Pi_N^{\tilde{\chi}}M_{F\circ\tilde{\chi}}\Pi_N^{\tilde{\chi}}\bar{\sigma}_N\right]. \tag{4.46}$$

Note that we may drop the factor of Π_N at the end when computing the trace. We use the shorthand

$$\tilde{y}_j := \tilde{\chi}(y_j), \quad y_j \in X.$$

Recall that σ_N denotes multiplication by the symbol σ_N, and the Szegő projectors have Schwartz kernels

$$\begin{aligned}
&\Pi_N^{\tilde{\chi}}(y_1,y_2) = \Pi_N(\tilde{y}_1,\tilde{y}_2),\\
&\Pi_N(y_1,y_2) = N\int_0^\infty\int_{S^1} e^{iN[-\theta+t\psi(r_\theta y_1,y_2)]}s(r_\theta y_1,y_2,Nt)\,d\theta dt.
\end{aligned}$$

The last equality is the Boutet de Monvel–Sjöstrand parametrix introduced in Section 4.2.3. Using Schwartz kernels, the trace (4.46) can be written as the following oscillatory integral

$$\frac{1}{d_N}\operatorname{Tr}[\tilde{L}_N^*\tilde{L}_N] = \frac{1}{d_N}\int_X(\tilde{L}_N^*\tilde{L}_N)(x,x)\,dx$$

$$= \frac{1}{d_N} \int_X \left(N^6 \int_{X^5 \times (S^1)^6 \times (\mathbb{R}_+)^6} A(x, \boldsymbol{y}, \boldsymbol{\theta}, \boldsymbol{t}) e^{iN\Psi(x,\boldsymbol{y},\boldsymbol{\theta},\boldsymbol{t})} \, d\boldsymbol{t} d\boldsymbol{\theta} d\boldsymbol{y} \right) dx,$$

where

$$\boldsymbol{y} = (y_1, \ldots, y_5) \in X^5, \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_6) \in (S^1)^6, \quad \boldsymbol{t} = (t_1, \ldots, t_6) \in (\mathbb{R}_+)^6$$

and the amplitude and phase function are given by

$$\begin{aligned} A &= s(r_{\theta_1} x, y_1, t_1 N) \bar{\sigma}_N(y_1) s(r_{\theta_2} \tilde{y}_1, \tilde{y}_2, t_2 N) \overline{F(\tilde{y}_2)} s(r_{\theta_3} \tilde{y}_2, \tilde{y}_3, t_3 N) \sigma_N(y_3) \\ &\quad \times s(r_{\theta_4} y_3, y_4, t_4 N) \sigma_N(y_4) s(r_{\theta_5} \tilde{y}_4, \tilde{y}_5, t_5 N) F(\tilde{y}_5) s(r_{\theta_6} \tilde{y}_5, \tilde{x}, t_6 N) \bar{\sigma}_N(x), \\ \Psi &= t_1 \psi(r_{\theta_1} x, y_1) - \theta_1 + t_2 \psi(r_{\theta_2} \tilde{y}_1, \tilde{y}_2) - \theta_2 + t_3 \psi(r_{\theta_3} \tilde{y}_2, \tilde{y}_3) - \theta_3 \\ &\quad + t_4 \psi(r_{\theta_4} y_3, y_4) - \theta_4 + t_5 \psi(r_{\theta_5} \tilde{y}_4, \tilde{y}_5) - \theta_5 + t_6 \psi(r_{\theta_6} \tilde{y}_5, \tilde{x}). \end{aligned}$$

The functions s and ψ come from the Boutet de Monvel–Sjöstrand parametrix (4.12), and σ_N comes from the quantization formula (4.10).

The method of stationary phase is used to compute the inner integral. The off-diagonal exponential decay estimate (4.13) for the Bergman kernel allows us to localize the X^5-space integral to the region $\{d(y_j, y_k) < N^{-1/3}\}$ and absorb the error in the remainder estimate for R_N. To locate the critical points of the phase function Ψ, recall from (4.11) that the function ψ has the form

$$\psi(x, y) = \frac{1}{i}\Big(1 - \Lambda(x, y)\Big) \quad \text{with} \quad \Lambda(x, y) := e^{-\frac{\varphi(z_1)}{2} - \frac{\varphi(z_2)}{2} + \varphi(z_1, \bar{z}_2)} e^{i(\tau_1 - \tau_2)},$$

from which it follows

$$\psi(r_\theta x, y) = \frac{1}{i}\Big(1 - e^{i\theta}\Lambda(x,y)\Big).$$

Therefore,

$$D_{t_1}\Psi = \psi(r_{\theta_1}x, y_1) = 0 \iff 1 = e^{i\theta_1}\Lambda(x, y_1).$$

The Schwarz inequality shows that a real critical point exists if and only if $x = y_1$. Similar computations for $D_{t_j}\Psi$ demand that $\tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3$, $y_3 = y_4$, and $\tilde{y}_4 = \tilde{y}_5 = \tilde{x}$. The real critical point of Ψ must therefore satisfy

$$x = y_1 = y_2 = y_3 = y_4 = y_5. \tag{4.47}$$

Consider now the θ_1 derivative:

$$D_{\theta_1}\Psi = -t_1 e^{i\theta_1}\Lambda(x, y_1) - 1 = 0 \iff 1 = -t_1 e^{i\theta_1}\Lambda(x, y_1).$$

From the constraint (4.47), we must have $x = (z_1, \tau_1) = (z_2, \tau_2) = y_1$, so $\Lambda(x, y_1) = 1$. It follows that $t_1 = -1$ and $\theta_1 = 0$. Similar computations for $D_{\theta_j}\Psi$ show that the real critical point of Ψ satisfies

$$\theta_1 = \cdots = \theta_6 = 0 \quad \text{and} \quad t_1 = \cdots = t_6 = -1. \tag{4.48}$$

Finally, we claim that $D_{y_j}\Psi$ automatically vanishes at the points satisfying (4.47) and (4.48). Indeed, at the critical point we have

$$D_{y_1}\Psi\Big|_{\substack{x=y_1=\cdots=y_5\\ \theta_j=0\\ t_j=-1}} = -D_{y_1}\psi(x,y_1)|_{y_1=x} - D_{y_1}\psi(\tilde{y}_1,\tilde{y}_2)|_{y_2=y_1=x}.$$

Recall, however, that along the diagonal of $X\times X$ we have

$$d_1\psi = -d_2\psi = \frac{1}{i}d\rho|_X = \alpha,$$

where α is the contact form. Here d_j refers to the derivative with respect to the jth slot of $\psi(\cdot,\cdot)$. The assumption that χ lifts to a contact transformation, that is, $\tilde{\chi}^*\alpha=\alpha$, implies

$$-D_{y_1}\psi(x,y_1)|_{y_1=x}-D_{y_1}\psi(\tilde{y}_1,\tilde{y}_2)|_{y_2=y_1=x} = \alpha(x)-\frac{1}{i}d\rho(\tilde{\chi}(x)) = \alpha(x)-\tilde{\chi}^*\left(\frac{1}{i}d\rho\right)(x) = 0.$$

Similar computations for $D_{y_j}\Psi$ show that the real critical points of Ψ are completely given by (4.47) and (4.48).

It is straightforward to verify that the Hessian at the critical point is a block matrix of the form

$$\operatorname{Hess}\Psi(x) = \begin{bmatrix} D_{\boldsymbol{tt}}\Psi = 0 & D_{\boldsymbol{t\theta}}\Psi = -\mathrm{Id} & D_{\boldsymbol{t1}}\Psi & D_{\boldsymbol{t2}}\Psi \\ D_{\boldsymbol{\theta t}}\Psi = -\mathrm{Id} & D_{\boldsymbol{\theta\theta}}\Psi = i\cdot\mathrm{Id} & D_{\boldsymbol{\theta 1}}\Psi & D_{\boldsymbol{\theta 2}}\Psi \\ D_{\boldsymbol{1t}}\Psi & D_{\boldsymbol{1\theta}}\Psi & D_{\boldsymbol{11}}\Psi & D_{\boldsymbol{12}}\Psi \\ D_{\boldsymbol{2t}}\Psi & D_{\boldsymbol{2\theta}}\Psi & D_{\boldsymbol{21}}\Psi & D_{\boldsymbol{22}}\Psi \end{bmatrix}$$

with

$$D_{\boldsymbol{t1}}\Psi = -D_{\boldsymbol{t2}}\Psi = \begin{bmatrix} \alpha(x) & 0 & 0 & 0 & 0 \\ -\alpha(x) & \alpha(x) & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -\alpha(x) & \alpha(x) \\ 0 & 0 & 0 & 0 & -\alpha(x) \end{bmatrix} = -(D_{\boldsymbol{2t}}\Psi)^t = (D_{\boldsymbol{1t}}\Psi)^t,$$

$$D_{\boldsymbol{\theta 1}}\Psi = -D_{\boldsymbol{\theta 2}}\Psi = \begin{bmatrix} -i\alpha(x) & 0 & 0 & 0 & 0 \\ i\alpha(x) & -i\alpha(x) & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & i\alpha(x) & -i\alpha(x) \\ 0 & 0 & 0 & 0 & i\alpha(x) \end{bmatrix} = -(D_{\boldsymbol{2\theta}})^t = (D_{\boldsymbol{2\theta}}\Psi)^t,$$

$$D_{\boldsymbol{11}}\Psi = \begin{bmatrix} -d\alpha(x) & d\alpha(x) & 0 & 0 & 0 \\ d\alpha(x) & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & d\alpha(x) \\ 0 & 0 & 0 & d\alpha(x) & -d\alpha(x) \end{bmatrix} = D_{\boldsymbol{22}}\Psi.$$

This Hessian matrix is invertible by the Schur complement formula (recall that $-id\rho = \alpha$ is non-vanishing in a neighborhood of X). The method of stationary phase shows that the Schwartz kernel $(\tilde{L}_N^* \tilde{L}_N)(x,x)$ along the diagonal has the expansion

$$(4.49)\quad (\tilde{L}_N^* \tilde{L}_N)(x,x) \sim \frac{N^6}{(N^{12+10m} \det \operatorname{Hess} \Psi(x))^{1/2}} \sum_{j,k,\ell,p,q,u,v \geq 0} N^{6m-j-k-\ell-p-q-u-v} \times L_j\Big(s_k(x,x)s_\ell(x,x)s_p(\tilde{x},\tilde{x})s_q(\tilde{x},\tilde{x})s_u(\tilde{x},\tilde{x})s_v(\tilde{x},\tilde{x})|\sigma_N(x)|^4|F(\tilde{x})|^2\Big),$$

where L_j are differential operators of order at most $2j$ that can be explicitly expressed in terms of s_k and the Hessian [**26**, Theorem 7.7.5].

Observe that the leading order term (obtained from the above expression by setting $j = k = \cdots = v = 0$) is of order $N^6(N^{12+10m})^{-1/2}N^{6m} = N^m$. The symbol σ_N is constructed to make $U_{\chi,N}$ unitary, i.e., $U_{\chi,N}^* U_{\chi,N} = \Pi_N$, and by taking the symbol of this equation it follows that

$$(4.50)\qquad (\det \operatorname{Hess} \Psi(x))^{-1/2} s_0(x,x)^2 s_0(\tilde{x},\tilde{x})^4 |\sigma_N(x)|^4 = 1.$$

Indeed, if we set $F \equiv 1$ so that $M_F = \mathrm{Id}$, then $\tilde{L}_N^* \tilde{L}_N = U_{\chi,N} U_{\chi,N}^* U_{\chi,N} U_{\chi,N}^* = \mathrm{Id}$. The identity (4.50) follows from plugging this particular choice of F into (4.49). Therefore, after dividing by $d_N \sim N^m$ (for N large enough), the leading order term of $d_N^{-1} \operatorname{Tr}[\tilde{L}_N^* \tilde{L}_N]$ is of order 0, and is equal to $\int |F(\tilde{x})|^2 = \int |F \circ \tilde{\chi}|^2$, which agrees with (4.45). The second order term (cf. [**26**, Theorem 7.7.5]) of $\tilde{L}_N^* L_N(x,x)$ is bounded above in sup norm by

$$C \sum_{|\alpha| \leq 2} \Big\| D^\alpha \Big((\det \operatorname{Hess} \Psi(x))^{-\frac{1}{2}} s_0(x,x)^2 s_0(\tilde{\chi}(x), \tilde{\chi}(x))^4 |\sigma_N(x)|^4 |F \circ \tilde{\chi}(x)|^2 \Big) \Big\|_\infty = C \sum_{|\alpha| \leq 2} \Big\| D^\alpha |F \circ \tilde{\chi}(x)|^2 \Big\|_\infty$$

$$\leq C\bigg(\sum_{|\alpha|\leq 2}\Big\|D^\alpha|F\circ\tilde{\chi}(x)|\Big\|_\infty\bigg)^2$$
$$\leq C\|F\|_{C^2}^2 e^{2\delta_0}.$$

for some constant C that depends on a fixed number of derivatives of the phase function Ψ (and hence on $\tilde{\chi}$) but is otherwise independent of N. Dividing through by $d_N \sim N^m$ yields the desired error estimate $\mathcal{O}(N^{-1}\|F\|_{C^2}^2 e^{2\delta_0})$. This completes the computation for $\tilde{L}_N$.

CHAPTER 5

Log-scale Equidistribution of Nodal Sets in Grauert Tubes

This chapter discusses small-scale equidistribution results, namely Theorem 5.1.1, Theorem 5.1.4, and Theorem 5.7.2, for complexified eigenfunctions on Grauert tubes. Throughout, let (M^n, g) be a compact, negatively curved, real analytic Riemannian manifold without boundary. By a well-known theorem of Bruhat–Whitney, M admits a complexification $M_{\mathbb{C}}$ into which it embeds as a totally real submanifold. The metric g on M induces a plurisubharmonic function ρ whose square root $\sqrt{\rho}\colon M_{\mathbb{C}} \to [0,\infty)$ is called the Grauert tube function. There exists a geometric constant $\tau_0 = \tau_0(M, g) > 0$ so that, for each $\tau \leq \tau_0$, the sublevel set

$$M_\tau := \{\zeta \in M_{\mathbb{C}} \colon \sqrt{\rho}(\zeta) < \tau\}$$

is a strictly pseudo-convex domain in $M_{\mathbb{C}}$. We call M_τ the *Grauert tube* of M of radius τ. The $(1,1)$-form $\omega := -i\partial\bar{\partial}\rho$ endows M_τ with a Kähler metric and $(M, g) \hookrightarrow (M_\tau, \omega)$ is an isometric embedding. (The unusual sign convention that makes the Kähler form negative is adopted from [22].) We write

$$(5.1) \qquad d\mu := \omega^n \quad \text{and} \quad d\mu_\tau := \frac{\omega^n}{d\sqrt{\rho}\mid_{\partial M_\tau}} = \frac{\omega^n}{d\tau}.$$

for the Kähler volume form on M_τ and the Liouville surface measure on ∂M_τ, respectively. There exists a diffeomorphism E, defined in (5.8), between M_τ and the co-ball bundle

$B^*_\tau M = \{(x,\xi) \in T^*M : |\xi|_{g_x} < \tau\}$. The Kähler form ω on M_τ is the pullback under E of the standard symplectic form on $B^*_\tau M$. Conversely, E endows $B^*_\tau M$ with a complex structure J_g adapted to g. Definitions and background are recalled in Section 5.2; see also [**23**, **38**].

