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Spectral Theory and Index Theorems for Stationary Spacetimes

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ABSTRACT

Spectral Theory and Index Theorems for Stationary Spacetimes

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We present both semiclassical asymptotics for the wave equation on a stationary Kaluza-Klein spacetime and an index theorem describing the difference of the positive-frequency spectral projectors for two stationary regions in a globally hyperbolic spacetime. The first result involves analyzing the restrictions of the wave trace to isotypic subspaces for the action of the structure group, and the asymptotic distribution of the frequencies as the representation corresponding to the isotypic subspace goes to infinity in the weight lattice. For the second result, it was previously known how to calculate the difference of these spectral projectors in the case where they are defined by spacetime regions on which the metric appears ultrastatic. Here we extend these techniques to handle the case where the defining regions merely appear stationary. As such a new formula for the relevant Feynman propagators is derived as one can no longer obtain spectral descriptions of the Feynman propagators of the square of the Dirac operator as is done in the case of local ultrastatic regions.

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Preface

A part of this thesis is based off of the article [**24**] by the author and portions of sections 1.3,3.1 and 1.2 are taken from this article. Chapter 1 consists of background on stationary globally hyperbolic spacetimes, together with a description of the dynamics of null geodesics in this setting. Chapter 2 then introduces the necessary background on normally hyperbolic and Dirac-type operators and introduces the relevant spaces and operators on which we will be doing spectral theory. Chapter 3 consists of the main original results of this thesis where sections 3.1 and 3.2 develop semiclassical asymptotics for the wave trace on stationary Kaluza-Klein spacetimes and 3.3 extends the index theory of [**4**] from spacetimes with two local ultrastatic regions to spacetimes with two local stationary regions.

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CHAPTER 1

Geometry of Stationary Spacetimes

Definition 1.0.1. A **Lorentzian manifold** is a smooth manifold M together with a smooth section g of $\mathrm{Sym}^2 T^*M$ such that for each $x \in M$, g_x is non-degenerate with signature $(-1, +1, ..., +1)$.

We give several examples of Lorentzian manifolds that we will return to throughout this chapter to illustrate concepts.

Example 1.0.2. The first example of a Lorentzian manifold one typically studies is flat **Minkowski space**: $M = \mathbb{R}^{n+1}$ with metric

$$g = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2.$$

More generally, one has the class of **ultrastatic spacetimes** which are manifolds of the form $M = \mathbb{R}_t \times \Sigma^n$ with metric

$$g = -dt^2 + h$$

where h is a fixed t-independent Riemannian metric on Σ.

Example 1.0.3. For $\alpha > 0$ we have the **de Sitter spacetime**

$$S_1^{n+1}(\alpha) := \{x \in \mathbb{R}^{n+2} \ : \ -x_0^2 + x_1^2 + \cdots + x_{n+1}^2 = \alpha^2\}$$

endowed with the pullback of the Minkowski metric on $\mathbb{R}^{n+2}$ to $S_1^{n+1}(\alpha)$.

Example 1.0.4. Similar to the previous example, we have spacetimes

$$H_1^{n+1}(\alpha) := \{x \in \mathbb{R}^{n+2} \ : \ -x_0^2 - x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = -\alpha^2\}$$

endowed with the pullback of the standard signature $(-1, -1, +1, ..., +1)$ metric on $\mathbb{R}^{n+2}$ to $H_1^{n+1}(\alpha)$. The universal cover $\widetilde{H}_1^{n+1}(\alpha)$ is called the **anti de Sitter spacetime**.

One can use the curvature computations of 1.2 that the Minkowski, de Sitter and anti de Sitter spacetimes are the Lorentzian analogues of the complete model spaces of constant curvature from Riemannian geometry.

Example 1.0.5. The **Schwarzschild spacetime** is

$$M = \mathbb{R}_t \times ((2m)^{2-n}, \infty)_r \times S^{n-1}$$

$$g = -(1 - 2mr^{2-n})dt^2 + (1 - 2mr^{2-n})^{-1}dr^2 + r^2 h_{S^{n-1}}$$

where $m > 0$ is some positive constant and $h_{S^{n-1}}$ is the round metric on the unit sphere S^{n-1}. This is used to model exterior regions of black holes, or even just regions of space of moderate distance away from a large object like a star.

Example 1.0.6. Slightly more general than the ultrastatic spacetimes are the **static spacetimes**. These are

$$M = \mathbb{R}_t \times \Sigma^n$$

$$g = -N^2 dt^2 + h$$

With h a fixed Riemannian metric on Σ and $N : \Sigma \to (0, \infty)$ a t-independent positive smooth function. If N is allowed to be t-dependent then the spacetime is simply called **conformally static**.

1.1. Standard Forms of Stationary Spacetimes

Definition 1.1.1. For (M, g) a Lorentzian manifold and $v \in TM$ we say that v is:

(1) **lightlike** (or **null**) if and only if $v \neq 0$ and $g(v, v) = 0$,

(2) **timelike** if and only if $g(v, v) < 0$,

(3) **spacelike** if and only if $g(v, v) > 0$ or $v = 0$, and

(4) **causal** if and only if v is either lightlike or timelike.

We use completely analogous terminology for covectors.

Definition 1.1.2. Let T_0M denote the set of null vectors in TM. This is a smooth fiber subbundle $T_0M \subseteq TM$ whose fibers each have exactly two connected components, both of which are diffeomorphic to $(0, \infty) \times S^n$. We call (M, g) **time orientable** if and only if T_0M can be written as a disjoint union

$$T_0M = T_+M \sqcup T_-M$$

of smooth subbundles such that for every $x \in M$, the fibers $(T_\pm M)_x$ are precisely the two connected components of $(T_0M)_x$. A choice of such a decomposition is called a **time orientation** with T_+M viewed as the forward time direction and T_-M as the backwards time direction.

All of the examples of spacetimes from the start of 1 are time-orientable, as can be seen by explicitly writing down a global causal vector field in each case. A standard example of a non-time-orientable spacetime **[18]** is the quotient of de Sitter spacetime $S_1^{n+1}(\alpha)$ by the action $x \mapsto -x$ on $\mathbb{R}^{n+1}$. Most Lorentzian manifolds admit time-orientable metrics, as the following lemma states.

Lemma 1.1.3. *[27] Every connected non-compact Lorentzian manifold is time-orientable.*

Definition 1.1.4. A **spacetime** is a connected, oriented and time-oriented Lorentzian manifold of dimension at least 3.

