Polygons, Companion Shapes, and the Construction of Polyhedra

WEINBERG COLLEGE OF ARTS & SCIENCES
Department of Mathematics

Yuxi Han
Advised by: Laura DeMarco   Northwestern University

Overview

- Given any polyhedron in $\mathbf{R}^3$, we can cut it open along its edges, flatten it out, and obtain a polygon in the plane $\mathbf{R}^2$.
- 70 years ago, a question raised by A. D. Alexandrov in Convex Surfaces:given a polygon in $\mathbf{R}^2$, what is the folding procedure to reconstruct the polyhedron in $\mathbf{R}^3$?
- Given a polygon with n vertices, we try to find its "companion shape" (another polygon with n vertices) so that, glued together and folded, we obtain a polyhedron where all n cone angles are equal.
- We then applied this general theory to other $\mathbf{R}^2$ developments, in particular the case of a polygon with n vertices, as n goes to infinity, where the curvature distribution approximates harmonic measure on a shape in the plane. We explored an iterative process of constructing the "harmonic caps".

Research Questions

- For what polygons with n vertices does there exist a companion polygon (also with n vertices) so that when glued together, we get a 3-d polyhedron with n equal cone angles?
- Given a polygon and its companion shape in $\mathbf{R}^2$, what is the folding procedure to reconstruct the polyhedron in $\mathbf{R}^3$?
- If the polygon has N (large) vertices, distributed according to harmonic measure, what does its cap look like; and what happens after iteratively constructing the harmonic cap?

Mathematical Approach

- We start from special cases where n=3 and n=4 to find patterns;
- We use Mathematica to write programs, produce companion shapes and conduct experiments of iteration through different algorithms;
- Practically, we analyze the folding lines by cutting, gluing and folding;
- Theoretically, we attempt to explain folding construction with mathematical proofs;
- Special cases of irregular shapes are studied;
- Detailed explanation is given on folding procedures of tetrahedra

Triangles and Quadrilaterals

n=3 case: Triangles and Tri-prisms

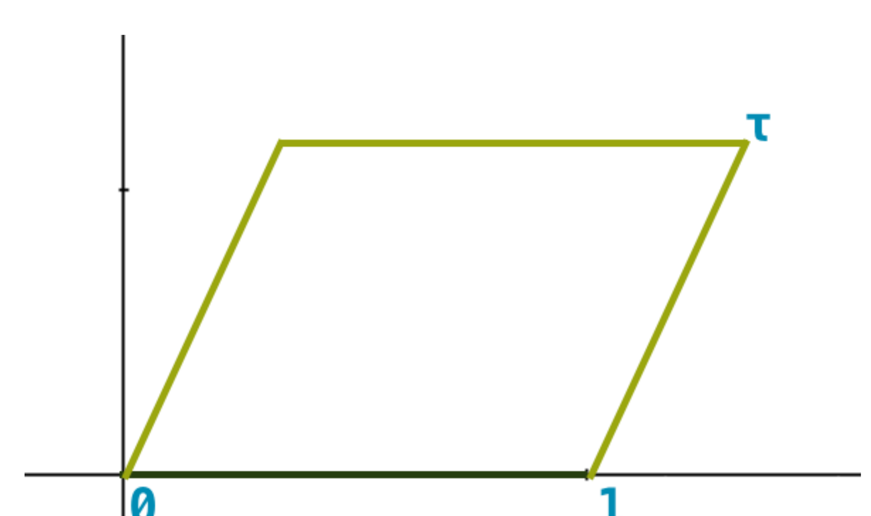
- ***In n=3 case, for which triangles does there exist a companion triangle that glued together, we get a polyhedron with equal cone angle 2 π/3 at each vertex?***
- The only possible shape whose companion shape is closed and thus can be folded into a degenerate polyhedron is the **equilateral triangle.**

n=4 case: Quadrilaterals and Tetrahedra

- ***In n=4 case, for which quadrilaterals does there exist a companion quadrilateral that glued together, we get a tetrahedron with equal cone angles π at each vertex?***

1. Given any parallelogram, the companion shape exists and is also a parallelogram.
2. In equal curvature case, if cap to a quadrilateral exists, then the quadrilateral is a parallelogram.