5.1. Main Results

Every eigenfunction φ_j on M admits an analytic extension $\varphi_j^{\mathbb{C}}$ to the maximal Grauert tube M_{τ_0}. The analytically continued eigenfunctions are smooth on the boundaries ∂M_τ for every $\tau \leq \tau_0$. The complex zero set of $\varphi_j^{\mathbb{C}}$ is the complex hypersurface

$$Z_j := \{\zeta \in M_{\tau_0} : \varphi_j^{\mathbb{C}}(\zeta) = 0\}.$$

The zero sets define currents $[Z_j]$ of integration in the sense that for every smooth $(n-1, n-1)$ test form $\eta \in \mathcal{D}^{n-1,n-1}(M_{\tau_0})$, we the pairing

$$\langle [Z_j], \eta\rangle := \int_{Z_j} \eta = \int_{M_{\tau_0}} \frac{i}{2\pi} \partial\bar{\partial} \log|\varphi_j^{\mathbb{C}}|^2 \wedge \eta \tag{5.2}$$

is a well-defined closed current[1]. In the special case $\eta = f\omega^{n-1}$, the zero set defines a positive measure $|Z_j|$ by

$$\langle |Z_j|, f\rangle := \int_{Z_j} f\omega^{n-1}, \qquad f \in C(M_{\tau_0}).$$

[1]Since Z_j may be singular, we include background on the last statement in Section 5.8.

The limit distribution of the zero currents (5.2) has been investigated in [71]. It was shown that on a compact, real analytic, negatively curved manifold, one has

$$\frac{1}{\lambda_{j_k}}[Z_{j_k}] \rightharpoonup \frac{i}{\pi}\partial\bar{\partial}\sqrt{\rho} \quad \text{weakly as currents on } M_{\tau_0} \tag{5.3}$$

along a density one subsequence of eigenvalues λ_{j_k}. The motivating problem of this article is to obtain a similar convergence theorem on balls in $M_{\tau_0}\backslash M$ with logarithmically shrinking radii of size

$$\varepsilon(\lambda_j) := (\log \lambda_j)^{-\alpha} \quad \text{for some fixed } \alpha > 0 \text{ to be specified.}$$

The parameter α depends only on the dimension, and is independent of the frequency λ_j. The resulting log-scale convergence theorems, Theorem 5.1.1 and Theorem 5.7.2, along with their proofs, are generalizations of those in [11] in the setting of eigensections of ample line bundles over a compact boundaryless Kähler manifold, but have several new features.

5.1.1. Log-scale equidistribution of zeros

THEOREM **5.1.1** (Equidistribution of complex zeros, Chang–Zelditch [10])**.** *Let (M, g) be a real analytic, negatively curved, compact manifold without boundary. Let $\omega := -i\partial\bar{\partial}\rho$ be the Kähler form on the Grauert tube M_{τ_0}. Assume that*

$$0 \leq \alpha < \frac{1}{2(3n-1)}, \quad \varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}.$$

Then there exists a full density subsequence of eigenvalues λ_{j_k} such that for any $f \in C(M_{\tau_0})$ and for any arbitrary but fixed $\zeta_0 \in M_{\tau_0} \setminus M_\tau$, we have

$$(5.4)\quad \left| \frac{1}{\lambda_{j_k}\varepsilon(\lambda_{j_k})^{2n-1}} \int_{Z_{j_k}\cap\mathcal{B}(\zeta_0,\varepsilon(\lambda_{j_k}))} f\omega^{n-1} \right. \\ \left. -\frac{1}{\varepsilon(\lambda_{j_k})^{2n-1}} \int_{\mathcal{B}(\zeta_0,\varepsilon(\lambda_{j_k}))} f\frac{i}{\pi}\partial\bar{\partial}|\Im(\zeta-\zeta_0)|_{g_0} \wedge \omega_0^{n-1} \right| = o(1).$$

Here, $\omega_0 := -i\partial\bar{\partial}|\Im(\zeta-\zeta_0)|^2_{g_0}$ denotes the flat Kähler form in local Kähler coordinates centered at ζ_0, with $|\cdot|_{g_0}$ the Euclidean distance. The $o(1)$ remainder is uniform for any ζ_0 lying in an 'annulus' $0 < \tau_1 \leq \sqrt{\rho}(\zeta_0) \leq \tau_0$.

Theorem 5.1.1 is deduced from a rescaled version given in Theorem 5.7.2. The latter theorem is stated using the holomorphic dilation introduced in Section 5.3.1. Briefly, define dilation operator $D^{\zeta_0}_{\varepsilon(\lambda_j)} \colon \zeta \mapsto \zeta_0 + \varepsilon(\lambda_j)(\zeta-\zeta_0)$ in Kähler normal coordinates around ζ_0. The zero currents $[Z_j]$ on shrinking balls $B(\zeta_0, \varepsilon(\lambda_j))$ pulls back to currents $D^{\zeta_0 *}_{\varepsilon(\lambda_j)}[Z_j]$ on a fixed unit ball $B(\zeta_0, 1) \subset \mathbb{C}^n$. The normalizing factors in Theorem 5.1.1 arise from homogeneity and rescaling: $\omega^{n-1}, \omega_0^{n-1}$ are homogeneous of degree $2n-2$ and $\frac{i}{\pi}\partial\bar{\partial}|\Im(\zeta-\zeta_0)|_{g_0}$ is homogeneous of degree 1. The scaling of the nodal current on the left side is the same as that of its limit current $\frac{i}{\pi}\partial\bar{\partial}|\Im(\zeta-\zeta_0)|_{g_0}$.

Remark **5.1.2.** *In the statement of Theorem 5.1.1, the center ζ_0 is arbitrary but fixed in the interior of $M_{\tau_0}\setminus M$ and only the radii of the balls are shrinking. Also, note that ζ_0 must lie away from the totally real submanifold M of M_{τ_0}, or equivalently the zero section 0_M of $B^*_{\tau_0}M$. Reasons are discussed in Section 5.1.4.*

Remark **5.1.3.** *The zero sets Z_j may be singular, but it is known that the singular set of the real nodal set is of real codimension four (see Section 5.8). For generic metrics, all of the nodal sets are regular [63].*

5.1.2. Log-scale equidistribution of masses

Knowledge of the log-scale L^2 masses of eigenfunctions is required to deduce Theorem 5.1.1. To state the relevant result, we need some more notation:

$$\Theta_j(\zeta) := \|\varphi_j^{\mathbb{C}} \,|_{\partial M_{\sqrt{\rho}(\zeta)}} \|_{L^2(\partial M_{\sqrt{\rho}(\zeta)})}, \qquad U_j(\zeta) := \frac{\varphi_j^{\mathbb{C}}(\zeta)}{\Theta_j(\zeta)}, \qquad (\zeta \in M_{\tau_0} \setminus M)$$

In words, the normalizing factor $\Theta_j(\zeta)$ is the L^2-norm (of the restriction $\varphi_j^{\mathbb{C}} \,|_{\partial M_{\sqrt{\rho}(\zeta)}}$) of $\varphi_j^{\mathbb{C}}$ along the boundary of the Grauert tube of radius $\sqrt{\rho}(\zeta)$. The function U_j is the (the unrestricted) complexified eigenfunction $\varphi_j^{\mathbb{C}}$ normalized by this L^2-norm. Finally, let

$$u_j^\tau(Z) := U_j(Z) \,|_{\partial M_\tau} = \frac{\varphi_j^{\mathbb{C}}(Z) \,|_{\partial M_\tau}}{\|\varphi_j^{\mathbb{C}} \,|_{\partial M_\tau} \|_{L^2(\partial M_\tau)}}, \qquad (Z \in \partial M_\tau,\ 0 < \tau \leq \tau_0)$$

be the restriction of U_j to the Grauert tube of radius $\sqrt{\rho}(\zeta) = \tau$. (We denote points by Z instead of ζ when working on a fixed slice ∂M_τ.) The global behavior of L^2 masses of U_j and u_j^τ are known. Specifically, Zelditch [71, Lemma 1.4, Lemma 4.1] proved the existence of a density one subsequence $\{\varphi_{j_k}\}$ of orthonormal basis such that

$$|U_{j_k}|^2 \,\omega^n \rightharpoonup \omega^n \quad \text{and} \quad |u_{j_k}^\tau|^2 \,d\mu_\tau \rightharpoonup d\mu_\tau \tag{5.5}$$

in the sense of weak* convergence on $C(M_{\tau_0})$ and on $C(\partial M_\tau)$ for each $0 < \tau \leq \tau_0$, respectively. (Recall (5.1) for the definitions.) Integrating over M_{τ_0} (resp. ∂M_τ) implies the L^2

masses of U_{j_k} (resp. $u^{\tau}_{j_k}$) become equidistributed in all of M_{τ_0} (resp. ∂M_τ). It is not known whether the convergence (5.5) holds at logarithmic length scales (i.e., simultaneously on all Kähler balls of logarithmically shrinking radii). Luckily, all that is needed for the proof of Theorem 5.1.1 is a uniform L^2 volume comparison theorem, which we presently state.

THEOREM **5.1.4** (Equidistribution of masses, Chang–Zelditch [**10**])**.** *Let (M, g) be a real analytic, negatively curved, compact manifold without boundary. Let $\omega := -i\partial\bar{\partial}\sqrt{\rho}$ denote the Kähler form on the Grauert tube M_{τ_0}. Assume that*

$$0 \leq \alpha < \frac{1}{2(3n-1)}, \quad \varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}.$$

Then there exists a full density subsequence of eigenvalues λ_{j_k} such that for arbitrary but fixed $\zeta_0 \in M_{\tau_0} \backslash M$, there is a uniform two-sided volume bound

$$c\mathrm{Vol}_\omega(\mathcal{B}(\zeta_0, \varepsilon(\lambda_{j_k}))) \leq \int_{\mathcal{B}(\zeta_0,\varepsilon(\lambda_{j_k}))} |U_{j_k}|^2 d\mu \leq C\mathrm{Vol}_\omega(\mathcal{B}(\zeta_0, \varepsilon(\lambda_{j_k}))). \tag{5.6}$$

The constants c, C are geometric constants depending only on $\sqrt{\rho}(\zeta_0)$; they are uniform for any ζ_0 lying in an 'annulus' $0 < \tau_1 < \sqrt{\rho}(\zeta_0) \leq \tau_0$.

REMARK **5.1.5.** *Only the lower bound in the statement of Theorem 5.1.4 – used crucially in a proof by contradiction argument for Proposition 5.7.5 around* (5.42)–(5.43) *– is needed to imply Theorem 5.1.1.*

Log-scale results of this kind, which we briefly recall in Section 5.5, were first proved in the real domain by Hezari–Rivière [**25**] and Han [**24**]. In the setting of a general compact, negatively curved, Kähler manifold (not necessarily real analytic), an analogous result can be found in [**11**, Theorem 2].

Remark **5.1.6.** *The semiclassical notation $h := \lambda^{-1}$ is also used throughout Section 5.4–Section 5.6, in which we write $\delta(h) = |\log h|^{-\alpha} = (\log \lambda)^{-\alpha} = \varepsilon(\lambda)$; see* (5.18).

5.1.3. Outline of proof

Theorem 5.1.4 is proved by expressing the L^2 mass of u_j^τ (resp. U_j) in terms of matrix elements of Szegő-Toeplitz operators on ∂M_τ for $0 < \tau \leq \tau_0$ (resp. Bergman-Toeplitz operators on M_{τ_0}). We show that a certain Poisson-FBI transform conjugates a (smoothed) characteristic function of the ball $\mathcal{B}(\zeta_0, \varepsilon(\lambda_j))$ to a semiclassical pseudodifferential operator acting on $L^2(M)$ whose symbol has the same properties as (but does not coincide with) the small-scale symbols used in [**24**]. This conjugation allows us to derive Proposition 5.6.2, a variance estimate for matrix elements in the complex domain, by relating it to the known variance estimate in the real domain of [**24**].

Once the variance estimate is proved, the comparability result of Theorem 5.1.4 follows the path in [**25, 24, 11**]. Namely, one chooses an appropriate covering of M_{τ_0} and extracts a subsequence of eigenvalues of density one for which one has simultaneous asymptotic log-scale QE for the balls in every cover. The balls are 'dense enough' that one obtains good upper and lower bounds for eigenfunction mass in any logarithmically shrinking ball.

Lastly, to derive Theorem 5.1.1 from Theorem 5.1.4, we follow the method of [**57, 11**] that uses well-known facts about plurisubharmonic functions. We begin by rewriting the zero current $[Z_j]$ as $\partial\bar{\partial}$ of plurisubharmonic functions using the Poincarè–Lelong formula (5.33). A standard compactness theorem yields the desired result.

5.1.4. Singular behavior along the real domain

We briefly discuss the reasons for requiring centers ζ_0 of balls to lie in $M_{\tau_0} \backslash M$.

The key tool in studying the mass and zeros in the complex domain is the complexified Poisson operator $P^\tau \colon L^2(M) \to \mathcal{O}^{\frac{n-1}{4}}(\partial M_\tau)$ defined in Section 5.2.3. By $\mathcal{O}^{-\frac{n-1}{4}}(\partial M_\tau)$ we mean the Hardy-Sobolev space of boundary values of holomorphic functions in M_τ with the designated Sobolev regularity. This Hilbert space is the quantization of the symplectic cone $\Sigma_\tau \subset T^*(\partial M_\tau)$ defined in Section 5.2.2, an $\mathbb{R}_+$-bundle $\Sigma_\tau \to \partial M_\tau$. The Poisson operator is a homogeneous Fourier integral operator with positive complex phase adapted to the homogeneous symplectic isomorphism $\iota_\tau \colon T^*M \setminus 0_M \to \Sigma_\tau$ of (5.10).

The homogeneous theory becomes singular along the zero section 0_M, or equivalently along the totally real submanifold M. This reflects the fact that the eigenfunctions φ_j microlocally concentrate on energy surfaces $\{|\xi|_g = \lambda_j\}$, the characteristic variety of $\Delta + \lambda_j^2$. In the semiclassical setting of $h^2\Delta + 1$ (with $h = \lambda_j^{-1}$), the eigenfunctions concentrate on S^*M. The energy level 1 is arbitrary here and depends on the choice of constant C in the semiclassical scaling $h_j = C\lambda_j^{-1}$. One may adjust it so that eigenfunctions concentrate on any energy surface $\partial B^*_\tau M \simeq \partial M_\tau$ with respect to semiclassical pseudodifferential operators $\mathrm{Op}_{h_j}(a)$. But this scaling breaks down on the zero section.

The singularity of the theory along the zero section may be seen in Theorem 5.4.1. When conjugated back to the real domain, the symbols become functions of $|\xi|$ and are singular when $\xi = 0$. It seems that the behavior on the zero section can be studied by using an adapted class of observables that smoothly interpolates between pseudodifferential operators when $\tau = 0$ and Toeplitz operators when $\tau > 0$. We hope to clarify this issue in the future.

5.2. Background: Microlocal Analysis on Grauert Tubes

5.2.1. Grauert tube and the co-ball bundle

The readers are referred to [22, 23, 38, 39] for the analysis of the complex Monge–Ampère equation, the Grauert tube function, the geometry of Grauert tubes and related topics. Here we provide only a brief summary of some notation and theorems needed for this paper, following [71, 73].