Definition 1.1.5. A curve $\gamma : I \to M$ with $I \subseteq \mathbb{R}$ an open interval, M a Lorentzian manifold, is called **lightlike** (respectively **timelike**, **spacelike** or **causal**) if and only if $\dot{\gamma}(s)$ is lightlike (respectively timelike, spacelike or causal) for every $s \in I$. If M is time-oriented then we say γ is **future-directed** (respectively **past-directed**) if and only if $\dot{\gamma}(s)$ is in the closed convex hull of $(T_+M)_{\gamma(s)}$ (respectively of $(T_-M)_{\gamma(s)}$) for every $s \in I$. Using this, given $A \subseteq M$ we denote by $I^{\pm}(A)$ the set of all points in M reachable from A by future (respectively past) directed timelike curves. Similarly $J^{\pm}(A)$ denotes the set of all points in M reachable from A by future (respectively past) directed causal curves.

The next condition on a spacetime is one we will almost always assume. It is stronger than the intuitive assumption that no closed timelike curves exist, but we make it nonetheless since it implies well-posedness for the Cauchy problem of any linear wave equation (with smooth coefficients).

Definition 1.1.6. A Lorentzian manifold M is called **globally hyperbolic** if and only if there exists a smooth spacelike hypersurface $\Sigma \subseteq M$ such that every smooth curve $\gamma : I \to M$ which is both inextendible and causal intersects Σ exactly once.

Example 1.1.7. The Schwarzschild and Kerr spacetimes are both globally hyperbolic. The constant-t hypersurfaces are Cauchy in both cases []. de Sitter spacetimes are also globally hyperbolic with Cauchy constant-t hypersurfaces, although the anti de Sitter spacetimes are not globally hyperbolic [**6**].

Below we list a nice consequence of global hyperbolicity. The Lorentzian distance function τ is defined on a spacetime (M, g) by setting $\tau(p, q)$ to be the supremum of all arc-lengths of piecewise smooth future-directed causal curves from p to q, if one such curve exists, and 0 otherwise. Notice that we take a supremum instead of an infimum since the Lorentzian arclength really measures proper time elapsed along a curve and, as such, it is sometimes called the **time separation function** instead.

Theorem 1.1.8. *[**6**] Let (M, g) be connected globally hyperbolic with time separation function $\tau(p, q)$. Then $\tau(p, q) < \infty$ for all $p, q \in M$.*

We are now prepared to introduce standard forms for the metrics we will be considering throughout this thesis. These will also allow us to produce a large class of globally hyperbolic metrics.

Theorem 1.1.9. *[**7**] Let (M^{n+1}, g) be a globally hyperbolic Lorentzian manifold and $\Sigma \subseteq M$ a Cauchy hypersurface. Then (M, g) is isometric to $\mathbb{R}_T \times \Sigma$ with a metric of the* *form*

$$-\alpha^2 dT^2 + k_T \tag{1.1}$$

for some smooth positive function $\alpha : M \to \mathbb{R}_{>0}$ *and some smooth family (in* T*) of Riemannian metrics* k_T *on* Σ*.*

It should be noted that such decompositions are highly non-unique and, given some fixed decomposition as above, not every Cauchy hypersurface in M will be of the form $\{\tau\} \times \Sigma$ for some τ. We will see several examples of this.

Definition 1.1.10. A Lorentzian manifold (M, g) is called **stationary** if and only if it admits a complete timelike Killing vector field Z.

As we are about to see, there are three standard forms for the metric of a globally hyperbolic stationary spacetime, and we will have need to use all three of them. The first arises from choosing a Cauchy hypersurface $\Sigma \subseteq M$ and using the flow of Z to obtain a diffeomorphism between M and a product $\mathbb{R}_t \times \Sigma$ with t being the flow parameter.

Lemma 1.1.11. *[22] Let* (M, g) *be a globally hyperbolic spacetime which is also stationary with respect to a complete timelike Killing vector field* Z*. Let* $\Sigma \subseteq M$ *be a Cauchy hypersurface. Then the flow of* Z *induced a diffeomorphism* $M \cong \mathbb{R}_t \times \Sigma$ *which places the metric* g *in the following standard form*

$$g = -(N^2 - |\eta|_h^2) dt \otimes dt + \eta \otimes dt + dt \otimes \eta + h \tag{1.2}$$

where h is a Riemannian metric on Σ, η a 1-form on Σ and N a smooth positive function on Σ such that $N^2 > |\eta|_h^2$ pointwise. Furthermore all of N, η, h are t-independent, Z becomes identified with ∂_t and each $\{t\} \times \Sigma$ is a Cauchy hypersurface.

We should comment here that the coefficient of dt^2 in the above form is often simply written as N^2 in the literature. Our choice to use $N^2 - |\eta|_h^2$ follows the conventions of [**34**]. One reason for adopting this convention is that we will later be making use of the inverse metric g^{-1} on T^*M more frequently than the metric g and with this convention the inverse metric g^{-1} takes on the simpler form

$$g^{-1} = -N^{-2}\partial_t \otimes \partial_t + N^{-2}\vec{\beta} \otimes \partial_t + N^{-2}\partial_t \otimes \vec{\beta} + \widetilde{h}^{-1}$$

where $\vec{\beta}$ is the vector field on Σ which is h-dual to the one-form η and $\widetilde{h}$ is the metric on Σ given by

$$\widetilde{h} = h + (N^2 - |\eta|_h^2)^{-1}\eta \otimes \eta$$

The inverse metric to $\widetilde{h}$ is easily computed as

$$\widetilde{h}^{-1} = h^{-1} - N^{-2}\vec{\beta} \otimes \vec{\beta}$$

The point of this being that the choice of convention for the dt^2-component of g determines whether one has the simpler expression for g or for g^{-1}.

Instead of using the flow of Z to identify M with a product, we could quotient M by the flow and identify it with the total space of a Riemannian submersion. This gives us our second standard form for globally hyperbolic stationary metrics.