- ***What are the folding lines for quadrilaterals and their caps to create a tetrahedron?***
- ***Different parallelograms may be folded into the same tetrahedron. What is the group pattern here?***

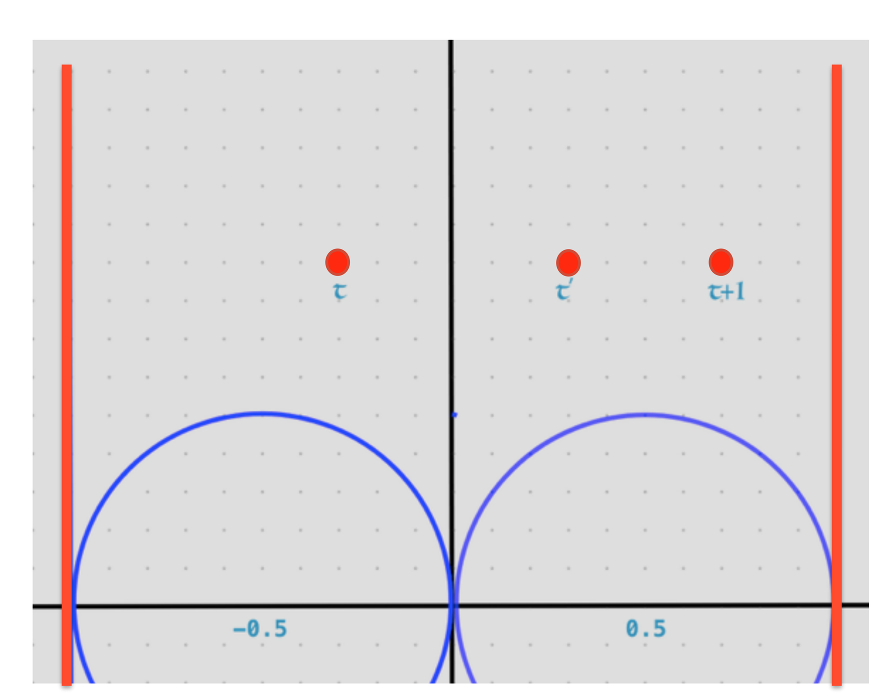


Given first edge fixed on *x*-axis from 0 to 1, third point τ>0.
Parallelograms with third point **τ, τ+1 and -1/τ** fold into the same tetrahedron even though the folding structure might be different. This property also holds for (aτ+b)/(cτ+d) where integers a,b,c,d satisfy ad-bc=1.

When **τ …**

- Falls on the red lines**(ie. |real τ|=1)**, the ending shape is **degenerate** by folding both the shorter diagonals and the original glued edge;
- Falls on the black y-axis**(ie. |real τ|=0)**,the ending shape is **degenerate** by folding only the original glued edge;
- Does not fall on lines but in the region above half circles and between two red lines(eg. **τ, τ' and -1/τ**), the ending tetrahedron is in $\mathbf{R}^3$ by folding lines are the shorter diagonals and the originally glued edge.
- Dots in the half circle correspond to **τ** from the family of **(aτ+b)/(cτ+d)**