A real analytic manifold (M, g) always possesses a complexification $M_{\mathbb{C}}$, that is, a complex manifold of which M is a totally real embedded submanifold. Let $\exp_x \colon T_x^*M \to M$ be the Riemannian exponential map, i.e., $\exp_x \xi = \pi \exp t\Xi_{|\xi|_g^2}$, where $\pi \colon T^*M \to M$ is the natural projection and $\Xi_{|\xi|_g^2}$ is the Hamiltonian flow of $|\xi|_g^2$. The analyticity of M implies that the exponential map admits an analytic extension

$$\exp_x^{\mathbb{C}} \colon U_x \subset T_x^*M \otimes \mathbb{C} \to M_{\mathbb{C}} \tag{5.7}$$

defined in a suitable domain $U_x \subset T_x^*M$. Its restriction to the imaginary axis (that is, the analytic extension in t of $\exp_x(t\xi)$ to imaginary time $t = i$) is denoted by

$$E \colon B_\tau^*M \to M_{\mathbb{C}}, \quad (x, \xi) \mapsto E(x, \xi) := \exp_x^{\mathbb{C}}(i\xi). \tag{5.8}$$

For all $\tau > 0$ sufficiently small, (5.8) is a diffeomorphism between the co-ball bundle $B_\tau^*M = \{(x, \xi) \in T^*M : |\xi|_{g_x} < \tau\}$ and the subset

$$M_\tau := \{\zeta \in M_{\mathbb{C}} : \sqrt{\rho}(\zeta) < \tau\} \subset M_{\mathbb{C}}.$$

Here, $\sqrt{\rho}$ is known as the Grauert tube function, and its sublevel set M_τ is known as the Grauert tube (of radius τ). The restriction $E\mid_{\partial B^*_\tau M}$ of (5.8) to the co-sphere bundle is a CR holomorphic diffeomorphism between the two strictly pseudo-convex CR manifolds $\partial B^*_\tau M$ and ∂M_τ.

The square ρ of the Grauert tube function is a strictly plurisubharmonic function uniquely determined by two conditions:

- It is a solution of the Monge-Ampère equation $(\partial\bar{\partial}\sqrt{\rho})^n = \delta_M$, where δ_M is the delta-function on the real manifold M with respect to the volume form dV_g;
- The Kähler form $\omega := -i\partial\bar{\partial}\rho$ restricts to g along M.

If we write $r(x,y)$ for the Riemannian distance function on M, then $r^2(x,y)$ is real analytic in a neighborhood of the diagonal in $M \times M$. It possesses an analytic continuation $r^2(\zeta,\bar{\zeta})$ for $\zeta \in M_{\mathbb{C}}$ in a sufficiently small neighborhood of the totally real submanifold M. The plurisubharmonic function is related to the Riemannian distance function by

$$\rho(\zeta) = -\frac{1}{4} r^2(\zeta,\bar{\zeta}).$$

For the trivial case $M = \mathbb{R}^n$, we have $M_{\mathbb{C}} = \mathbb{C}^n$ and $\sqrt{\rho}(\zeta) = \sqrt{-\frac{1}{4}(\zeta - \bar{\zeta})^2} = |\Im \zeta|$. More examples are found in [**71**].

5.2.2. Szegő projector

Let $\mathcal{O}(\partial M_\tau)$ denote the space of CR holomorphic functions on ∂M_τ. We use the notation

$$\mathcal{O}^{s+\frac{n-1}{4}}(\partial M_\tau) := W^{s+\frac{n-1}{4}}(\partial M_\tau) \cap \mathcal{O}(\partial M_\tau)$$

for the subspace of the Sobolev space $W^{s+\frac{n-1}{4}}(\partial M_\tau)$ consisting of CR holomorphic functions. The inner product is taken with respect to the Liouville surface measure (5.1). The Szegő projector

$$\Pi_\tau \colon L^2(\partial M_\tau) \to \mathcal{O}^0(\partial M_\tau) \tag{5.9}$$

is the orthogonal projection onto boundary values of holomorphic function. It is well-known (cf. [**5, 49, 23**]) that Π_τ is a complex Fourier integral operator of positive type, whose real canonical relation is the graph of the identity map on the symplectic cone

$$\Sigma_\tau = \{(Z; r d^c\sqrt{\rho}(Z)) \in T^*(\partial M_\tau) : Z \in \partial M_\tau,\ r > 0\}$$

spanned by the contact form $d^c\sqrt{\rho} = -i(\partial - \bar{\partial})\sqrt{\rho}$ on ∂M_τ. Since Σ_τ is an $\mathbb{R}_+$-bundle over ∂M_τ, we can define the symplectic equivalence of cones:

$$\iota_\tau \colon T^*M \setminus 0 \to \Sigma_\tau, \quad \iota_\tau(x,\xi) := \left(E\Big(x, \tau\frac{\xi}{|\xi|}\Big), |\xi| d^c\sqrt{\rho}_{E(x,\tau\frac{\xi}{|\xi|})} \right). \tag{5.10}$$

5.2.3. Poisson-wave operator

A key object in our analysis is the Poisson-wave operator

$$P^\tau \colon L^2(M) \to \mathcal{O}^{\frac{n-1}{4}}(\partial M_\tau).$$

(Unlike for the Szegő projector (5.9), τ appears as a superscript here because we will be considering semiclassical Poisson-wave operators, which are denoted by P_h^τ.) The Poisson-wave operator is obtained from the half-wave operator by analytic extension in the time and spatial variables. Specifically, recall that the half-wave operator is given by

$U(t) := e^{it\sqrt{-\Delta}}$. When $t = i\tau$ lies in the positive imaginary axis, $P^\tau := U(i\tau) = e^{-\tau\sqrt{-\Delta}}$ is a complex Fourier integral operator known as the Poisson-wave operator. As discussed in [**4, 23, 36**], for $0 < \tau \leq \tau_0$ and $y \in M$ fixed, the Poisson kernel $P^\tau(\,\cdot\,, y) = U(i\tau, \cdot\,, y)$ extends to a holomorphic function on M_τ.

Take for concreteness the wave kernel on $\mathbb{R}^n$ as an example. The Euclidean wave kernel

$$U(t,x,y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y\rangle}\, d\xi$$

analytically continues to $(i\tau, x+ip) \in \mathbb{C}_+ \times \mathbb{C}^n$ by the integral formula

$$P^\tau(x+ip, y) = \int_{\mathbb{R}^n} e^{-\tau|\xi|} e^{i\langle \xi, x-y+ip\rangle}\, d\xi,$$

which converges absolutely for $|p| < \tau$.

On a general Riemannian manifold there exists a similar Lax–Hörmander parametrix for the wave kernel:

$$U(t,x,y) = \int_{T_y^*M} e^{it|\xi|_y} e^{i\langle \xi, \exp_y^{-1}(x)\rangle} A(t,x,y,\xi)\, d\xi, \tag{5.11}$$

where $|\cdot|_y$ is the metric norm function at y, and where $A(t,x,y,\xi)$ is a polyhomogeneous amplitude of order 0. The holomorphic extension $x \mapsto \zeta$ to the Grauert tube M_{τ_0} at time $t = i\tau$ is a Fourier integral operator with complex phase of the form

$$P^\tau(\zeta, y) = \int_{T_y^*M} e^{-\tau|\xi|_y} e^{i\langle \xi, (\exp_y^{\mathbb{C}})^{-1}(\zeta)\rangle} A(t,\zeta,y,\xi)\, d\xi. \tag{5.12}$$

The complexified exponential map $\exp_y^{\mathbb{C}}$ appearing in the phase function of the parametrix above is the local holomorphic extension of the Riemannian exponential map as defined in (5.7). It is easy to see that the integral converges absolutely for $\sqrt{\rho}(\zeta) < \tau$. We refer to [**62**, **35**, **72**] for proofs and background. The following result is stated by Boutet de Monvel [**4**]; proofs are given in [**72**, **35**].

THEOREM **5.2.1.** *Let $\iota_\tau \colon T^*M \setminus 0 \to \Sigma_\tau$ be the symplectic equivalence defined by* (5.10)*. Then the Poisson-wave operator $P^\tau \colon L^2(M) \to \mathcal{O}(\partial M_\tau)$ with the parametrix given by* (5.12) *is a complex Fourier integral operator of order $-\frac{n-1}{4}$ associated to the positive complex canonical relation*

$$\Gamma := \{(y, \eta, \iota_\tau(y, \eta)\} \subset T^*M \times \Sigma_\tau.$$

Moreover, for any s,

$$P^\tau \colon W^s(M) \to \mathcal{O}^{s+\frac{n-1}{4}}(\partial M_\tau)$$

is a continuous isomorphism.

It is helpful to introduce the framework of *adapted Fourier integral operators*. This notion is defined and discussed in the [**3**, Appendix A.2]. If X, X' are two smooth real manifolds, and $\Sigma \subset T^*X \setminus 0$, $\Sigma' \subset T^*X' - 0$ are two symplectic cones, then a Fourier integral operator F with complex phase is adapted to a homogeneous symplectic diffeomorphism $\chi \colon \Sigma \to \Sigma'$ if the canonical relation of F is a positive complex canonical relation whose real points consist of the graph of χ and if the symbol of F is elliptic. Theorem 5.2.1 may be reformulated in this language as follows: P^τ is a Fourier integral operator with complex phase of order $-\frac{n-1}{4}$ adapted to the symplectic isomorphism $\iota_\tau\colon T^*M \setminus 0 \to \Sigma_\tau$ given by (5.10). The point of the reformulation is that one may identify the graph of ι_τ with the graph of $G^{i\tau}$, where $G^t(x,\xi) = |\xi| G^t(x, \frac{\xi}{|\xi|})$ is the homogeneous geodesic flow defined on $T^*M \setminus 0$. Its analytic continuation in t is also homogeneous, so we have

$$G^{i\tau}(x,\xi) = |\xi| G^{i\tau}\Big(x, \frac{\xi}{|\xi|}\Big).$$

It is observed in [73] that $\iota_\tau(y,\eta) = G^{i\tau}(y,\eta)$. Thus, $G^{i\tau}$ gives a homogeneous symplectic isomorphism $G^{i\tau}\colon T^*M \setminus 0 \to \Sigma_\tau$.

In light of Theorem 5.2.1 and the calculus of FIOs, the operator

$$A^\tau := (P^{\tau *} P^\tau)^{-\frac{1}{2}}\colon L^2(M) \to L^2(M). \tag{5.13}$$

is an elliptic, self-adjoint pseudodifferential operator of order $\frac{n-1}{4}$ with principal symbol $|\xi|^{\frac{n-1}{4}}$. Equivalently, $P^{\tau *} P^\tau$ is a pseudodifferential operator of order $-\frac{n-1}{2}$ with principal symbol $|\xi|^{-\frac{n-1}{2}}$. An immediate consequence of Theorem 5.2.1, (5.13) and the symbol calculus of FIOs is the following.

Proposition **5.2.2.** *The operator $V^\tau := P^\tau A^\tau\colon L^2(M) \to \mathcal{O}^0(\partial M_\tau)$ is unitary (of order 0) with an approximate left inverse given by $V^{\tau *} A^\tau P^{\tau *}$. Moreover,*

$$(A^\tau)^2 P^{\tau *}\colon \mathcal{O}^0(\partial M_\tau) \to L^2(M)$$

is an approximate left inverse to P^τ.

5.2.4. Analytic continuation of eigenfunctions via the Poisson-wave kernel

Let $\{\varphi_j\}$ be an orthonormal basis of Laplacian eigenfunctions on (M, g) with eigenvalue $-\lambda_j^2$. Then the half-wave kernel $U(t, x, y) := e^{it\sqrt{-\Delta}}(x, y)$ admits the eigenfunction expansion

$$U(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_j(x)\overline{\varphi_j(y)}.$$

It follows that the holomorphic extension to $M_\tau \times M$ of the Poisson kernel is given by

$$P^\tau(\zeta, y) = U(i\tau, \zeta, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j^{\mathbb{C}}(\zeta)\overline{\varphi_j(y)}, \qquad (\zeta, y) \in M_\tau \times M.$$

We therefore obtain a formula for the analytic extension $\varphi_j^{\mathbb{C}}$ of an eigenfunction φ_j to the Grauert tube. Specifically, if $Z \in \partial M_\tau$ (so in particular $\sqrt{\rho}(Z) = \tau$), then

$$\varphi_j^{\mathbb{C}}(Z) = e^{\tau\lambda_j}(P^\tau \varphi_j)(Z) = e^{\sqrt{\rho}(Z)\lambda_j}(P^\tau \varphi_j)(Z), \qquad Z \in \partial M_\tau. \tag{5.14}$$

5.2.5. Szegő–Toeplitz multiplication operators

Let M_{τ_0} be a Grauert tube of some fixed radius τ_0. For $0 < \tau \leq \tau_0$ we consider operators of the form

$$\Pi_\tau a \Pi_\tau \colon \mathcal{O}^0(\partial M_\tau) \to \mathcal{O}^0(\partial M_\tau), \tag{5.15}$$

where by an abuse of notation we write a for multiplication by the symbol $a \in C^\infty(\partial M_\tau)$. The operator (5.15) is an example of a Szegő–Toeplitz operator. More generally, such an operator of order s acting on $H^2(\partial M_\tau)$ is of the form $\Pi_\tau Q \Pi_\tau$, with Q a pseudodifferential operator of order s. For this article it suffices to take $Q = a$ to be a multiplication operator. A Szegő–Toeplitz operator might be homogeneous or semiclassical depending on the nature of Q.

5.2.6. Poisson conjugation of Szegő–Toeplitz operators

The conjugation of a Toeplitz multiplication operator by the Poisson-wave FIO is studied in [**71**, Lemma 3.1] and in [**73**, Section 4.1]

Lemma **5.2.3.** *Let $a \in C^\infty(M_{\tau_0})$ and let P^τ be the Poisson-wave operator defined by* (5.12). *Then the conjugation*

$$P^{\tau *}\Pi_\tau a \Pi_\tau P^\tau \in \Psi^{-\frac{n-1}{2}}(M)$$

is a pseudodifferential operator with principal symbol equal to (the homogeneous extension of) $a(x,\xi)|\xi|_g^{-\frac{n-1}{2}}$. Moreover, let V^τ be the unitary operator defined in Proposition 5.2.2, then

$$V^{\tau *}\Pi_\tau a \Pi_\tau V^\tau \in \Psi^0(M)$$

with principal symbol equal to (the homogeneous extension of) $a(x,\xi)$.

Note that

$$V^{\tau *}\Pi_\tau a \Pi_\tau V^\tau = A^\tau P^{\tau *}\Pi_\tau a \Pi_\tau P^\tau A^\tau,$$

so that the second statement follows from Proposition 5.2.2 or from the first by (5.13).

Remark **5.2.4.** *The factors of* Π_τ *are redundant here because, by Theorem 5.2.1,* P^τ *maps into the range of* Π_τ*.*

5.3. Balls and Dilation in Grauert Tubes

The purpose of this section is to introduce the balls and local dilation that are relevant to the calculus of pseudodifferential operators with log-scale symbols.

Definition **5.3.1.** *We define Kähler balls* $\mathcal{B}(\zeta_0, \varepsilon(\lambda_j))$ *in the Grauert tube to be balls with respect to the Kähler metric* $\omega = -i\partial\bar{\partial}\rho$*. For reasons discussed in Section 5.1.4, we consider Kähler balls whose centers* $\zeta_0 \in M_{\tau_0} \backslash M$ *do not lie on the totally real submanifold* M*. The radii* $\varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}$ *shrinks logarithmically relative the frequency parameter* λ_j*.*

We also need to introduce local dilation centered at points $\zeta_0 \in M_{\tau_0}$. When working with holomorphic or plurisubharmonic functions, we always use local holomorphic dilation. But when working with dilated symbols we may use more general dilation that are more convenient. A technical point to address is that the local dilation does not preserve the family of Kähler balls. But for centers close enough to the real domain M, the metric is almost Euclidean on logarithmically shrinking balls.