Lemma 1.1.12. *Let (M, g) be a globally hyperbolic spacetime which is also stationary with respect to a complete timelike Killing vector field Z, and let $\Sigma \subseteq M$ be a Cauchy hypersurface. We will make use of N, η, h from 1.1.11. For any $x \in M$ we let $\pi(x) \in \Sigma$ denote the unique point in Σ through which the integral curve of Z through x passes. Then*

$$\pi : (M, g) \to (\Sigma, \widetilde{h})$$

is a Riemannian submersion and if

$$\theta := dt - (N^2 - |\eta|_h^2)^{-1}\eta$$

then

$$g = -(N^2 - |\eta|_h^2)\theta \otimes \theta + \pi^*\widetilde{h}$$

The 1-form θ above defines a connection 1-form for the principal $\mathbb{R}$-bundle $\pi : M \to \Sigma$ with $\mathbb{R}$-action given by the flow of Z. Indeed, by construction

$$\theta = \frac{1}{g(\partial_t, \partial_t)} g(\partial_t, -)$$

so $\theta(Z) = 1$, θ is invariant under the flow of Z, and the set of horizontal vectors

$$\{X \in TM \ : \ \theta(X) = 0\}$$

determined by θ is precisely the g-orthogonal complement to the vertical bundle $\ker \pi_*$.

Let's give a simple application of this standard form 1.1.12. A less-trivial application will be given in the next section.

Theorem 1.1.13. *[1] Let (M, g) be globally hyperbolic stationary with metric in the standard form 1.1.12. Then (M, g) is geodesically complete if and only if $(\Sigma, \widetilde{h})$ is geodesically complete.*

While it is fairly simple to go between the product 1.1.11 and Riemannian submersion 1.1.12 standard forms of a globally hyperbolic stationary spacetime (M, g), it is not so easy to start with a product or Riemannian submersion standard form, and obtain from it the standard form 1.1.9

$$g = -\alpha^2 dT^2 + k_T$$

which we know exists due to (M, g) being globally hyperbolic.

Lemma 1.1.14. *Let (M, g) be globally hyperbolic stationary with metric of the standard form 1.1.11 and let $\Psi_T : \Sigma \to \Sigma$ denote the flow of $\vec{\beta}$. Assume that this flow is complete. Then the map*

$$F : \mathbb{R}_T \times \Sigma \to \mathbb{R}_t \times \Sigma = M$$

$$(T, x) \mapsto (T, \Psi_{-T}(x))$$

satisfies

$$(F^* g)_{(T,x)} = -N(\Psi_{-T}(x))^2 dT^2 + (\Psi^*_{-T} h)_x$$

and so if we set $k_T := \Psi^*_{-T} h$ and $\alpha(T, x) := N(\Psi_{-T}(x))$ then our metric is placed in the standard form 1.1.9 by F.

Remark 1.1.15. If one instead uses geodesic normal flow out of Σ then one can place the metric in the standard form 1.1.9 with $\alpha \equiv 1$.

We end this section with a converse describing when metrics of the form 1.1.11 are globally hyperbolic.

Theorem 1.1.16. *[31] Let $M = \mathbb{R}_t \times \Sigma$ with metric*

$$g = -(N^2 - |\eta|_h^2)dt \otimes dt + \eta \otimes dt + dt \otimes \eta + h$$

for η a smooth t-dependent family of 1-forms on Σ, $N : \mathbb{R}_t \times \Sigma \to \mathbb{R}_{>0}$ a smooth function such that $N^2 > |\eta|_h^2$ pointwise everywhere and h a smooth t-dependent family of Riemannian metrics on Σ. Fix a reference Riemannian metric k on Σ such that

$$h_{t,x}(-,-) = k(\alpha_{t,x}(-),-) \ \text{ for every } (t,x) \in \mathbb{R} \times \Sigma$$

and let $\lambda : M \to \mathbb{R}_{>0}$ denote the continuous function whose value at (t,x) is the minimum eigenvalue of $\alpha_{t,x}$. Suppose there exists a sequence of smooth functions $f_\ell : \Sigma \to \mathbb{R}_{>0}$ such that

- *$k(\eta,\eta) + (\lambda(N^2 - |\eta|_h^2) + |\eta|_k^2)^{1/2} \leq \lambda f_\ell$ pointwise on $[-\ell,\ell] \times \Sigma$, and*
- *the metrics $f_\ell^{-2}k$ on Σ are all complete.*

Then (M,g) is globally hyperbolic and each $\{t\} \times \Sigma$ is a Cauchy hypersurface.

The above theorem greatly simplifies in the case that N, η, h are all t-independent, especially when we allow some compactness assumptions.

Definition 1.1.17. A globally hyperbolic spacetime is called **spatially compact** if and only if it admits a compact Cauchy hypersurface (and therefore all of its Cauchy hypersurfaces are compact).

Corollary 1.1.18. *If Σ is compact then any metric of the standard form 1.1.11 is globally hyperbolic. More generally, instead of assuming Σ is compact we can simply assume that N is uniformly bounded above and away from 0, h is complete, and that η has compact support.*

Proof. We include here for the reader's convenience a specialization of the proof of 1.1.16 from **[31]** to the case where our metric is of the form 1.1.11 with Σ compact since we will use this later in the thesis.

Let $\gamma : I \to M = \mathbb{R} \times \Sigma$ be an arbitrary inextendible causal curve, parametrized with respect to t so that $\gamma(t) = (t, \gamma_0(t))$ for some $\gamma_0 : I \to \Sigma$. Write $I = (a, b)$ for $b \in (-\infty, \infty]$ and $a \in [-\infty, b)$. Since γ is causal it follows from the form 1.1.11 of our metric that

$$-N(\gamma_0(t))^2 + |\eta|_h^2(\gamma_0(t)) + 2\eta(\gamma_0'(t)) + h(\gamma_0'(t), \gamma_0'(t)) \leq 0$$

for all $t \in (a, b)$. Thus by Cauchy-Schwarz we have:

$$|\gamma_0'|_h^2 - 2|\eta|_h|\gamma_0'|_h - N^2 + |\eta|_h^2 \leq 0$$

and thus

$$\left(|\gamma_0'|_h^2 - |\eta|_h^2\right)^2 \leq N^2$$

Suppose for contradiction that $b < \infty$ (the $-\infty < a$ case is completely analogous). Setting

$$C := \sup_{[-|b|-1, b+1] \times \Sigma} (N + |\eta|_h)$$

and noting that $C < \infty$ by our assumptions we then have

$$|\gamma_0'|_h \leq C \quad \text{on} \quad [-|b|-1, b]$$

hence the curve γ must be extendible beyond time b, a contradiction. $\square$

With the above theorem, we obtain a massive collection of examples of globally hyperbolic stationary spacetimes. We'll see in section 2.2 that one has analogues of Riemannian spectral theory for wave and Dirac-type equations on such spacetimes, however we should first verify that these spacetimes are actually of interest in the sense that one has solutions to Einstein's equations among such metrics.