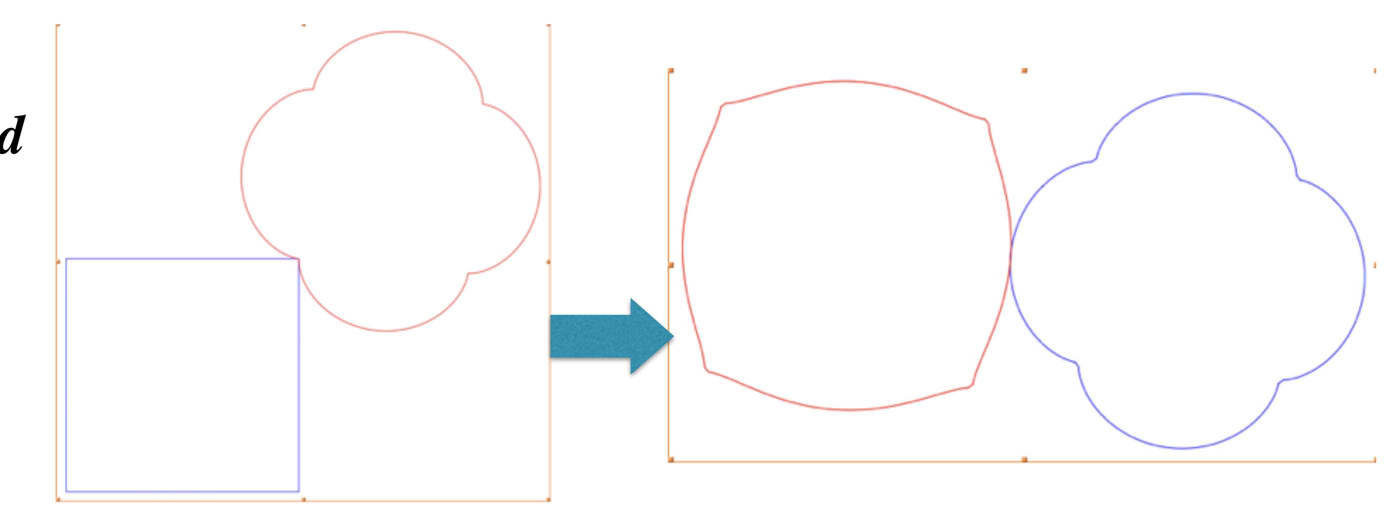
τ in fundamental domain



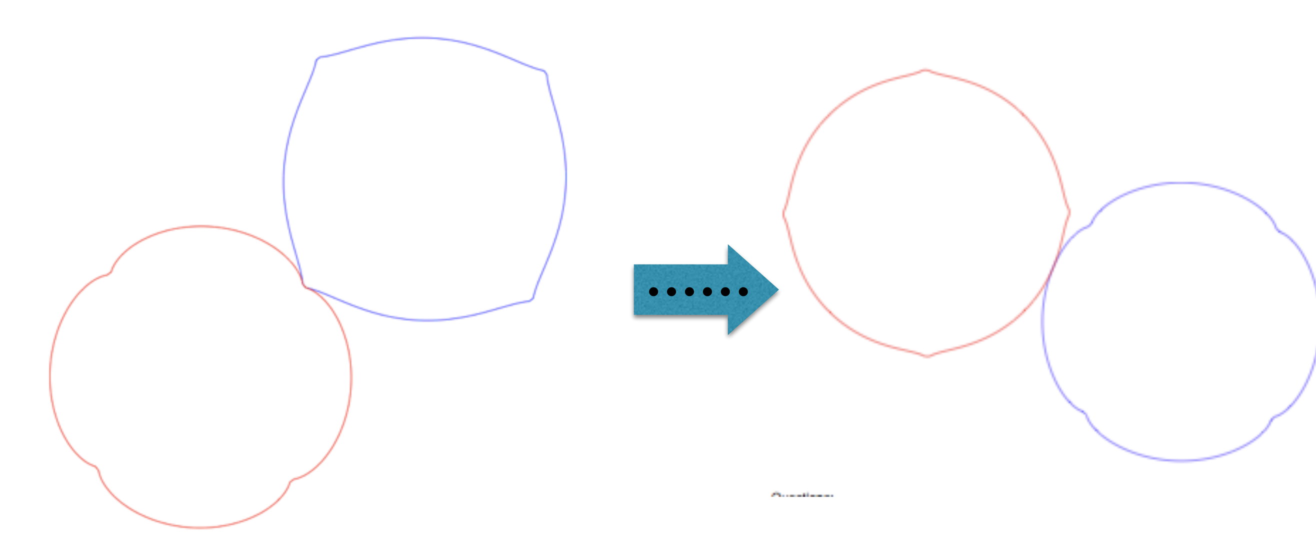
Harmonic caps

Conjecture:
By taking harmonic cap of a shape and iterating this process (use the cap, which is the output, as input for the next step), the shape we get is getting rounder so that the limiting shape is a circle.

Example: Square shape and its harmonic caps (using DM* algorithm)



Step 1 **Step2**



Step3 **Step6**

*Note; Riemann map, **Don Marshall's** program Zipper

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