5.3.1. Holomorphic dilation

Let $\zeta_0 = E(x_0, \xi_0) \in M_{\tau_0}$ be fixed and consider a local Kähler normal coordinate chart around ζ_0 [20]. In such a chart, the Kähler potential satisfies $\rho(\zeta, \overline{\zeta}) = |\Im(\zeta - \zeta_0)|^2 + O(|\Im(\zeta - \zeta_0)|^4)$, so that $\partial\bar{\partial}\rho = g_0 + O(|\Im(\zeta - \zeta_0)|^2)$, where g_0 is the standard Euclidean

Hermitian metric. We denote the unit ball centered at ζ_0 in this local Euclidean metric by $B(\zeta_0, 1)$.

The local holomorphic dilation of $B(\zeta_0, 1)$ in Kähler normal coordinates ζ centered at $\zeta_0 \in M_{\tau_0} \setminus M$ is defined by

$$D^{\zeta_0}_{\varepsilon(\lambda)} \colon B(\zeta_0, 1) \to B(\zeta_0, \varepsilon(\lambda)), \qquad \zeta \mapsto \zeta_0 + \varepsilon(\lambda)(\zeta - \zeta_0). \tag{5.16}$$

This choice of local dilation is not adapted to Grauert tube geometry in that sense that the ε-dilate of a point in ∂M_τ is not necessarily a point in $\partial M_{\varepsilon\tau}$. But since the metric and tube function are almost Euclidean in shrinking balls one has constants $c_g, C_g > 0$ so that

$$c_g \varepsilon(\lambda) \sqrt{\rho}(\zeta) \leq \sqrt{\rho}(D^{\zeta_0}_{\varepsilon(\lambda)} \zeta) \leq C_g \varepsilon(\lambda) \sqrt{\rho}(\zeta)$$

provided $\sqrt{\rho}(\zeta)$ is small enough. Indeed, it suffices to verify the inequalities for the Euclidean metric, where $\sqrt{\rho}(\zeta) = |\Im \zeta|$ and where $C_g = c_g = 1$.

5.3.2. Phase space dilation

Theorem 5.4.3 introduces another type of dilation, which is more conveniently expressed in terms of the usual cotangent coordinates (x, ξ). The dilation in local coordinates centered at $(x_0, \xi_0) \in \partial B^*_\tau M$ is of the form

$$(x, \xi) \mapsto \left(x_0 + \frac{x - x_0}{\varepsilon(\lambda)}, \xi_0 + \frac{\tau\hat{\xi} - \xi_0}{\varepsilon(\lambda)} \right), \qquad (x_0, \xi_0) \in \partial B^*_\tau M. \tag{5.17}$$

Note that the unit vector $\hat{\xi} := \xi/|\xi|$ is scaled by the parameter $\tau = |\xi_0|_{x_0}$, with (x_0, ξ_0) the fixed center of dilation.

This is closely related to, but not identical to, the dilation introduced in [24]. In that article one fixes a point $(x_0, \xi_0) \in S^*M = \partial B_1^* M$ in the unit co-sphere bundle and dilates by

$$(x, \xi) \mapsto \left(x_0 + \frac{x - x_0}{\varepsilon(\lambda)}, \xi_0 + \frac{\hat{\xi} - \xi_0}{\varepsilon(\lambda)} \right), \qquad (x_0, \xi_0) \in S^*M.$$

Both types of dilation are homogeneous in ξ. The one essential difference is that in (5.17), we allow $|\xi_0|_{x_0} = \tau$ and $\tau\hat{\xi}$ to be any positive numbers bounded away from zero; they need not be the same. Thus, we are not only localizing in the direction of co-vectors but also in their norms.

5.4. Poisson Conjugation of Semiclassical Toeplitz Operators to Semiclassical Pseudodifferential Operators

In this section, we generalize the conjugation result of Lemma 5.2.3 in two ways. On one hand, we let the symbol depend on the frequency λ, similar to the $\delta(h)$-(micro)localized symbols (5.26) in the Riemannian setting. On the other hand, we consider Bergman-Toeplitz operators, realized as direct integrals of Szegő-Toeplitz operators. We show that conjugation by the FBI transform takes a decomposable, log-scale Bergman-Toeplitz operator to a semiclassical pseudodifferential operator with a log-scale symbol.

It is convenient to introduce the semiclassical parameter

$$h := \lambda^{-1}, \quad h^{-2} E_j = \lambda_j^2, \quad \delta(h) := |\log h|^{-\alpha} = (\log \lambda)^{-\alpha} = \varepsilon(\lambda). \tag{5.18}$$

In this semiclassical notation, the Laplacian eigenfunctions satisfy $\Delta\varphi_j = h^{-2}E_j\varphi_j = \lambda_j^2\varphi_j$.

5.4.1. Semiclassical Poisson-wave operator

The Poisson kernel (5.12) may be realized as a semiclassical Fourier integral operator with the introduction of a semiclassical parameter h. In the Euclidean case, we define the semiclassical Poisson kernel to be

$$P_h^\tau(x,y) = h^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} e^{-\tau|\xi|/h}\,d\xi.$$

Here, we use the semiclassical Fourier transform

$$\mathcal{F}_h u(y) = h^{-n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle y,\xi\rangle} f(y)\,dy,$$

to diagonalize $P^\tau = e^{-\tau\sqrt{-\Delta}}$. It is evident that $P_h^\tau = P^\tau$ by changing variables $\xi \to \xi/h$. Indeed,

$$P_h^\tau e^{i\langle x,k\rangle/h} = e^{-\tau|k|/h} e^{i\langle x,k\rangle/h}.$$

Thus P_h^τ is still the homogeneous Poisson operator $e^{-\tau\sqrt{-\Delta}}$.

The same change of variables is valid in the manifold setting (5.11) and we continue to denote the Poisson operator in semiclassical form by P_h^τ. The semiclassical version of the zeroth order unitary operator V^τ from Proposition 5.2.2 is denoted

$$V_h^\tau := P_h^\tau (P_h^{\tau *} P_h^\tau)^{-\frac{1}{2}} \colon L^2(M) \to \mathcal{O}^0(\partial M_\tau).$$

5.4.2. Log-scale symbols and semiclassical pseudodifferential operators

Let $0 \le a \le 1$ be a smooth cutoff function that is equal to 1 on $B(0,1) \subset \mathbb{C}^n$ and vanishes outside $B(0,2) \subset \mathbb{C}^n$. We use (5.8) to identify M_{τ_0} with $B^*_{\tau_0}M$. Using local coordinates induced by $\exp^{\mathbb{C}}_{x_0} \colon T^*_{x_0}M \otimes \mathbb{C} \to M_\tau$, consider symbols that, near $(x_0,\xi_0) \in \partial B^*_\tau M$, are locally of the form

$$a^{(x_0,\xi_0)}_{\delta(h)}(x,\xi) := a\left(x_0 + \frac{x-x_0}{\delta(h)}, \xi_0 + \frac{\xi-\xi_0}{\delta(h)}\right). \tag{5.19}$$

Symbols of the type (5.19) satisfy the estimate

$$|D^\beta a^{(x_0,\xi_0)}_{\delta(h)}| \le C_\beta \delta(h)^{-|\beta|}, \tag{5.20}$$

and are said to belong to the symbol classes $S^0_{\delta(h)}$. More generally, a function $b \in C^\infty(T^*M)$ belongs to the symbol class $S^k_{\delta(h)}$ if

$$\sup_{(x,\xi)\in T^*M} |\partial_x^\beta \partial_\xi^\gamma b| \le C_{\beta,\gamma} \delta(h)^{-|\beta|-|\gamma|}(1+|\xi|_x^2)^{(k-|\beta|)/2} \tag{5.21}$$

for some constant $C_{\beta,\gamma}$ independent of h.

The semiclassical pseudodifferential operator quantizing a symbol a is defined by the usual local (semiclassical) Fourier transform formula

$$\mathrm{Op}_h(a)(x,y) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle \xi, x-y\rangle} a(x,\xi,h)\, d\xi.$$

The quantization of a symbol $b \in S^k_{\delta(h)}$ is denoted by $\mathrm{Op}_h(b) \in \Psi^k_{\delta(h)}$. We refer to [24] for a discussion of the symbol classes $S^k_{\delta(h)}$ and [75] for symbol classes and quantizations in general.

5.4.3. Semiclassical Poisson conjugation of log-scale Toeplitz operators

THEOREM **5.4.1.** *Let $(x_0,\xi_0)\in\partial B^*_\tau M$ be fixed. For symbols $a^{(x_0,\xi_0)}_{\delta(h)}\in C^\infty(M_{\tau_0})$ of the form (5.19), we have*

(5.22)

$$P_h^{\tau *}\Pi_\tau a^{(x_0,\xi_0)}_{\delta(h)}\Pi_\tau P_h^\tau = \mathrm{Op}_h\left(h^{\frac{n-1}{2}}|\xi|^{-\frac{n-1}{2}}a\left(x_0+\frac{x-x_0}{\delta(h)},\xi_0+\frac{\tau\hat{\xi}-\xi_0}{\delta(h)}\right)\right)\in\Psi^{-\frac{n-1}{2}}_{\delta(h)}(M)$$

modulo $h\delta(h)^{-2}\Psi^{-\frac{n-1}{2}}_{\delta(h)}(M)$ and

$$V_h^{\tau *}\Pi_\tau a^{(x_0,\xi_0)}_{\delta(h)}\Pi_\tau V_h^\tau = \mathrm{Op}_h\left(a\left(x_0+\frac{x-x_0}{\delta(h)},\xi_0+\frac{\tau\hat{\xi}-\xi_0}{\delta(h)}\right)\right)\in\Psi^{0}_{\delta(h)}(M)$$

modulo $h\delta(h)^{-2}\Psi^0_{\delta(h)}(M)$. Note that the τ-scaling affects only $\hat{\xi}:=\xi/|\xi|$.

REMARK **5.4.2.** *Note that the factors of Π_τ are redundant because P^τ maps into the range of Π_τ. We prove only (5.22) as the second conjugation statement may be proved using the first statement and the composition rule for pseudodifferential operators.*

PROOF OF THEOREM 5.4.1. The proof is essentially the same as in Lemma 5.2.3, since the dilation has no effect on the properties of the conjugation. Indeed, conjugation by the Fourier integral operator P_h^τ preserves the symbol class $S^*_{\delta(h)}$. Since $a^{(x_0,\xi_0)}_{\delta(h)}$ is a function on ∂M_τ, it defines a *homogeneous symbol* of order zero on Σ_τ in the fiber direction. Under conjugation by P_h^τ it goes over to a pseudodifferential operator of order zero on M whose symbol is the transport $a^{(x_0,\xi_0)}_{\delta(h)}(\iota_\tau(x,\xi))$ to $T^*M\setminus 0_M$, with ι_τ given by (5.10). If $\pi_\tau\colon\Sigma_\tau\to\partial M_\tau$ is the natural projection then

$$\iota_\tau^* a^{(x_0,\xi_0)}_{\delta(h)}(x,\xi)=a^{(x_0,\xi_0)}_{\delta(h)}(E(x,\tau\hat{\xi})),\qquad \hat{\xi}=\frac{\xi}{|\xi|}.$$

For $\tau, \delta(h)$ small enough we may use the Euclidean approximation to the distance function. If we center the local coordinates at (x_0, ξ_0) then the cutoff as a function on T^*M has the form

$$a_{\delta(h)}^{(x_0,\xi_0)}(\iota_\tau(x,\xi)) = a\left(x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau\hat{\xi} - \xi_0}{\delta(h)}\right), \qquad \hat{\xi} = \frac{\xi}{|\xi|}. \tag{5.23}$$

Thus, $P_h^{\tau*}\Pi_\tau a_{\delta(h)}^{(x_0,\xi_0)}\Pi_\tau P_h^\tau$ is a homogeneous pseudodifferential operator with dilated symbol.

We now provide more details. Since the calculation is local we first provide a proof in the Euclidean case.

5.4.3.1. Euclidean case. Write $Z = x_1 + i\tau p$ with $|p| = 1$ and centering the dilation at $Z_0 = x_0 + i\xi_0$. We do not assume $\tau = |\xi_0|$. The composition has the form

$$\begin{aligned} &P_h^{\tau*}\Pi_\tau a_{\delta(h)}^{(x_0,\xi_0)}\Pi_\tau P_h^\tau(x,y) \\ &\qquad = h^{-2n}\tau^{n-1}\int_{\mathbb{R}^n\times\mathbb{R}^n\times S^{n-1}\times\mathbb{R}^n} e^{\Psi_0/h} a\left(x_0 + \frac{x_1 - x_0}{\delta(h)}, \xi_0 + \frac{\tau p - \xi_0}{\delta(h)}\right) d\xi_1 d\xi_2 d\sigma(p) dx_1, \end{aligned}$$

where $d\sigma(p)$ is the standard surface area measure on S^{n-1}. The phase is

$$\Psi_0(\xi_1, \xi_2, x_1, p; x, y, \tau) = -\tau(|\xi_1| + |\xi_2|) + i\langle \xi_1, x_1 + i\tau p - y\rangle - i\langle \xi_2, x - (x_1 - i\tau p)\rangle$$

We note that

$$\Re\Psi_0 = -\tau(|\xi_1| + |\xi_2|) - \tau\langle \xi_1 - \xi_2, p\rangle \leq 0$$

with equality if and only if $\hat{\xi}_1 = -\hat{\xi}_2 = \pm p$, that is, the Schwartz kernel integral is of smooth and of order $O(h^\infty)$. We absorb the factor apply the complex stationary phase method to the $dx_1 d\xi_2 d\sigma(p)$ integral. The critical point equations for $\Im\Psi$ in (x_1, ξ_2) are

$$\begin{cases} d_{x_1}\Im\Psi_0 = 0 \iff \xi_1 = -\xi_2, \\ \\ d_{\xi_2}\Im\Psi_0 = 0 \iff x_1 = x \end{cases}$$

The extra dp integral localizes at the above point. Since the $dx_1 d\xi_2$ integral has a non-degenerate Hessian, we may eliminate the $dx_1 d\xi_2$ integrals by stationary phase, obtaining a simpler oscillatory integral

$$h^{-2n+n}\tau^{n-1}\int_{\mathbb{R}^n\times S^{n-1}} e^{\Psi_1/h} a\left(x_0 + \frac{x-x_0}{\delta(h)}, \xi_0 + \frac{\tau p - \xi_0}{\delta(h)}\right) d\xi_1 d\sigma(p),$$

with

$$\Psi_1(\xi_1, p; x, y, \tau) = -2\tau|\xi_1| - 2\tau\langle\xi_1, p\rangle + i\langle\xi_1, x-y\rangle.$$

Applying the method of stationary phase (steepest descent) to the integral over S^{n-1} gives the critical point equation $p = -\hat{\xi}_1$, i.e., the point where the phase is maximal. It follows that

$$\begin{aligned} &P_h^{\tau *}\Pi_\tau a_{\delta(h)}^{(x_0,\xi_0)}\Pi_\tau P_h^\tau(x,y) \\ &\qquad = h^{-2n+n+\frac{n-1}{2}}\tau^{n-1-\frac{n-1}{2}}\int_{\mathbb{R}^n} e^{i\langle\xi_1, x-y\rangle/h} a\left(x_0 + \frac{x-x_0}{\delta(h)}, \xi_0 + \frac{\tau\hat{\xi}_1 - \xi_0}{\delta(h)}\right) d\xi_1 \end{aligned}$$

modulo terms of order $h\delta(h)^{-2}$ (since each derivative of the symbol pulls out a factor of $\delta(h)^{-1}$).