1.2. Geodesics and Curvature of Stationary Spacetimes

Throughout this section, we fix a globally hyperbolic stationary spacetime (M^{n+1}, g) in the standard form $M = \mathbb{R}_t \times \Sigma$,

$$\begin{aligned} g &= -(N^2 - |\eta|_h^2)dt^2 + dt \otimes \eta + \eta \otimes dt + h \\ &= -(N^2 - |\eta|_h^2)\theta \otimes \theta + \widetilde{h} \\ g^{-1} &= -N^{-2}\partial_t \otimes \partial_t + N^{-2}\partial_t \otimes \vec{\beta} + N^{-2}\vec{\beta} \otimes \partial_t + \widetilde{h}^{-1} \end{aligned}$$

with

$$\begin{aligned} \widetilde{h} &= h + (N^2 - |\eta|_h^2)^{-1}\eta \otimes \eta = (h^{-1} - N^{-2}\vec{\beta} \otimes \vec{\beta})^{-1} \\ \theta &= dt - (N^2 - |\eta|_h^2)^{-1}\eta \end{aligned}$$

We follow the usual convention that Greek indices run from 0 to n, Roman indices run from 1 to n, and $x^0 = t$ with x^j local coordinates along Σ.

The geodesic equations in Lorentzian geometry have the same form as those from Riemannian geometry: $\nabla_{\dot\gamma}\dot\gamma = 0$ and so $g(\dot\gamma,\dot\gamma)$ is constant along a geodesic γ. Thus geodesics which are lightlike, spacelike or null at some point will remain lightlike, spacelike or null respectively. One can obtain fairly explicit forms for the geodesic equations on globally hyperbolic stationary spacetimes by explicitly computing the Christoffel symbols in terms of the components of g in the standard form 1.1.11.

Lemma 1.2.1. *Let g have standard form 1.1.11 and denote by ${}^h\Gamma$ the Christoffel symbols of the Riemannian metric h. Then the Christoffel symbols of g are given by:*

$$
\begin{aligned}
\Gamma^a_{ij} &= \frac{1}{2}N^{-2}\beta^a(\partial_j\eta_i + \partial_i\eta_j) - N^{-2}\beta^a\eta_k {}^h\Gamma^k_{ij} + {}^h\Gamma^a_{ij}\\
\Gamma^a_{00} &= \frac{1}{2}(h^{ab} - N^{-2}\beta^a\beta^b)\partial_b(N^2 - |\eta|^2_h)\\
\Gamma^a_{0j} &= \frac{1}{2}(h^{ab} - N^{-2}\beta^a\beta^b)(\partial_j\eta_b - \partial_b\eta_j) - \frac{1}{2}N^{-2}\beta^a\partial_j(N^2 - |\eta|^2_h)\\
\Gamma^0_{j0} &= \frac{1}{2}N^{-2}\partial_j(N^2 - |\eta|^2_h) + \frac{1}{2}N^{-2}\beta^a(\partial_j\eta_a - \partial_a\eta_j)\\
\Gamma^0_{ij} &= -\frac{1}{2}N^{-2}(\partial_j\eta_i + \partial_i\eta_j) + N^{-2}\eta_k {}^h\Gamma^k_{ij}\\
\Gamma^0_{00} &= \frac{1}{2}N^{-2}\beta^i\partial_i(N^2 - |\eta|^2_h)
\end{aligned}
$$

Proof. One simply uses $\Gamma^\gamma_{\mu\nu} = \frac{1}{2}g^{\gamma\theta}(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\mu\theta} - \partial_\theta g_{\mu\nu})$, the explicit descriptions of g, g^{-1} from the start of this section. $\square$

We now list two commonly-used consequences of this when computing with Dirac-type equations on these spacetimes.

Corollary 1.2.2. *On a standard stationary spacetime of the form 1.1.11 one has*

$$\Gamma^0_{j0} - \beta^i \Gamma^0_{ji} = \partial_j \log N \quad \textit{and}$$

$$\Gamma^\mu_{0\mu} = 0$$

(with the second equation simply being a restatement of the fact that Killing fields are divergence-free).

Later when working with Dirac-type equations it will be useful to note that $\Gamma^\gamma_{0\mu}$ vanishing for all γ, μ is equivalent to $N^2 - |\eta|^2_h$ being constant and $d\eta = 0$. These assumptions also allow for a simplification of the geodesic equations.

Suppose that we were given a geodesic in our standard stationary spacetime 1.1.11

$$z : [0,1] \to M \quad \text{with} \quad z(0) = (x^0, \vec{x}),\ z(1) = (y^0, \vec{y}).$$

Since ∂_t is a Killing vector field in our stationary manifolds, the $0'th$ component of the velocity $\dot{z}^0(s)$ is constant in s. Thus our initial and final conditions on $z(s)$ tell us that z has the form

$$z(s) = \big((y^0 - x^0)s + x^0, \gamma(s)\big) \tag{1.3}$$

with $\gamma : [0,1] \to \Sigma$ a curve satisfying $\gamma(0) = \vec{x}$ and $\gamma(1) = \vec{y}$. Making use of the geodesic equations

$$\frac{d^2 z^\mu}{ds^2} + \Gamma^\mu_{\nu\theta} \frac{dz^\nu}{ds} \frac{dz^\theta}{ds} = 0$$

we arrive at the following lemma.

Lemma 1.2.3. *Let $z : [0,1] \to M$ be a geodesic with $z(0) = (x^0, \vec{x})$ and $z(1) = (y^0, \vec{y})$. Then*

$$z(s) = \big((y^0 - x^0)s + x^0, \gamma(s)\big)$$

for a curve $\gamma : [0,1] \to \Sigma$ satisfying $\gamma(0) = \vec{x}$, $\gamma(1) = \vec{y}$ and

$$\begin{aligned}
\frac{d^2\gamma^a}{ds^2} &+ \left(\frac{1}{2}N^{-2}\beta^a(\partial_j \eta_i + \partial_i \eta_j) - N^{-2}\beta^a \eta_k {}^h\Gamma^k_{ij} + {}^h\Gamma^a_{ij}\right) \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} \\
&= \left(N^{-2}\beta^a \partial_j (N^2 - |\eta|^2_h) - (h^{ab} - N^{-2}\beta^a\beta^b)(\partial_j \eta_b - \partial_b \eta_j)\right)(y^0 - x^0)\frac{d\gamma^j}{ds} \\
&\quad + \frac{1}{2}(y^0 - x^0)^2 (h^{ab} - N^{-2}\beta^a\beta^b)\partial_b (N^2 - |\eta|^2_h)
\end{aligned}$$

From this we see that in the case relevant to Dirac-type equations we have a massive simplification:

$$\frac{d^2\gamma^a}{ds^2} + \left(\frac{1}{2}N^{-2}\beta^a(\partial_j \eta_i + \partial_i \eta_j) - N^{-2}\beta^a \eta_k {}^h\Gamma^k_{ij} + {}^h\Gamma^a_{ij}\right) \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} = 0.$$

Before returning to the geodesic equations, we state the results of [**1**] and [**11**] on the curvature tensors of such metrics. To do this it is convenient to switch to the standard form 1.1.12 obtained by interpreting M as the total space of a Riemannian submersion. The following two tensors, Riemannian submersion analogues of the fundamental forms [**28**].

Definition 1.2.4. For (M,g) in the standard form 1.1.12 we define two 2-tensors A, T via their values on horizontal vectors X, Y and the vertical vector ∂_t:

$$T_X :\equiv 0 \qquad\qquad A_{\partial_t} :\equiv 0$$

$$T_{\partial_t}\partial_t := \nabla_{\partial_t}\partial_t - \theta(\nabla_{\partial_t}\partial_t) \qquad\qquad A_X\partial_t := \nabla_X\partial_t - \theta(\nabla_X\partial_t)\partial_t$$

$$T_{\partial_t}X := \theta(\nabla_{\partial_t}X)\partial_t \qquad\qquad A_XY := \theta(\nabla_XY)\partial_t$$

From our computation of the Christoffel symbols of g we immediately have that A is completely expressible in terms of $d\theta$ via:

$$A_XY = -\frac{1}{2}(d\theta)(X,Y)\partial_t \quad \text{and} \quad A_X\partial_t = -\frac{1}{2}(N^2 - |\eta|_h^2)X \lrcorner d\theta$$

meanwhile T is completely expressible in terms of $d(N^2 - |\eta|_h^2)$ via:

$$T_{\partial_t}\partial_t = \frac{1}{2}\nabla(N^2 - |\eta|_h^2) \quad \text{and} \quad T_{\partial_t}X = \frac{1}{2}(N^2 - |\eta|_h^2)^{-1}\left(\mathcal{L}_X(N^2 - |\eta|_h^2)\right)\partial_t$$

For the next result, we note that our conventions for curvature tensors are that $R(X,Y)$ is given by $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, which is the negative of the conventions used in [**11**] and [**28**].

Lemma 1.2.5. *For (M,g) in the standard form 1.1.12 and X, Y, V, W horizontal vector fields we have*

$$\begin{aligned} g(R_g(X,\partial_t)Y,\partial_t) = &-(N^2 - |\eta|_h^2)\theta(\nabla_{\partial_t}X)\theta(\nabla_{\partial_t}(Y)) - g(A_X\partial_t, A_Y\partial_t) \\ &- g((\nabla_XT)_{\partial_t}\partial_t, Y) - g((\nabla_{\partial_t}A)_XY, \partial_t) \end{aligned}$$

$$g(R_g(X,Y)V,\partial_t) = -(N^2-|\eta|_h^2)\left[\theta(\nabla_Y V)\theta(\nabla_{\partial_t}X)+\theta(\nabla_V X)\theta(\nabla_{\partial_t}Y)-\theta(\nabla_X Y)\theta(\nabla_{\partial_t}V)\right]$$
$$- g((\nabla_V A)_X Y,\partial_t)$$
$$g(R_g(X,Y)V,W) = g(R_{\widetilde{h}}(X,Y)V,W)$$
$$+ (N^2-|\eta|_h^2)\big[\theta(\nabla_V X)\theta(\nabla_Y W)+\theta(\nabla_Y V)\theta(\nabla_X W)$$
$$- 2\theta(\nabla_X Y)\theta(\nabla_V W)\big]$$

From the previous two lemmas the covariant derivatives of T, A and then the Ricci tensor of g are computed by [**11**]. It should be noted that I follow the conventions of [**29**] for the definition of the sectional and Ricci tensors whereas [**11**] follows the negative of these conventions. Combined with their opposite sign on the full curvature tensor, we obtain the same result as [**11**] for the Ricci tensor as the sign differences cancel.

Corollary 1.2.6. *For g in the standard form 1.1.12 and for X, Y horizontal we have:*

$$\mathrm{Ric}_g(\partial_t,\partial_t) = \frac{1}{2}\Box_g(N^2-|\eta|_h^2) - \frac{1}{4}(N^2-|\eta|_h^2)^{-1}|d(N^2-|\eta|_h^2)|_{\widetilde{h}}^2 + \frac{1}{2}(N^2-|\eta|_h^2)^2|d\theta|_{\widetilde{h}}^2$$
$$\mathrm{Ric}_g(X,\partial_t) = \frac{1}{2}(N^2-|\eta|_h^2)\left[(d^*d\theta)(X) + \frac{3}{2}(N^2-|\eta|_h^2)^{-1}(d\theta)(\nabla(N^2-|\eta|_h^2),X)\right]$$
$$\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_{\widetilde{h}}(X,Y) - (N^2-|\eta|_h^2)^{-1/2}\,\mathrm{Hess}_g((N^2-|\eta|_h^2)^{1/2})(X,Y)$$
$$+ \frac{1}{2}(N^2-|\eta|_h^2)\widetilde{h}^{-1}(X\llcorner d\theta, Y\llcorner d\theta)$$

We should remark here on the appearance of both metrics h and $\widetilde{h}$ on Σ in the above. When taking norms of vectors, multivectors, etc. using g, it follows from 1.1.11 that if these vectors are tangent to Σ then the g-norm of the vector is equal to its h-norm. However, from our expression for g^{-1} we see that the g-norm of a differential form on Σ

is instead equal to its $\widetilde{h}$ norm. One has to take great care, however, since $\vec{\beta}$ is defined as the h-dual of the 1-form η, not the g-dual, and so $\vec{\beta} \llcorner \eta = |\eta|_h^2$ not $|\eta|_{\widetilde{h}}^2$.

Corollary 1.2.7. *For (M, g) in the standard for 1.1.12 the scalar curvature is given by*

$$\mathrm{Scal}_g = \mathrm{Scal}_{\widetilde{h}} + \frac{3}{2}(N^2 - |\eta|_h^2)|d\theta|_{\widetilde{h}}^2$$

Proof. Since the vertical and horizontal subspaces are orthogonal with respect to g, we complete $(N^2 - |\eta|_h^2)^{-1/2}\partial_t$ to a local orthonormal frame by locally choosing an orthonormal frame $e_1, ..., e_n$ for the horizontal subbundle. $\square$

These equations all greatly simplify in the case relevant to Dirac-type operators we will see later: $d(N^2 - |\eta|_h^2) = 0 = d\eta$.