5.4.3.2. General Riemannian manifold. The proof is similar on any real analytic Riemannian manifold. In place of the integral over $\mathbb{R}^n \times S^{n-1}$ we now have an integral over $Z \in \partial M_\tau$ or $(x_1, sp) \in \partial B^*_\tau M$ with $|p| = 1$ under the map $Z = E(x_1, sp)$. Using the parametrix (5.12), we have

$$\begin{aligned} &P_h^{\tau *} \Pi_\tau a_{\delta(h)}^{(x_0,\xi_0)} \Pi_\tau P_h^\tau (x,y) \\ &\qquad = h^{-2n} \tau^{n-1} \int_{T_x^* M \times T_y^* M \times \partial M_\tau} e^{\Psi/h} a\left(x_0 + \frac{x_1 - x_0}{\delta(h)}, \xi_0 + \frac{sp - \xi_0}{\delta(h)} \right) A \overline{A}\, d\xi_1 d\xi_2 d\mu_\tau(Z) \end{aligned}$$

with

$$\Psi = -\tau \left(|\xi_1|_x + |\xi_2|_y \right) + i \langle \xi_1, (\exp_y^{\mathbb{C}})^{-1}(Z) \rangle - i \langle \xi_2, (\exp_x^{\mathbb{C}})^{-1}(\bar{Z}) \rangle.$$

The phase is only well-defined when Z is sufficiently close to x and to y, but the phase is non-stationary and the integral is exponentially decaying otherwise. The only points for which the integral is not exponentially decaying are those Z satisfying $\Im \langle \xi_1, (\exp_y^{\mathbb{C}})^{-1}(Z) \rangle = \tau |\xi_1|$ (and a similar condition holds with y replaced by x and ξ_1 replaced by ξ_2). Note that $(\exp_x^{\mathbb{C}})^{-1}(Z) \in U_x \subset T_x^* M \otimes \mathbb{C}$.

The critical set C_Ψ of the phase is defined by

$$C_\Psi = \{ (x, y, \tau; \xi_1, \xi_2, Z) : d_{\xi_1, \xi_2, Z} \Psi = 0 \}.$$

The associated canonical relation is defined by the embedding

$$\iota_\Psi \colon C_\Psi \to T^*M \times T^*M, \quad (x, y, \tau; \xi_1, \xi_2, Z) \to (x, d_x \Psi, y, -d_y \Psi). \tag{5.24}$$

The composite operator is manifestly a Fourier integral operator with complex phase, and is a pseudodifferential operator if and only if $C_\Psi = \Delta_{T^*M\times T^*M}$ (the diagonal).

Let $Z = E(x_1, \tau p)$. Then the critical point equations are

(i) $d_{\xi_1}\Psi = 0 \iff (\exp_y^{\mathbb{C}})^{-1}(Z) = -i\tau\hat{\xi}_1 \iff x_1 = y,\ p = -i\tau\hat{\xi}_2,$

(ii) $d_Z\Psi = d_Z\left(\langle(\exp_y^{\mathbb{C}})^{-1}(Z), \xi_2\rangle - (\exp_x^{\mathbb{C}})^{-1}(\bar{Z}), \xi_1\rangle\right) = 0,$

(iii) $d_{\xi_2}\Psi = 0 \iff (\exp_x^{\mathbb{C}})^{-1}(\bar{Z}) = -i\tau\hat{\xi}_2.$

Equations (i) and (iii) show that

$$Z = \exp_x^{\mathbb{C}}(i\tau\hat{\xi}_2) = \exp_y^{\mathbb{C}}(-i\tau\hat{\xi}_1).$$

This implies that $Z \in \pi_\tau^{-1}(x) \cap \pi_\tau^{-1}(y)$, where $\pi_\tau\colon \partial M_\tau \to M$. Of course, these fibers are disjoint unless $x = y$, so only in that case does there exist a solution of the critical point equation. It then follows that $\hat{\xi}_1 = -\hat{\xi}_2$.

To see that $\xi_1 = -\xi_2$ on the critical point set, we use further use (ii). There only exists a solution of the critical point equations when $x = y$, and then we may write $Z = u + iv \in T_x^*M \otimes \mathbb{C}$ and study the restricted critical point equation

$$d_Z\Psi = 0 \iff d_{u,v}\left(\langle u + iv, \xi_2\rangle - \langle u + iv, \xi_1\rangle\right) = 0.$$

Just using $u \in T_x^*M$ already shows that $\xi_1 = \xi_2$ on the critical set.

To calculate (5.24) we may use the Euclidean approximation to the phase based at (x, ξ_1) because on C_Ψ only the first order terms in the Taylor expansion of Ψ contribute. But then it is evident that $d_x\Psi = \xi_2 = -d_y\Psi|_{y=x} = \xi_1$, proving that the canonical relation is the diagonal.

The principal symbol of $P_h^{\tau *}\Pi_\tau P_h^\tau(x,y)$ is calculated in [71] and the principal symbol of $P_h^{\tau *}\Pi_\tau a_{\delta(h)}^{(x_0,\xi_0)}\Pi_\tau P_h^\tau(x,y)$ is the same multiplied by the value of $a_{\delta(h)}^{(x_0,\xi_0)}$ at the critical point. Note that because of the symbol class we are working with, the sub-leading term is of order $h\delta(h)^{-2}$ as each derivative of the symbol pulls out a factor of $\delta(h)^{-1}$. If we use V_h^τ in place of P_h^τ as in Proposition 5.2.2 then the principal symbol is the one stated in Theorem 5.4.1. $\square$

5.4.4. Comparison of symbols

We note that symbols of the form (5.23) are not quite the same as the log-scaled symbols $a_{z_0}^b(x,\xi;h)$ of (5.26) considered in [24]. However, as long as (x_0,ξ_0) are fixed at a positive distance from the real domain M, the same symbol estimates (5.20) are valid. Also note that it is not necessary to multiply by a cutoff $\varphi(|\xi|)$ to S^*M since the cutoff $a_{z_0}^b(x,\xi;h)$ is supported in a shrinking Kähler ball around $E(x_0,\xi_0)$. In fact, we define the sequence h_j so that eigenfunctions concentrate on the energy surface ∂M_{τ_0} with $|\xi_0|_{x_0} = \tau_0$. There is no difficulty as long as $\tau_0 > 0$. We continue to use the notation $\mathrm{Op}_h(a)$ for semiclassical pseudodifferential operators with symbols of the form (5.23).

5.4.5. Decomposable Poisson–FBI transform and Bergman–Toeplitz operators

In this section we introduce a Poisson FBI transform taking $L^2(M)$ to a weighted Hilbert space of holomorphic functions on M_τ rather than to CR-holomorphic functions on ∂M_τ. As explained in Section 5.4.6, it is defined in a novel way by a direct integral of Poisson transforms P^s, and therefore all of its main properties flow from those established above for the Poisson kernel. The main result is the conjugation Theorem 5.4.3.

5.4.6. Weighted Bergman space and Poisson–FBI transform

The Poisson kernel endows $\mathcal{O}^0(M_\tau)$ with a plurisubharmonic weight $e^{-\sqrt{\rho}/h}$. We define

$$A^2(M_\tau, h^{-\frac{n-1}{2}} e^{-2\sqrt{\rho}/h} d\mu)$$

to be the Hilbert space of holomorphic functions on M_τ that lie in $L^2(M_\tau, e^{-2\sqrt{\rho}/h} d\mu)$. It is isometric to the Hilbert space

$$H_{\sqrt{\rho}} := \{f h^{-\frac{m-1}{4}} e^{-\sqrt{\rho}/h} : f \in A^2(M_\tau\} \subset L^2(M_\tau, d\mu)$$

endowed with the inner product of $L^2(M_\tau, d\mu)$.

It is useful to regard $H_{\sqrt{\rho}}$ as a direct integral

$$H_{\sqrt{\rho}} = \int_{[0,\tau_0]}^{\oplus} H^2(\partial M_\tau)\, d\tau$$

of Hilbert spaces $H^2(\partial M_\tau)$. Here, $\int_{[0,\tau_0]}^{\oplus} H^2(\partial M_\tau)\, d\tau$ denotes the space of L^2 sections $f(\tau) \in H^2(\partial M_\tau)$ of the Hilbert bundle, and the direct integral formula follows from Fubini's theorem,

$$\|f\|^2 = \int_0^{\tau_0} \left(\int_{\partial M_\tau} |f(Z)|^2\, d\mu_\tau(Z) \right) d\tau.$$

We then define the 'moving Poisson operator' or FBI transform by

$$T_h f(\zeta) = P^{\sqrt{\rho}(\zeta)} f(\zeta) = \int_M P^{\sqrt{\rho}(\zeta)}(\zeta, y) f(y)\, dV(y), \qquad \zeta \in M_{\tau_0}.$$

We claim that $T_h\colon L^2(M) \to H_{\sqrt{\rho}}$ is a unitary operator. To see this, we use that P^τ is unitary from $L^2(M)$ to each integrand, and observe that

$$T_h = \int_{[0,\tau_0]}^{\oplus} P_h^\tau \, d\tau$$

is the direct integral of a family of unitary operators index by τ.

5.4.7. FBI conjugation theorem

Next we define Bergman–Toeplitz operators. For $a \in C^\infty(M_{\tau_0})$ define

$$\widetilde{\mathrm{Op}}_h(a) = \int_{[0,\tau_0]}^{\oplus} \Pi_\tau (a|_{\partial M_\tau}) \Pi_\tau \, d\tau.$$

Implicitly $H^2(\partial M_\tau) \perp H^2(\partial M_\sigma)$ if $\tau \neq \sigma$. This is a decomposable operator.

THEOREM **5.4.3.** *For symbols $a_{\delta(h)}^{(x_0,\xi_0)} \in C^\infty(M_{\tau_0})$ of the form (5.19), we have*

$$\begin{aligned} (5.25)\quad & T_h^* \widetilde{\mathrm{Op}}_h(a_{\delta(h)}^{(x_0,\xi_0)}) T_h \\ &= \mathrm{Op}_h \left(\int_0^{\tau_0} h^{\frac{n-1}{2}} |\xi|^{-\frac{n-1}{2}} a\left(x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau\hat{\xi} - \xi_0}{\delta(h)} \right) d\tau \right) \in \Psi_{\delta(h)}^{-\frac{n-1}{2}}(M). \end{aligned}$$

Note that (5.25) follows from (5.22) thanks to the identity

$$T_h^* \widetilde{\mathrm{Op}}_h(a) T_h = \int_0^{\tau_0} P_h^{\tau *} \widetilde{\mathrm{Op}}_h(a) P_h^\tau \, d\tau.$$

Indeed, a multiplication operator is automatically decomposable and the Schwartz kernel is

$$\int_{M_{\tau_0}} P_h^*(x,\zeta)a(\zeta)P_h(\zeta,y)\,d\mu(\zeta) = \int_0^{\tau_0}\left(\int_{\partial M_\tau} P_h^{\tau *}(x,Z)a(Z)P_h^\tau(Z,y)\,d\mu_\tau(Z)\right)d\tau.$$

By Theorem 5.4.1, each integrand of the $d\mu_\tau(Z)$ integral in the expression above is a semiclassical pseudodifferential operator by (5.22). The entire $d\tau$ integral is therefore an integral of an analytic family (in τ) of semiclassical pseudodifferential operators on M with the prescribed principal symbol.

5.5. Log-scale Quantum Ergodicity in the Real Domain

A key part of our analysis is to relate log-scale quantum variance estimates in the complex domain to those in the real domain, and reduce variance estimates to the small-scale quantum ergodicity results on negatively curved Riemannian manifolds due to Hezari–Rivière [25] and Han [24]. We briefly review their results in preparation for the next section.

As before, let $\delta(h) = |\log h|^{-\alpha}$, with the semiclassical parameter given by (5.18). Consider compactly supported smooth functions that, near $z_0 = (x_0, \xi_0) \in S^*M$, can be locally expressed as

$$a_{z_0}^b(x,\xi;h) := b\left(x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\hat{\xi} - \hat{\xi}_0}{\delta(h)}\right)\varphi(|\xi|_x) \in S^0_{\delta(h)}, \tag{5.26}$$

where $b \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ is some compactly supported smooth function and where $\varphi \in C_c^\infty((1-1/2, 1+1/2))$ is a smooth cutoff function that is identically 1 on $(1-1/4, 1+1/4)$.[2]

[2]There is a misprint in [24] where the support is said to be $(-\frac{1}{2}, \frac{1}{2})$ around the zero section 0_M. In fact, it needs to be around S^*M.

It is easy to see that such a function belongs to the symbol class $S^0_{\delta(h)}$ by verifying the symbol estimate (5.21). The following result pertains to $\delta(h)$-microlocalized symbols (5.26).

THEOREM **5.5.1** (Han [**24**, Theorem 1.6]). *Let (M^n, g) be negatively curved (not necessarily real analytic). Let*

$$0 < \alpha < \frac{1}{2(2n-1)}, \; 0 \leq \beta < 1 - 2\alpha(2n-1) \quad \text{or} \quad \alpha = 0, \; \beta = 1.$$

Set $\delta(h) = |\log h|^{-\alpha}$. Then for any orthonormal basis $\{\varphi_j\}$ of $h^2\Delta$, we have

$$h^{n-1} \sum_{E_j \in [1,1+h]} \left| \langle \mathrm{Op}_h(a^b_{z_0})\varphi_j, \varphi_j \rangle - \fint_{S^*M} a^b_{z_0} \, d\mu_L \right|^2 = \mathcal{O}(\delta(h)^{2(2n-1)} |\log h|^{-\beta}).$$

Here, Op_h is a suitable semiclassical quantization, and $d\mu_L$ is the Liouville measure.

A covering argument using balls of inverse logarithmic radii implies the next volume comparison result.

THEOREM **5.5.2** ([**24**, Corollary 1.9]; see also [**25**, Lemma 3.1]). *Let (M^n, g) be negatively curved (not necessarily real analytic). Let*

$$0 < \alpha < \frac{1}{3n} \quad \text{and} \quad r(\lambda) = (\log \lambda)^{-\alpha}.$$

Ten, there exists a full density subsequence such that

$$c\mathrm{Vol}(B(x, r_{j_k})) \leq \int_{B(x, r_{j_k})} |\varphi_{j_k}|^2 \, dV \leq C\mathrm{Vol}(B(x, r_{j_k}))$$

uniformly for all $x \in M$, where $c, C > 0$ depends only on (M, g).

Remark **5.5.3.** *An important technical point for this article is that the proofs of the theorems hold for symbols in $S^0_{\delta(h)}$; the precise form of $a^b_{z_0}$ is not relevant.*

5.6. Proof of Theorem 5.1.4: Log-scale QE in Grauert Tubes

We introduce some notation. Let

$$\Theta_j(\zeta) := \left\| \varphi_j^{\mathbb{C}} \mid_{\partial M_{\sqrt{\rho}(\zeta)}} \right\|_{L^2(\partial M_{\sqrt{\rho}(\zeta)})}$$

denote the L^2-norm of $\varphi_j^{\mathbb{C}}$ restricted to the boundary of the Grauert tube of radius $\sqrt{\rho}(\zeta)$. Let

$$U_j(\zeta) := \frac{\varphi_j^{\mathbb{C}}(\zeta)}{\Theta_j(\zeta)}$$

denote the normalized complexified eigenfunction. We will also consider its restriction to ∂M_τ for each $0 < \tau \leq \tau_0$ fixed:

$$u_j^\tau(Z) := U_j(Z) \mid_{\partial M_\tau} = \frac{\varphi_j^{\mathbb{C}}(Z) \mid_{\partial M_\tau}}{\left\| \varphi_j^{\mathbb{C}} \mid_{\partial M_\tau} \right\|_{L^2(\partial M_\tau)}}, \qquad (Z \in \partial M_\tau). \tag{5.27}$$

Note that the denominator in (5.27) is a constant (depending on τ), and the numerator is a CR-holomorphic function on ∂M_τ.