Corollary 1.2.8. *Suppose (M, g) is in standard form 1.1.12 and we additionally have $d\eta = 0$ and $N^2 - |\eta|_h^2$ constant. Then we also have $d\theta = 0$, $\mathrm{Ric}_g(\partial_t, -) = 0$ and for X, Y horizontal we have $\mathrm{Ric}_g(X, Y) = \mathrm{Ric}_{\widetilde{h}}(X, Y)$. Thus the Einstein tensor $G = \mathrm{Ric}_g - \frac{1}{2}\mathrm{Scal}_g \cdot g$ of (M, g) is given by*

$$\mathrm{Ric}_{\widetilde{h}} - \frac{1}{2}\mathrm{Scal}_{\widetilde{h}} \cdot \widetilde{h} + \frac{1}{2}(N^2 - |\eta|_h^2)\theta \otimes \theta$$

We now state the main theorem of **[1]**, **[11]**. Later we will almost exclusively consider spatially compact spacetimes and so, by the below theorem, we cannot expect them to solve the vacuum Einstein equations. Nevertheless, we are studying matter fields in these spacetimes so perhaps one should not expect them to solve these equations.

Theorem 1.2.9. *[1] [11] Let (M, g) be a 4-dimensional stationary globally hyperbolic spacetime in the standard form 1.1.12 and assume that $(N^2 - |\eta|_h^2)\widetilde{h}$ is a complete Riemannian metric on Σ. If g solves the vacuum Einstein equations then $d\theta = 0$, $N^2 - |\eta|_h^2$ is constant and $\widetilde{h}$ is flat.*

Returning to our geodesic equations, we will need a Hamiltonian/symplectic description of the dynamics of geodesics in (M, g). Recall that the relativistic description of the phase space of a system is simply the space of solutions to the equations of motion, and the identification with a cotangent bundle arises from the equations typically being second order ODE and so solutions correspond to initial data. For this reason we introduce the following.

Definition 1.2.10. Let

$$\mathcal{N}_a := \{ \text{ all future-directed, inextendible null geodesics in } (M, g)\}$$

The subscript "a" stands for "affinely parametrized" and it is a common convention (see **[34]** for example) to include it. For us, geodesics are defined as solutions to $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ hence they are already "affinely parametrized".

Since M is globally hyperbolic and the elements of $\mathcal{N}_a$ are in particular inextendible causal curves they necessarily intersect each Cauchy hypersurface exactly once. An invariant way of dealing with the fact that such curves γ need not have $\gamma(0)$ in our chosen Cauchy hypersurface is to define

$$\mathcal{N} := \mathcal{N}_a/\mathbb{R}$$

where $b \in \mathbb{R}$ acts on $\mathcal{N}_a$ by $\gamma(s) \mapsto \gamma(s+b)$. Notice additionally that the $\mathbb{R}_{>0}$-action on $\mathcal{N}_a$ where $a \in \mathbb{R}_{>0}$ acts by $\gamma(s) \mapsto \gamma(as)$ descends to an $\mathbb{R}_{>0}$-action on the quotient $\mathcal{N}$.

The above set $\mathcal{N}$ is naturally a symplectic manifold and one can view the next lemma as saying that there are $\mathbb{R}_{>0}$-equivariant Cauchy data symplectomorphisms between $\mathcal{N}$ and a cotangent bundle. Instead, we will simply take the next lemma as the definition of the smooth manifold and symplectic structures on $\mathcal{N}$. For this, we will need the following remark.

Remark 1.2.11. Recall that T_0^*M is the sub-cone-bundle of $T^*M \setminus 0$ consisting of all null covectors and, since our spacetimes are assumed to be time oriented, we have a decomposition

$$T_0^*M = T_+^*M \sqcup T_-^*M$$

into the cone subbundles $T_\pm^*M \subseteq T^*M \setminus 0$ of future/past directed null covectors. If we fix a Cauchy hypersurface $\Sigma \subseteq M$ then, as mentioned in **[34]**, there are natural isomorphisms of bundles over Σ:

$$T^*\Sigma \setminus 0 \cong T_+^*M|_\Sigma \quad \text{and} \quad T^*\Sigma \setminus 0 \cong T_-^*M|_\Sigma$$

given by $T^*\Sigma \setminus 0 \ni \zeta \mapsto \zeta \pm \sqrt{g(\zeta,\zeta)}\widehat{\nu}$ where $\widehat{\nu}$ is the future-directed unit conormal to Σ. These are homogeneous symplectomorphisms when $T_\pm^*M|_\Sigma$, a submanifold of $T^*M \setminus 0$, is given the pullback of the symplectic form on $T^*M \setminus 0$.

Lemma 1.2.12. *Let (M,g) be a globally hyperbolic spacetime with $\Sigma \subseteq M$ a Cauchy hypersurface. Each equivalence class in $\mathcal{N}$ has a unique representative $\gamma : \mathbb{R} \to M$ satisfying $\gamma(0) \in \Sigma$. Identifying elements of $\mathcal{N}$ with these representatives gives us an*

$\mathbb{R}_{>0}$*-equivariant bijection*

$$\mathcal{N} \xrightarrow{\cong} T_+^*M|_\Sigma$$

$$\gamma \mapsto (\gamma(0), \dot{\gamma}(0))$$

where $\dot{\gamma}(0)$ *is identified with a covector by* g*.*

Proof. This follows immediately from the definition of a future-directed inextendible null geodesic and the existence and uniqueness of solutions to ODE. □

Occasionally we will need the explicit form of the inverse of the above bijection and so we introduce the following.

Definition 1.2.13. Let G_s denote the Hamiltonian flow on $T^*M \setminus 0$ of the Hamiltonian $\zeta \mapsto \frac{1}{2}g^{-1}(\zeta, \zeta)$. The restriction of G_s to T_0^*M is called the **null bicharacteristic flow**.

The inverse of our bijection $\mathcal{N} \cong T_+^*M|_\Sigma$ can now be expressed as mapping $\zeta \in T_+^*M|_\Sigma$ to the projection of the curve $s \mapsto G_s(\zeta)$ down to M.