5.6.1. Variance estimates in Grauert tubes

We begin with a log-scale variance estimate for symbols on ∂M_τ, which parallels [11, Theorem 4]. Using the E map (5.8) to identify $B^*_{\tau_0}M$ with M_{τ_0}, we henceforth write

$$a^{\zeta_0}_{\delta(h)} := a^{(x_0,\xi_0)}_{\delta(h)} \in C^\infty(M_{\tau_0}), \qquad \zeta_0 = E(x_0,\xi_0)$$

for small-scale symbols of the form (5.19). We write Z in place of ζ when restricting to the boundary ∂M_τ, so for instance

$$a^{\zeta_0}_{\delta(h)}(\zeta)\mid_{\partial M_\tau} = a^{\zeta_0}_{\delta(h)}(Z), \qquad Z \in \partial M_\tau.$$

Proposition **5.6.1.** *Let (M^n,g) be negatively curved and real analytic. Let*

$$0 < \alpha < \frac{1}{2(2n-1)},\ 0 \le \beta < 1 - 2\alpha(2n-1) \quad \text{or} \quad \alpha = 0,\ \beta = 1.$$

Set $\delta(h) = |\log\delta|^{-\alpha}$ as in (5.18). Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions for Δ. Then for every $0 < \tau \le \tau_0$ and every $\zeta_0 \in M_\tau \backslash M$, we have

$$h^{n-1} \sum_{E_j \in [1,1+h]} \left| \int_{\partial M_\tau} a^{\zeta_0}_{\delta(h)}(Z) |u^\tau_j(Z)|^2 \, d\mu_\tau(Z) - \frac{1}{\mu_\tau(\partial M_\tau)} \int_{\partial M_\tau} a^{\zeta_0}_{\delta(h)}(Z)\, d\mu_\tau \right|^2$$
$$= \mathcal{O}(\delta(h)^{2(2n-1)} |\log h|^{-\beta}).$$

The remainder is uniform for any ζ_0 in an 'annulus' $0 < \tau_1 \le \sqrt{\rho}(\zeta_0) \le \tau_0$.

Proof of Proposition 5.6.1. We use Theorem 5.4.1 to transport matrix elements on ∂M_τ to matrix elements of pseudodifferential operators on $L^2(M)$. Since the restriction

$\varphi_h^{\mathbb{C}}(Z)$ to ∂M_τ is a CR-holomorphic function, it satisfies $\Pi_\tau \varphi_j^{\mathbb{C}}(Z) = \varphi_j^{\mathbb{C}}(Z)$. Moreover, $e^{-2\sqrt{\rho}(Z)/h} = e^{-2\tau/h}$ on ∂M_τ. Therefore,

$$\begin{aligned}
\int_{\partial M_\tau} a_{\delta(h)}^{\zeta_0}(Z)|u_j^\tau(Z)|^2\, d\mu_\tau(Z) &= \|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\tau)}^{-2} \left\langle a_{\delta(h)}^{\zeta_0} \Pi_\tau \varphi_j^{\mathbb{C}}, \Pi_\tau \varphi_j^{\mathbb{C}} \right\rangle_{L^2(\partial M_\tau)} \\
&= e^{2\tau/h}\|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\tau)}^{-2} \left\langle a_{\delta(h)}^{\zeta_0} \Pi_\tau P_h^\tau \varphi_j, \Pi_\tau P_h^\tau \varphi_j \right\rangle_{L^2(M)} \\
&= \frac{\langle P_h^{\tau *} \Pi_\tau a_{\delta(h)}^{\zeta_0} \Pi_\tau P_h^\tau \varphi_j, \varphi_j \rangle_{L^2(M)}}{\langle P_h^{\tau *} \Pi_\tau P_h^\tau \varphi_j, \varphi_j \rangle_{L^2(M)}}. \qquad (5.28)
\end{aligned}$$

The last equality follows from setting $a_{\delta(h)}^{\zeta_0} \equiv 1$, which implies

$$1 = e^{2\tau/h}\|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\tau)}^{-2} \langle P_h^{\tau *} \Pi_\tau P_h^\tau \varphi_j, \varphi_j \rangle_{L^2(M)}.$$

By Theorem 5.4.1, $P_h^{\tau *} \Pi_\tau a_{\delta(h)}^{\zeta_0} \Pi_\tau P_h^\tau$ is an h-pseudodifferential operator with principal symbol

$$h^{\frac{n-1}{2}} |\xi|^{-\frac{n-1}{2}} a\bigg(x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau\hat{\xi} - \xi_0}{\delta(h)} \bigg).$$

By taking $a_{\delta(h)}^{\zeta_0} \equiv 1$ in Theorem 5.4.1, the denominator $P_h^{\tau *} \Pi_\tau P_h^\tau = P_h^{\tau *} P_h^\tau$ is found to be an h-pseudodifferential operator with principal symbol $h^{\frac{n-1}{2}}|\xi|^{-\frac{n-1}{2}}$. The quotient (5.28) may be rewritten using Theorem 5.4.1:

$$\begin{aligned}
&\int_{\partial M_\tau} a_{\delta(h)}^{\zeta_0}(Z)|u_j^\tau(Z)|^2\, d\mu_\tau(Z) \\
&\qquad = \frac{\left\langle \mathrm{Op}_h \left(h^{\frac{n-1}{2}} |\xi|^{-\frac{n-1}{2}} a \left(x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau\hat{\xi} - \xi_0}{\delta(h)} \right) \right) \varphi_j, \varphi_j \right\rangle_{L^2(M)} + \mathcal{O}(h\delta(h)^{-2})}{\left\langle \mathrm{Op}_h \left(h^{\frac{n-1}{2}} |\xi|^{-\frac{n-1}{2}} \right) \varphi_j, \varphi_j \right\rangle_{L^2(M)} + \mathcal{O}(h\delta(h)^{-2})}
\end{aligned}$$

$$= \left\langle \mathrm{Op}_h \left(a\left(x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau\hat{\xi} - \xi_0}{\delta(h)} \right) \right) \varphi_j, \varphi_j \right\rangle_{L^2(M)} + \mathcal{O}(h\delta(h)^{-2})$$

$$= \left\langle V_h^{\tau *} \Pi_\tau a_{\delta(h)}^{\zeta_0} \Pi_\tau V_h^\tau \varphi_j, \varphi_j \right\rangle_{L^2(M)} + \mathcal{O}(h\delta(h)^{-2}). \tag{5.29}$$

As noted in Remark 5.5.3, Theorem 5.5.1 applies to symbols in the symbol class $S^0_{\delta(h)}$. But $V_h^{\tau *} \Pi_\tau a_{\delta(h)}^{\zeta_0} \Pi_\tau V_h^\tau \in \Psi^0_{\delta(h)}(M)$, so the proof is complete. □

Proposition **5.6.2.** *With the same notation and assumptions as in Proposition 5.6.1: For every $\zeta_0 \in M_\tau \setminus M$ and $a_{\delta(h)}^{\zeta_0}$, we have*

$$h^{n-1} \sum_{E_j \in [1, 1+h]} \left| \int_{M_{\tau_0}} a_{\delta(h)}^{\zeta_0}(\zeta) |U_j(\zeta)|^2 \, d\mu(\zeta) - \int_0^{\tau_0} \int_{\partial M_\tau} \frac{a_{\delta(h)}^{\zeta_0}(Z)}{\mu_\tau(\partial M_\tau)} \, d\mu_\tau(Z) d\tau \right|^2$$
$$= \mathcal{O}(\delta(h)^{4n} |\log h|^{-\beta}).$$

The remainder is uniform for any ζ_0 in an 'annulus' $0 < \tau_1 \leq \sqrt{\rho}(\zeta_0) \leq \tau_0$.

Proof of Proposition 5.6.2. Rewrite the integral over M_{τ_0} as an iterated integral:

$$\int_{M_{\tau_0}} a_{\delta(h)}^{\zeta_0}(\zeta) |U_j(\zeta)|^2 \, d\mu(\zeta) = \int_0^{\tau_0} \int_{\partial M_\tau} a_{\delta(h)}^{\zeta_0}(Z) |u_j^\tau(Z)|^2 \, d\mu_\tau(Z) d\tau.$$

We make two observations. First, for the outer integral it suffices to integrate over $\tau \in [\sqrt{\rho}(\zeta_0) - 2\delta(h), \sqrt{\rho}(\zeta_0) + 2\delta(h)]$ thanks to the choice (5.19) of symbols. Second, the inner integral may be replaced by matrix elements of $V_h^{\tau *} \Pi_\tau a_{\delta(h)}^{\zeta_0} \Pi_\tau V_h^\tau$ at the cost of $\mathcal{O}(h\delta(h)^{-2})$

in light of (5.29):

$$\int_{M_{\tau_0}} a^{\zeta_0}_{\delta(h)}(\zeta)|U_j(\zeta)|^2\, d\mu(\zeta) = \int_{\sqrt{\rho}(\zeta_0)-2\delta(h)}^{\sqrt{\rho}(\zeta_0)+2\delta(h)} \left(\left\langle V_h^{\tau *}\Pi_\tau a^{\zeta_0}_{\delta(h)}\Pi_\tau V_h^\tau \varphi_j, \varphi_j\right\rangle d\tau + \mathcal{O}(h\delta(h)^{-2})\right)$$
$$= \int_{\sqrt{\rho}(\zeta_0)-2\delta(h)}^{\sqrt{\rho}(\zeta_0)+2\delta(h)} \left\langle V_h^{\tau *}\Pi_\tau a^{\zeta_0}_{\delta(h)}\Pi_\tau V_h^\tau \varphi_j, \varphi_j\right\rangle d\tau + \mathcal{O}(h\delta(h)^{-1}).$$

We now subtract $\int_0^{\tau_0}\int_{\partial M_\tau} \frac{a^{\zeta_0}_{\delta(h)}(Z)}{\mu_\tau(\partial M_\tau)}\, d\mu_\tau(Z)d\tau$ from both sides of the equality and then square both sides. The error is then of order $h^2\delta(h)^{-2}$, which we move to the left-hand side of the equality to conserve space:

$$\left|\int_{M_\tau} a^{\zeta_0}_{\delta(h)}(\zeta)|U_{\lambda_j}(\zeta)|^2\, d\mu(\zeta) - \int_0^{\tau_0}\int_{\partial M_\tau} \frac{a^{\zeta_0}_{\delta(h)}(Z)}{\mu_\tau(\partial M_\tau)}\, d\mu_\tau(Z)d\tau\right|^2 + \mathcal{O}(h^2\delta(h)^{-2})$$

$$= (4\delta(h))^2\left|\int_{\sqrt{\rho}(\zeta_0)-2\delta(h)}^{\sqrt{\rho}(\zeta_0)+2\delta(h)} \left(\left\langle V_h^{\tau *}\Pi_\tau a^{\zeta_0}_{\delta(h)}\Pi_\tau V_h^\tau \varphi_j, \varphi_j\right\rangle - \int_{\partial M_\tau} \frac{a^{\zeta_0}_{\delta(h)}(Z)}{\mu_\tau(\partial M_\tau)}\, d\mu_\tau(Z)\right)\frac{d\tau}{4\delta(h)}\right|^2$$

$$\leq (4\delta(h))^2\int_{\sqrt{\rho}(\zeta_0)-2\delta(h)}^{\sqrt{\rho}(\zeta_0)+2\delta(h)} \left|\left\langle V_h^{\tau *}\Pi_\tau a^{\zeta_0}_{\delta(h)}\Pi_\tau V_h^\tau \varphi_j, \varphi_j\right\rangle - \int_{\partial M_\tau} \frac{a^{\zeta_0}_{\delta(h)}(Z)}{\mu_\tau(\partial M_\tau)}\, d\mu_\tau(Z)\right|^2\frac{d\tau}{4\delta(h)}$$

$$= 4\delta(h)\int_{\sqrt{\rho}(\zeta_0)-2\delta(h)}^{\sqrt{\rho}(\zeta_0)+2\delta(h)} \left|\left\langle V_h^{\tau *}\Pi_\tau a^{\zeta_0}_{\delta(h)}\Pi_\tau V_h^\tau \varphi_j, \varphi_j\right\rangle - \int_{\partial M_\tau} \frac{a^{\zeta_0}_{\delta(h)}(Z)}{\mu_\tau(\partial M_\tau)}\, d\mu_\tau(Z)\right|^2 d\tau.$$

For the inequality we used that $\frac{d\tau}{4\delta(h)}$ is a probability measure on the interval $[\sqrt{\rho}(\zeta_0) - 2\delta(h), \sqrt{\rho}(\zeta_0) + 2\delta(h)]$, so Jensen's inequality applies. Performing the Cesàro sum and using Proposition 5.6.1, we find

$$h^{n-1}\sum_{E_j\in[1,1+h]}\left|\int_{M_\tau} a^{\zeta_0}_{\delta(h)}(\zeta)\ |U_j(\zeta)|^2\,d\mu(\zeta)-\int_0^{\tau_0}\int_{\partial M_\tau}\frac{a^{\zeta_0}_{\delta(h)}(Z)}{\mu_\tau(\partial M_\tau)}\,d\mu_\tau(Z)d\tau\right|^2$$

$$\le 4\delta(h)\int_{\sqrt{\rho}(\zeta_0)-2\delta(h)}^{\sqrt{\rho}(\zeta_0)+2\delta(h)} C\delta(h)^{2(2n-1)}|\log h|^{-\beta}\,d\tau$$

$$=\mathcal{O}(\delta(h)^{4n}|\log h|^{-\beta})+\mathcal{O}(h^2\delta(h)^{-2}).$$

This completes the proof. □

5.6.2. Proof of Theorem 5.1.4 using Proposition 5.6.2

We now have enough tools to tackle the key volume comparison estimate Theorem 5.1.4, which is a Grauert tube analogue of Theorem 5.5.2. The proof uses the covering argument of [**25**, §3.2], [**24**, §5.2], [**11**, §4.2]. In what follows we revert to using λ-notation. Recall from (5.18) that the semiclassical h-notation in Proposition 5.6.1 and Proposition 5.6.2; in particular, we have $\delta(h)=|\log h|^{-\alpha}=(\log\lambda)^{-\alpha}=\varepsilon(\lambda)$.

Proof of Theorem 5.1.4. Let τ_0,τ_1 be fixed with $0<\tau_1<\tau_0$. In what follows we work with centers ζ_k that lie in the fixed 'annulus' $M_{\tau_0}\setminus M_{\tau_1}$, on which the errors remain uniform estimates. As in [**24**, Lemma 5.1], for every $\varepsilon(\lambda)$, there exists a *log-good* cover

$$\mathcal{U}_\lambda := \{\mathcal{B}(\zeta_k,\varepsilon(\lambda))\}_{k=1}^{R(\varepsilon(\lambda))}$$

of $M_{\tau_0}\setminus M_{\tau_1}$ by balls of radii $c\varepsilon(\lambda)$ such that

(i) The number $R(\varepsilon(\lambda))$ of elements in the covering satisfies $c_1\varepsilon(\lambda)^{-2n}\le R(\varepsilon(\lambda))\le c_2\varepsilon(\lambda)^{-2n}$, where c_1,c_2 are independent of $\varepsilon(\lambda)$.