1.3. Kaluza-Klein Spacetimes

In [**38**], the classical limit of a massive quantum particle in an external classical Yang-Mills field was determined to satisfy the equations of motion

$$m\ddot{x}^\mu = \mathrm{Tr}(qF_A^{\mu\nu})\dot{x}_\nu, \quad \dot{x}^2 = -1$$

where $q \in \mathfrak{u}(k)$ is a conserved quantity describing the internal degrees of freedom of the system (a generalization of charge). This is an analogue of the Lorentz force law, generalized to connections with structure groups G other than $\mathrm{U}(1)$. Some references for the study of these equations on curved non-relativistic space are [**33**], [**37**]. Of relevance to us is that these are precisely the geodesic equations on the total space of a principal G-bundle over spacetime when it is equipped with a special metric built from the metric on the base spacetime and the connection A.

Throughout this section we fix a spacetime (M, g), $G \subseteq \mathrm{SO}(k)$ a compact Lie group and $\pi : M \to G$ a principal G-bundle.

Definition 1.3.1. Recalling that $(X, Y) \mapsto -\operatorname{Tr}(XY)$ is a positive definite Ad-invariant inner product on the Lie algebra $\mathfrak{g} \subseteq \mathfrak{so}(k)$ of G, given any connection 1-form ω on P we obtain an induced **Kaluza-Klein metric**:

$$g_\omega := \pi^* g - \operatorname{Tr}(\omega(-)\omega(-))$$

on the total space P. This is again a Lorentzian metric of signature $(-1, +1, ..., +1)$. We endow (P, g_ω) with the time orientation induced by that of (M, g). Given $\xi \in \mathfrak{g}$ we denote by $\widehat{\xi}$ the vertical vector field on P induced by ξ.

Since the spacetimes we are considering in this thesis tend to be topologically products $\mathbb{R} \times \Sigma$, so too will be the case for our principal bundles. An interesting example of such a bundle arises by taking M to be the Schwarzschild spacetime with $n = 3$ so that

$$M = \mathbb{R}_t \times ((2m)^{2-n}, \infty)_r \times S^2$$

and to take the pullback of the Hopf fibration $S^3 \to S^2$ as our principal bundle. $G = \mathrm{U}(1)$ in this case.

Lemma 1.3.2. *Suppose (M, g) is a stationary spacetime with complete timelike Killing vector field Z, $\pi : P \to M$ a principal G-bundle with connection ω and Z^ω the horizontal lift of Z to P with respect to ω. Assuming the curvature F_ω of ω satisfies $Z \llcorner F_\omega = 0$, it follows that Z^ω is complete timelike Killing for g_ω and furthermore:*

$$[Z^\omega, \widehat{\xi}] = 0 \quad \textit{for all} \quad \xi \in \mathfrak{g}.$$

Proof. The flow of Z^ω on P is the horizontal lift of the flow of Z on M (see [**8**] section 10.1, for example), thus the completeness of Z^ω follows immediately from the completeness of Z. Since Z is Killing for g it follows that $\pi^* g$ is invariant under the flow of Z^ω and $Z \llcorner F_\omega$ implies

$$\mathcal{L}_{Z^\omega} \operatorname{Tr}(\omega(-)\omega(-)^T) = 0$$

thus Z^ω is indeed Killing for g_ω. Finally, Z^ω is invariant under pushforward along the G-action (see [**8**] section 2.2, for example) and therefore $[Z^\omega, \widehat{\xi}] = 0$ for all $\xi \in \mathfrak{g}$ as well. $\square$

Combining the above lemma with the corollary 1.1.18 we obtain the result that, if M is stationary globally hyperbolic and $Z \llcorner F_\omega = 0$, then so is P.

Lemma 1.3.3. *Suppose (M, g) is stationary spatially compact globally hyperbolic with complete timelike Killing vector field Z and $\pi : P \to M$ a principal G-bundle with connection ω such that $Z \llcorner F_\omega = 0$. Then (P, g_ω) is stationary globally hyperbolic with complete* *timelike Killing vector field Z^ω. If M has standard form 1.1.11 then each $\pi^{-1}(\{t\} \times \Sigma)$ is a Cauchy hypersurface for (P, g_ω).*

Proof. Place (M, g) in the standard form 1.1.11, so $M = \mathbb{R}_t \times \Sigma$, and write $P_t := \pi^{-1}(\{t\} \times \Sigma)$. Flowing along Z^ω induces a diffeomorphism

$$\mathbb{R}_t \times P_0 \to P$$

intertwining the right G-actions. We abuse notation and continue to use the notation ω for the pullback of the connection ω along this flow map and note that $\omega(Z^\omega) = 0$ implies that the pullback of ω has been placed in temporal gauge with respect to $\mathbb{R}_t \times P_0$ and $0 = Z \llcorner F_\omega = \partial_t \omega$. Thus our metric g_ω pulls back to the metric

$$-\left((N \circ \pi)^2 - |\pi^*\eta|^2_{\pi^*h}\right) dt^2 + dt \otimes (\pi^*\eta) + (\pi^*\eta) \otimes dt + \pi^*h - \mathrm{Tr}(\omega(-)\omega(-))$$

Since Σ, G are assumed to be compact, so is P_0 and therefore the assumptions of 1.1.18 are satisfied. Hence g is isometric to the above globally hyperbolic metric and is therefore globally hyperbolic with Cauchy hypersurfaces $\pi^{-1}(\{t\} \times \Sigma)$ corresponding to the Cauchy hypersurfaces $\{t\} \times P_0$ of the above metric obtained from 1.1.18. $\square$

As we mentioned at the start of this section, Wong's equations for massive particles in a background Yang-Mills field can be expressed in terms of the geodesic equations on a Kaluza-Klein spacetime. Despite describing massive particles, the relevant type of geodesics in (P, g_ω) will be the null geodesics, as the below lemma explains.

Lemma 1.3.4. *Let γ be a geodesic in a Kaluza-Klein spacetime (P, g_ω). Then the value of $\omega(\dot\gamma(s))$ is constant. Furthermore its projection $\pi \circ \gamma$ to M has $g((\pi \circ \gamma)', (\pi \circ \gamma)')$*

constant and if γ is causal then $\pi \circ \gamma$ is timelike unless γ is both null and horizontal, in which case $\pi \circ \gamma$ is null.