(ii) Any $\mathcal{B}(\zeta', \varepsilon(\lambda)) \subset M_{\tau_0} \setminus M_{\tau_1}$ is covered by at most c_3 (independent of $\varepsilon(\lambda)$) number of elements of $\mathcal{U}_\lambda$.

(iii) Any $\mathcal{B}(\zeta', \varepsilon(\lambda)) \subset M_{\tau_0} \setminus M_{\tau_1}$ contains at least one element of $\{\mathcal{B}(\zeta_k, \frac{1}{3}\varepsilon(\lambda))\}_{k=1}^{R(\varepsilon(\lambda))}$.

We proceed to provide the extraction argument. For each

$$\lambda_j \in [\lambda, \lambda+1], \quad 1 \le k \le R(\varepsilon(\lambda)), \tag{5.30}$$

Set

$$X_{j,k} := \left| \int_{M_{\tau_0}} a^{\zeta_k}_{\varepsilon(\lambda)}(\zeta) |U_j|^2 \, d\mu - \int_0^{\tau_0} \int_{\partial M_\tau} \frac{a^{\zeta_k}_{\varepsilon(\lambda_j)}(\zeta)}{\mu_\tau(\partial M_\tau)} \, d\mu_\tau d\tau \right|^2.$$

(The two subscripts j, k correspond to the subscript j for the eigenvalue λ_j and the subscript k for the points ζ_k.) Also, let $\beta' > 0$ be a parameter to be chosen later and define 'exceptional sets' by

$$\Lambda_k := \left\{ j \colon \lambda_j \in [\lambda, \lambda+1],\ X_{j,k} \ge \varepsilon(\lambda)^{4n} (\log \lambda)^{-\beta'} \right\}.$$

We claim

$$\frac{\#\Lambda_k}{\lambda^{n-1}} \le C(\log \lambda)^{-\beta+\beta'}. \tag{5.31}$$

Indeed, this follows from Markov's inequality $\mathbb{P}(X_{j,k} \ge x) \le x^{-1}\mathbb{E}X_{j,k}$. We view $X_{j,k}$ as real-valued random variables index by j. The probability measure is the normalized counting measure on the set of indices j satisfying (5.30). Thanks to Proposition 5.6.2, for all such j the expected value of this random variable is

$$\mathbb{E}X_{j,k} = \mathcal{O}(\varepsilon(\lambda)^{4n}(\log\lambda)^{-\beta}),$$

with the error is uniform in $\zeta_k \in M_{\tau_0} \setminus M_{\tau_1}$ for $k = 1, 2, \ldots, R(\varepsilon(\lambda))$. Finally, setting $x = \varepsilon(\lambda)^{4n}(\log\lambda)^{-\beta'}$ in the inequality yields (5.31).

Moreover, the union

$$\Lambda := \bigcup_{k=1}^{R(\varepsilon(\lambda))} \Lambda_k$$

of the exceptional sets satisfies

$$\frac{\#\Lambda}{\lambda^{n-1}} \le CR(\varepsilon(\lambda))(\log\lambda)^{-\beta+\beta'} = C\varepsilon(\lambda)^{-2n}(\log\lambda)^{-\beta+\beta'} = C(\log\lambda)^{2n\alpha-\beta+\beta'}. \tag{5.32}$$

Recall from Proposition 5.6.2 that $0 < \beta < 1-2\alpha(2n-1)$, so $\beta' > 0$ can always be chosen small enough such that the quantity (5.32) tends to zero whenever $2n\alpha-(1-2\alpha(2n-1)) < 0$. This corresponds to the range of α in the statement of Theorem 5.1.4.

Consider now the 'generic set'

$$\Sigma := \{j\colon \lambda_j \in [\lambda, \lambda+1]\} \setminus \Lambda,$$

which is by construction a subsequence of full density:

$$\frac{\#\Sigma}{\lambda^{n-1}} \ge 1 - C\varepsilon(\lambda)^{-2n}(\log\lambda)^{-\beta+\beta'} \to 1.$$

If $j \in \Sigma$, then we must have

$$\left| \int_{M_{\tau_0}} a^{\zeta_k}_{\varepsilon(\lambda_j)}(\zeta) |U_j|^2 \, d\mu - \int_0^{\tau_0} \int_{\partial M_\tau} \frac{a^{\zeta_k}_{\varepsilon(\lambda_j)}(\zeta)}{\mu_\tau(\partial M_\tau)} \, d\mu_\tau d\tau \right|^2 \leq \varepsilon(\lambda)^{4n} (\log \lambda)^{-\beta'}$$

simultaneously for all $k = 1, 2, \ldots, R(\varepsilon(\lambda))$, that is,

$$\int_{M_{\tau_0}} a^{\zeta_k}_{\varepsilon(\lambda_j)}(\zeta) |U_j|^2 \, d\mu \leq C \mathrm{Vol}_\omega(\mathcal{B}(\zeta_k, \varepsilon(\lambda_j))) + o(\varepsilon(\lambda)^{2n} (\log \lambda)^{-\beta'/2}).$$

If $\zeta' \in M_\tau \setminus M$ is an arbitrary point, then the ball $\mathcal{B}(\zeta', \varepsilon(\lambda_j))$ is contained in at most c_2 number (independent of λ) of elements of the log-good cover $\mathcal{U}_\lambda$, whence we obtain the upper bound

$$\int_{\mathcal{B}(\zeta', \varepsilon(\lambda_j))} |U_j|^2 \, d\mu \leq C \sum_{\ell=1}^{c_2} \mathrm{Vol}_\omega(\mathcal{B}(\zeta_{k_\ell}, \varepsilon(\lambda_j))) + o(\varepsilon(\lambda)^{2n} (\log \lambda)^{-\beta'/2}) \leq C \mathrm{Vol}(\mathcal{B}(\zeta', \varepsilon(\lambda_j)).$$

The constant $C = C(M, g)$ is independent of ζ' throughout.

It remains to extract another full density subsequence Σ' using symbols of the form $b^{\zeta_0}_\varepsilon(\zeta) := b(\zeta/\varepsilon)$ in local coordinates centered at ζ_0. Here, $0 \leq b \leq 1$ is taken to be a smooth cut-off function that equals 1 on $B(0, 1/6) \subset \mathbb{C}^n$ and vanishes outside $B(0, 1/3) \subset \mathbb{C}^n$. Repeating the same arguments, we see that for $j \in \Sigma'$, we have

$$\int_{\mathcal{B}(\zeta_k, \varepsilon(\lambda_j)/3)} |U_j|^2 \, d\mu \geq c \mathrm{Vol}(\mathcal{B}(\zeta_k, \varepsilon(\lambda_j)/6)) - o(|\log \lambda|^{-\beta'/2})$$

simultaneously for all $k = 1, 2, \ldots, R(\varepsilon(\lambda))$. Let $\zeta' \in M_\tau \setminus M$ be arbitrary. Every ball $\mathcal{B}(\zeta', \varepsilon(\lambda_j))$ contains at least one element $\mathcal{B}(\zeta', \varepsilon(\lambda_j)/3) \in \mathcal{U}_\lambda$ of the log-good cover, whence

$$\int_{\mathcal{B}(\zeta', \varepsilon(\lambda_j))} |U_j|^2 \, dV \geq c \mathrm{Vol}(\mathcal{B}(\zeta_{k_0}, \varepsilon(\lambda_j)/3)) \geq c \mathrm{Vol}(\mathcal{B}(\zeta', \varepsilon(\lambda_j))).$$

Again, it is easy to verify that $c = c(M,g)$ is independent of ζ'. This is the statement of the volume lower bound.

The intersection $\Gamma = \Sigma \cap \Sigma'$ is again a full density subsequence. By construction, every $j \in \Gamma$ satisfies the two-sided bound:

$$c\mathrm{Vol}_\omega(\mathcal{B}(\zeta', \varepsilon(\lambda_j))) \leq \int_{\mathcal{B}(\zeta', \varepsilon(\lambda_j))} |U_j|^2 \, d\mu \leq C\mathrm{Vol}(\mathcal{B}(\zeta', \varepsilon(\lambda_j))) \quad \text{for all } \zeta' \in M_\tau \setminus M.$$

This completes the proof of Theorem 5.1.4. □

5.7. Proof of Theorem 5.1.1: Log-scale Equidistribution of Complex Zeros

Recall from the previous section the two key objects of study:

$$\Theta_j(\zeta) := \|\varphi_j^{\mathbb{C}}\,|_{\sqrt{\rho}(\zeta)}\,\|_{L^2(M_{\sqrt{\rho}(\zeta)})} \qquad \text{and} \qquad U_j(\zeta) := \frac{\varphi_j^{\mathbb{C}}(\zeta)}{\Theta_j(\zeta)}.$$

By the Poincaré–Lelong formula [20, p.388, Lemma], the current of integration $[Z_j]$ over the zero set $Z_j = \{\zeta \in M_{\tau_0} : \varphi_j^{\mathbb{C}}(\zeta) = 0\}$ is given by the identity

$$\frac{i}{2\pi}\partial\bar{\partial}\log|U_j|^2 = \frac{i}{2\pi}\partial\bar{\partial}\log|\varphi_j^{\mathbb{C}}|^2 - \frac{i}{2\pi}\partial\bar{\partial}\log\Theta_j^2 = [Z_j] - \frac{i}{2\pi}\partial\bar{\partial}\log\Theta_j^2. \tag{5.33}$$

To study the currents $[Z_j]$ at logarithmic length scales, let $D^{\zeta_0 *}_{\varepsilon(\lambda_j)}$ denote the corresponding pullback operator corresponding to the local holomorphic dilation map (5.16). This allows us to work not on shrinking balls $B(\zeta_0, \varepsilon(\lambda_j))$ but on a fix-sized ball $B(\zeta_0, 1)$, which is more convenient. The (normalized) small-scale version of (5.33) becomes

$$\frac{i}{2\pi\lambda_j\varepsilon(\lambda_j)}\partial\bar{\partial}D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\log|U_j|^2 \tag{5.34}$$

$$= \frac{1}{\lambda_j \varepsilon(\lambda_j)} D^{\zeta_0 *}_{\varepsilon(\lambda_j)} [Z_j] - \frac{i}{2\pi \lambda_j \varepsilon(\lambda_j)} \partial\bar{\partial} D^{\zeta_0 *}_{\varepsilon(\lambda_j)} \log \Theta_j^2 \quad \text{as currents on } B(\zeta_0, 1).$$

We used the fact that the local dilation map $D^{\zeta_0}_{\varepsilon(\lambda_j)}$, being holomorphic, commutes with $\partial\bar{\partial}$.

Remark **5.7.1.** *The λ_j^{-1} normalization is already present in* (5.3)*, due to* [**71**]*. Here there is an additional factor of $\varepsilon(\lambda_j)^{-1}$, which comes from the proof of Proposition 5.7.4, specifically* (5.41)*.*

5.7.1. Proof of Theorem 5.1.1 using Theorem 5.1.4

We rescale the convergence statement (5.4) as in (5.34), so that the various objects are defined on a fixed-sized ball $B(\zeta_0, 1)$ that does not change with respect to the frequency λ.

We point out a subtlety involving the parameter $\alpha > 0$ in the proof of Theorem 5.1.1 using Theorem 5.1.4. Namely, if a full density subsequence satisfies volume comparison (5.6) at length scale $\varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}$, then it satisfies the zeros distribution result (5.4) at a coarser length scale $\varepsilon'(\lambda_j) := (\log \lambda_j)^{-\alpha'}$ for any $\alpha' < \alpha$. This inequality is strict – see the argument around (5.42)–(5.43). To emphasize the role of the two scales, we restate Theorem 5.1.1 and Theorem 5.1.4 as follows.

Theorem **5.7.2.** *Let (M, g) be a real analytic, negatively curved, compact manifold without boundary. Let $\omega := -i\partial\bar{\partial}\rho$ denote the Kähler form on the Grauert tube M_{τ_0}. Assume that*

$$0 \leq \alpha' < \frac{1}{2(3n-1)}, \quad \varepsilon'(\lambda_j) = (\log \lambda_j)^{-\alpha'}.$$

Then there exists a full density subsequence of eigenvalues λ_{j_k} such that for arbitrary but fixed $\zeta_0 \in M_{\tau_0} \backslash M$, there is a uniform two-sided volume bound

$$c\mathrm{Vol}_\omega(\mathcal{B}(\zeta_0, \varepsilon'(\lambda_{j_k}))) \leq \int_{\mathcal{B}(\zeta_0,\varepsilon'(\lambda_{j_k}))} |U_{j_k}|^2 d\mu \leq C\mathrm{Vol}_\omega(\mathcal{B}(\zeta_0, \varepsilon'(\lambda_{j_k}))). \tag{5.35}$$

The constants c, C are geometric constants depending only on $\sqrt{\rho}(\zeta_0)$; they are uniform for ζ_0 lying in an 'annulus' $0 < \tau_1 \leq \sqrt{\rho}(\zeta_0) \leq \tau_0$.

Moreover, for any α satisfying

$$0 \leq \alpha < \alpha' < \frac{1}{2(3n-1)}, \quad \varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha},$$

the full density subsequence satisfying (5.35) also satisfies

$$\frac{1}{\lambda_{j_k}\varepsilon(\lambda_{j_k})} D^{\zeta_0 *}_{\varepsilon(\lambda_j)}[Z_{\lambda_{j_k}}] \rightharpoonup \frac{i}{\pi}\partial\bar{\partial}|\Im(\zeta - \zeta_0)|_{g_0} \quad \text{as currents on } B(\zeta_0, 1). \tag{5.36}$$

Here, $D^{\zeta_0 *}_{\varepsilon(\lambda_j)}$ denote pullback by the local holomorphic dilation (5.16) and g_0 denotes the flat metric. Equivalently, for every test form $\eta \in \mathcal{D}^{(n-1,n-1)}(B(\zeta_0, 1))$,

$$\int_{B(\zeta_0,1)} \eta \wedge \frac{1}{\lambda_{j_k}\varepsilon(\lambda_{j_k})} D^{\zeta_0 *}_{\varepsilon(\lambda_j)}[Z_{\lambda_{j_k}}] = \int_{B(\zeta_0,1)} \eta \wedge \frac{i}{\pi}\partial\bar{\partial}|\Im(\zeta - \zeta_0)|_{g_0} + o(1).$$

REMARK **5.7.3.** *By a partition of unity argument, Theorem 5.7.2 for general test forms supported on Kähler balls implies Theorem 5.1.1 for test forms on M_{τ_0} of the form $f\omega^{n-1}$ with $f \in C(M_{\tau_0})$.*

The volume comparison (5.35) has already been proved in the previous section. Comparing what is left to prove – namely (5.36) – with the identity (5.34), we see that it suffices to establish Proposition 5.7.4 and Proposition 5.7.5 below.

Proposition **5.7.4.** *For the entire sequence of eigenvalues λ_j, for every $\zeta_0 \in M_{\tau_0}\backslash M$, we have*

$$\frac{i}{2\pi\lambda_j\varepsilon(\lambda_j)}\partial\bar{\partial}D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\log\Theta_j^2 \to \frac{i}{\pi}\partial\bar{\partial}|\Im(\zeta-\zeta_0)|_{g_0} \quad \textit{as currents on } B(\zeta_0,1).$$

Here, $|\cdot|_{g_0}$ denotes the Euclidean distance.

Proposition **5.7.5.** *There exists a full density subsequence of eigenvalues λ_{j_k} such that, for every $\zeta_0 \in M_{\tau_0}\backslash M$, we have*

(i) $(\lambda_{j_k}\varepsilon(\lambda_{j_k}))^{-1}\log D^{\zeta_0 *}_{\varepsilon(\lambda_{j_k})}|U_{j_k}|^2 \to 0$ *strongly in* $L^1(B(\zeta_0,1))$;

(ii) $(\lambda_{j_k}\varepsilon(\lambda_{j_k}))^{-1}\partial\bar{\partial}\log D^{\zeta_0 *}_{\varepsilon(\lambda_{j_k})}|U_{j_k}|^2 \rightharpoonup 0$ *weakly in* $\mathcal{D}^{(n-1,n-1)'}(B(\zeta_0,1))$.

5.7.2. Proof of Proposition 5.7.4 using pseudodifferential operators

Using (5.14), we see

$$\varphi_j^{\mathbb{C}}(\zeta) = e^{\lambda_j\sqrt{\rho}(\zeta)}(P^{\sqrt{\rho}(\zeta)}\varphi_j)(\zeta), \qquad \zeta \in M_{\tau_0}. \tag{5.37}$$

Therefore,

$$\begin{aligned}
D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\Theta_j(\zeta)^2 &= D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\left\|\varphi_j^{\mathbb{C}}\,|_{\partial M_{\sqrt{\rho}(\zeta)}}\right\|^2_{L^2(\partial M_{\sqrt{\rho}(\zeta)})} \\
&= \left\|\varphi_j^{\mathbb{C}}\,|_{\partial M_{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}}\right\|^2_{L^2\left(\partial M_{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}\right)} \\
&= \left\langle \Pi_{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}\varphi_j^{\mathbb{C}}, \Pi_{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}\varphi_j^{\mathbb{C}}\right\rangle_{L^2\left(\partial M_{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}\right)} \\
&= e^{2\lambda_j D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}\left\langle P^{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)*}\Pi_{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}P^{D^{\zeta_0 *}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}\varphi_j, \varphi_j\right\rangle_{L^2(M)}.
\end{aligned} \tag{5.38}$$

The last equality follows from (5.37).

The operators

$$A(\varepsilon(\lambda_j),\sqrt{\rho}(\zeta)) := P^{D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)*}\Pi_{D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}P^{D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)} \in \Psi^{-\frac{n-1}{2}}(M)$$

forms an analytic family in the parameter $\sqrt{\rho}(\zeta) \in (0,\tau_0]$ with $A(\varepsilon(\lambda_j),\sqrt{\rho}(\zeta)) \to \mathrm{Id}$ as $\sqrt{\rho}(\zeta) \to 0$. It is easy to see using the Schur-Young test that $(1+\Delta)^{-\frac{n+1}{2}}A(\varepsilon) \in \Psi^{-n}(M)$ is a uniformly upper bounded family of operators on $L^2(M)$ (see [71, (34)]). Therefore, writing $A(\varepsilon(\lambda_j),\sqrt{\rho}(\zeta)) = (1+\lambda_j)^{\frac{n+1}{2}}(1+\Delta)^{-\frac{n+1}{2}}A(\varepsilon(\lambda_j),\sqrt{\rho}(\zeta))$, we find

$$\left|\frac{1}{\lambda_j}\log\left\langle P^{D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)*}\Pi_{D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}P^{D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta)}\varphi_j,\varphi_j\right\rangle_{L^2(M)}\right| \le C\frac{\log\lambda_j}{\lambda_j} \tag{5.39}$$

for some C independent of ε. Combining (5.38) and (5.39) gives

$$\frac{1}{2\pi\lambda_j\varepsilon(\lambda_j)}\log D^{\zeta_0*}_{\varepsilon(\lambda_j)}\Theta_j(\zeta)^2 = \frac{1}{\pi\varepsilon(\lambda_j)}D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta) + \mathcal{O}(\lambda_j^{-1}\log\lambda_j). \tag{5.40}$$

Recall from Section 5.2 that the Grauert tube function ρ is related to the complexified Riemannian distance function r on $M_{\mathbb{C}}\times\overline{M}_{\mathbb{C}}$ by

$$\rho(\zeta) = -\frac{1}{4}r^2(\zeta,\bar{\zeta}), \qquad \zeta = \exp_x^{\mathbb{C}}(i\xi) \in M_{\tau_0}.$$

Taylor expanding the metric yields $\sqrt{\rho}(\zeta) = |\Im(\zeta-\zeta_0)|_{g_0} + O(|\Im(\zeta-\zeta_0)|^2_{g_0})$, in which $|\cdot|_{g_0}$ denotes the flat metric. This gives rise to the $\lambda_j\to\infty$ asymptotics

$$D^{\zeta_0*}_{\varepsilon(\lambda_j)}\sqrt{\rho}(\zeta) = \varepsilon(\lambda_j)|\Im(\zeta-\zeta_0)|_{g_0} + O(\varepsilon(\lambda_j)^2), \qquad \zeta = \exp_x^{\mathbb{C}}(i\xi) \in M_{\tau_0}. \tag{5.41}$$

The statement of Proposition 5.7.4 is now an immediate consequence of (5.40) and (5.41).

5.7.3. Proof of Proposition 5.7.5 using subharmonic function theory

Proposition 5.7.5 is modeled after arguments that have appeared in [**57, 71, 11**]. Given $\zeta_0 \in M_{\tau_0} \backslash M$, consider the family of plurisubharmonic functions

$$v_j := \frac{1}{\lambda_j \varepsilon(\lambda_j)} \log D^{\zeta_0 *}_{\varepsilon(\lambda_j)} |\varphi_j^{\mathbb{C}}|^2 \in \mathrm{PSH}(B(\zeta_0, 1)).$$

(The functions v_j are indeed subharmonic because $\varphi_j^{\mathbb{C}}$ are holomorphic by construction.) We claim

(i) $\{v_j\}$ is uniformly bounded above on $B(\zeta_0, 1)$;

(ii) $\limsup_{j\to\infty} v_j(\zeta) \leq 2\sqrt{\rho}(\zeta)$ on $B(\zeta_0, 1)$.

Notice $\sup_{B(\zeta_0,1)} D^{\zeta_0 *}_{\varepsilon(\lambda_j)} |U_j|^2 = \sup_{\mathcal{B}(\zeta_0, \varepsilon(\lambda))} |U_j|^2$. To prove the first statement, it suffices to obtain a uniform upper bound on each slice $\partial M_\tau \cap \overline{\mathcal{B}(\zeta_0, \varepsilon(\lambda_j))}$ that is independent of τ. Since $u_j^\tau \in \mathcal{O}^{\frac{n-1}{4}}(\partial M_\tau)$, we see (cf. [**71**, §5.1])

$$\sup_{\partial M_\tau \cap \overline{B(\zeta_0, \varepsilon(\lambda_j))}} |U_j|^2 \leq \sup_{\partial M_\tau} |u_j^\tau|^2 \leq \lambda_j^n \|u_j^\tau\|_{L^2(\partial M_\tau)} = \lambda_j^n.$$

Rewriting the left-hand side as $U_j = \varphi_j^{\mathbb{C}} / \|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_{\sqrt{\rho}})}$, taking the logarithm, dividing by λ_j, and finally using the limit formula of Proposition 5.7.4 finishes the proof of (i) and (ii).

It follows from a standard compactness theorem on plurisubharmonic functions [**26**, Theorem 4.1.9] that either $v_j \to -\infty$ locally uniformly, or there exists a subsequence that is convergent in $L^1_{\mathrm{loc}}(B(\zeta_0, 1))$. The first possibility is easily ruled out. Indeed, if it were true, then

$$\frac{1}{\lambda_j \varepsilon(\lambda_j)} \log D^{\zeta_0 *}_{\varepsilon(\lambda_j)} |U_j|^2 \leq -1 \quad \text{on } B(\zeta_0, 1) \text{ for all } \lambda_j \gg 1$$

$$\iff |U_j|^2 \leq e^{-\lambda_j \varepsilon(\lambda_j)} \quad \text{on } \mathcal{B}(\zeta_0, \varepsilon(\lambda_j)) \text{ for all } \lambda_j \gg 1,$$

contradicting the mass comparison assumption (5.35).

Remark **5.7.6.** *By a covering argument similar to the proof of Theorem 5.1.4, it is easy to see that if a sequence $\{U_j\}$ satisfies volume comparison (5.35), then it satisfies volume comparison at all coarser length scales $\varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}$ for $\alpha' < \alpha < \frac{1}{2(3n-1)}$.*

Therefore, v_j has a subsequence, which we continue to denote by v_j, that converges in L^1 to $v \in L^1(B(\zeta_0, 1))$. By passing to yet another subsequence if necessary, we may assume that the convergence to v is pointwise almost everywhere. The upper-semicontinuous regularization

$$v^*(\zeta) := \limsup_{\eta \to \zeta} v(\eta) \leq 2\sqrt{\rho}(\zeta)$$

of v is then a plurisubharmonic function on $B(\zeta_0, 1)$ and $v_j \to v^*$ pointwise almost everywhere.[3] The upper bound of $2\sqrt{\rho}(\zeta)$ follows from claim (ii) above.

Set

$$\psi := v^* - 2\sqrt{\rho} \leq 0 \quad \text{on } B(\zeta_0, 1).$$

[3]A similar argument is used in [**57**, Lemma 1.4], which gives further details. See also [**32**] for background.

Assume for purposes of a contradiction that $\|\lambda_j^{-1}\varepsilon(\lambda_j)^{-1}\log D^{\zeta_0 *}_{\varepsilon(\lambda_j)}|U_j|^2\|_{L^1(B(\zeta_0,1))} \geq \delta > 0$. It follows that

$$W_\delta := \{\zeta \in B(\zeta_0, 1) : \psi(\zeta) < -\delta/2\} \tag{5.42}$$

is an open set with nonempty interior. The shape of W_δ is unknown – it may have a very small inradius – but it is a fixed (independent of λ_j) open set. To gain control over this unknown set W_δ, we make use of the volume comparison assumption (5.35) that takes place at the finer scale $\varepsilon'(\lambda_j) = (\log \lambda_j)^{-\alpha'}$ for $\alpha' < \alpha$. From this assumption we know

$$\int_{B(\zeta',\varepsilon'(\lambda_j))} |U_j|^2 \omega^n \geq c\mathrm{Vol}_\omega(\mathcal{B}(\zeta_0, \varepsilon'(\lambda_j))) \quad \text{for all } \zeta' \in M_{\tau_0} \setminus M.$$

Rescaling yields

$$\int_{B(\zeta',\varepsilon'(\lambda_j)\varepsilon^{-1}(\lambda_j))} D^{\zeta_0}_{\varepsilon(\lambda_j)}|U_j|^2 \omega^n \geq c\mathrm{Vol}_\omega(\mathcal{B}(\zeta_0, \varepsilon'(\lambda_j)\varepsilon^{-1}(\lambda_j))). \tag{5.43}$$

Notice in the above integral the radii $\varepsilon'(\lambda_j)\varepsilon^{-1}(\lambda_j) = \log(\lambda_j)^{-(\alpha'-\alpha)}$ of the domain of integration shrinks to 0. Therefore, there exists $\zeta' \in M_{\tau_0} \setminus M$ for which $B(\zeta', \varepsilon'(\lambda_j)\varepsilon^{-1}(\lambda_j)) \subset W_\delta$ for all λ_j sufficiently large.

On one hand, from the definition (5.42), we know that on all of W_δ – and in particular on $B(\zeta', \varepsilon'(\lambda_j)\varepsilon^{-1}(\lambda_j))$ – we have the upper bound $\lambda_j^{-1}\varepsilon(\lambda_j)^{-1}\log D^{\zeta' *}_{\varepsilon(\lambda_j)}|U_j|^2 < -\delta/2$, i.e.,

$$D^{\zeta_0 *}_{\varepsilon(\lambda_j)}|U_j(\zeta)|^2 \leq e^{-\delta\lambda_j\varepsilon(\lambda_j)}, \quad \zeta \in B(\zeta', \varepsilon'(\lambda_j)\varepsilon^{-1}(\lambda_j)), \quad \lambda_j \gg 1. \tag{5.44}$$

Clearly, the exponential decay upper bound (5.44) is incompatible with the logarithmic lower bound (5.43) as $\lambda_j \to \infty$. This shows by way of contradiction that the original assumption

$$\|\lambda_j^{-1}\varepsilon(\lambda_j)^{-1}\log D^{\zeta_0 *}_{\varepsilon(\lambda_j)}|U_j|^2\|_{L^1(B(\zeta_0,1))} \geq \delta > 0$$

does not hold, thereby proving Proposition 5.7.5 (i), from which Proposition 5.7.5 (ii) is an immediate consequence. Combining (5.34), Proposition 5.7.4, and Proposition 5.7.5 (ii), we obtain the zeros distribution statement of Theorem 5.7.2:

$$\frac{1}{\lambda_{j_k}\varepsilon(\lambda_{j_k})} D^{\zeta_0 *}_{\varepsilon(\lambda_j)}[Z_{\lambda_{j_k}}] \rightharpoonup \frac{i}{\pi}\partial\bar{\partial}|\Im(\zeta-\zeta_0)|_{g_0} \quad \text{as currents on } B(\zeta_0,1)$$

for a full density subsequence satisfying volume comparison at the finer scale α'. This concludes the proof of Theorem 5.1.1.

5.8. Appendix: Currents of Integration over Singular Varieties

In general, the zero set X of a holomorphic function on a complex manifold V is called a complex analytic variety (which could also be the common zeros of finitely many holomorphic functions). See for instance [**65**]. It has a decomposition into a regular set $R(X)$ and a lower-dimensional singular set $S(X)$, i.e., $X = R(X) \cup S(X)$ where $R(X)$ is a manifold and $\dim S(X) < \dim X$ (see [**31**, Theorem 2.1.8]). In [**31**, Theorem 3.1.1] it is proved that if X a k-dimensional complex subvariety of a complex manifold V and $u \in A^{2k}_c(V)$ is a smooth $(2k)$-form then

$$[X](u) := \int_X u = \int_{R(X)} \iota^* u$$

is a closed current (due to Lelong [**37**]). King used Federer's geometric measure theory [**18**] to study such currents. A modern exposition can be found in [**15**, Example 1.16].

5.8.1. Shiffman's Appendix

We asked B. Shiffman for further references on currents of integration over singular analytic varieties. He wrote the following addition to the Appendix, and refers to [**56**, Lemma A.2] for an elementary proof.

Here is a simpler way to show that $[X] = [Z_f]$ is a well-defined current: It suffices to show that the set $R(X)$ of smooth points has finite volume in a neighborhood U of a singular point z_0. By the Weierstrass preparation theorem applied to f, it follows that projections from $X \cap U$ to coordinate hyperplanes have finite fibers of bounded cardinality (for good coordinates) and therefore $\mathrm{Vol}(R(X) \cap U) = \int_{R(X)\cap U} \omega^{n-1} < \infty$.

The fact that Poincaré–Lelong holds at the singular points follows from the fact that the singular set $S(X)$ has Hausdorff $(2n-3)$-dimensional measure 0, and therefore $\|\partial\bar{\partial} \log |f|\|(S) = 0$, since the total variation measure of a current of order zero and dimension p vanishes on sets of Hausdorff p-measure zero. (In fact, $S(X)$ is a subvariety of real codimension 4).

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