Proof. That $\omega(\dot{\gamma})$ is constant can be found as theorem 10.1.5 in [**8**]. Since γ is a geodesic it follows that

$$g_\omega(\dot{\gamma}, \dot{\gamma}) = g((\pi \circ \gamma)', (\pi \circ \gamma)') - \operatorname{Tr}(\omega(\dot{\gamma})\omega(\dot{\gamma}))$$

is also constant hence $g((\pi \circ \gamma)', (\pi \circ \gamma)')$ is constant. Finally, since $-\operatorname{Tr}(\omega(\dot{\gamma})\omega(\dot{\gamma})) \geq 0$ (and is zero if and only if $\omega(\dot{\gamma}) = 0$) we see that

$$g_\omega(\dot{\gamma}, \dot{\gamma}) \leq 0 \;\text{ implies } g((\pi \circ \gamma)', (\pi \circ \gamma)') \leq 0$$

thus causal curves in (P, g_ω) project to causal curves in (M, g), with the projection being null if and only if $\omega(\dot{\gamma}) = 0$ and γ is null. $\square$

Instead of writing-out the full computation of the Christoffel symbols of g_ω in terms of g and ω we simply defer to theorem 10.1.6 in [**8**] for the proof of the below corollary, although we have stated it using our notational conventions.

Corollary 1.3.5. *Let γ be a null geodesic in (P, g_ω) with projection $x := \pi \circ \gamma$, a curve in M, and $q \in x^* \operatorname{Ad}(P)$ the section induced by the constant $\omega(\dot{\gamma})$ along γ. Let A denote* *the connection on* $\mathrm{Ad}(P)$ *induced by* ω*. Then the curve* x *in* M *satisfies*

$$\begin{cases} \nabla_{\dot{x}}\dot{x} & = & -\dot{x} \llcorner \mathrm{Tr}(qF_A) \\ (x^*\nabla^A)q & = & 0 \\ g(\dot{x},\dot{x}) & = & \textit{constant} \end{cases}$$

If $M = \mathbb{R}_t \times \mathbb{R}^n$ *is flat Minkowski space together with the trivial principal* G*-bundle* $P = M \times G$ *then these are precisely Wong's equations.*

Given a null geodesic γ in (P, g_ω), we would like to think of the constant $\omega(\dot{\gamma})$ as the "charge". Unfortunately, unlike the abelian case of the Lorentz force law, different lifts of solutions to Wong's equations in M to geodesics in (P, g_ω) will have different charges. Indeed, if the two lifts of our curve in M are related by the right action of $g \in G$ on P then the charges of the two lifts will be related by Ad_g. Identifying the charge q with $-\mathrm{Tr}(q(-)) \in \mathfrak{g}^*$ we arrive at the following gauge invariant definition of charge.

Definition 1.3.6. Let γ be a null geodesic in (P, g_ω) and $\xi_0 := -\mathrm{Tr}(\omega(\dot{\gamma})(-)) \in \mathfrak{g}^*$. The **charge** of γ is defined to be the coadjoint orbit:

$$\mathcal{O} := \{\mathrm{Ad}_g^* \xi_0 \ : \ g \in G\} \subseteq \mathfrak{g}^*.$$

Just as in the flat case, Wong's equations on a curved spacetime will arise as classical limits of the quantum system. One consequence of this will be *charge quantization.*

For now, let's proceed to the Hamiltonian description of the dynamics of these null geodesics. Similar to our earlier discussion of geodesics in M, the following results and definitions can been seen as relativistic versions of the results on the phase space for Wong's equations from [**33**], [**37**].

Definition 1.3.7. We again denote by G_s the Hamiltonian flow of the Hamiltonian $\xi \mapsto \frac{1}{2} g_\omega^{-1}(\xi, \xi)$, now on $T^*P \setminus 0$. Its restriction to T_0^*P will again be called the null bicharacteristic flow.

Lemma 1.3.8. *Let Φ_s^Z and Φ_s^ξ respectively denote the flows on $T^*P \setminus 0$ given by the derivatives of the flows of Z^ω and $\widehat{\xi}$ ($\xi \in \mathfrak{g}$) on P. Then Φ_s^Z and Φ_s^ξ commute with G_s for every $\xi \in \mathfrak{g}$.*

Proof. Since Φ_s^Z and Φ_s^ξ are derivatives of flows on P they are a 1-parameter family of canonical transformations on $T^*P \setminus 0$ and therefore, by the Hamiltonian version of Noether's theorem, it suffices to show that the Hamiltonian $\xi \mapsto \frac{1}{2} g_\omega^{-1}(\xi, \xi)$ is invariant under the flows Φ_s^Z, Φ_s^ξ in order to prove that they commute with G_s. But this is immediate from both Z^ω and $\widehat{\xi}$ being Killing vector fields for the metric g_ω. $\square$

Recall now the definitions of $\mathcal{N}_a, \mathcal{N}$ from section 1.2. We will continue to use this notation, only our manifold (M, g) will now be replaced by (P, g_ω).

Lemma 1.3.9. *$g \in G$ has a right action on $\mathcal{N}$ induced by its right action on $\mathcal{N}_a$ given by $\gamma(s) \mapsto \gamma(s)g$ (via the right action on P). The bijection in 1.2.12 intertwines this right action with the right action on $T_+^*P|_{P_0}$ given by dualizing (using g_ω) the action of pushing forward by right multiplication by g on P.*

Proof. This is immediate from the explicit form of our isomorphism $\mathcal{N} \cong T_+^*P|_{P_0}$ and the fact that G acts by isometries and therefore leaves $T_+^*P|_{P_0}$ invariant. $\square$

Lemma 1.3.10. *The flows Φ_s^Z and Φ_s^ξ on $\mathcal{N}$ induced by 1.2.12 are Hamiltonian flows with respective Hamiltonians:*

$$H_Z(\gamma) = Z^\omega \llcorner (\gamma(0), \dot\gamma(0)) \quad \textit{and} \quad H_\xi(\gamma) = \widehat{\xi} \llcorner (\gamma(0), \dot\gamma(0))$$

where again we have chosen representative geodesics γ with $\gamma(0) \in P_0$. Furthermore, the Φ_s^ξ's arise (through the exponential map) from the natural right-action of G on $\mathcal{N}$ hence this G-action is Hamiltonian.

As the above right G-action is Hamiltonian, we can consider its **moment-map**:

$$\mu : \mathcal{N} \to \mathfrak{g}^*$$

$$\langle \mu(\gamma), \xi \rangle = H_\xi(\gamma).$$

Lemma 1.3.11. *Under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by our* Ad*-invariant inner product on $\mathfrak{g}$, the moment map is given by*

$$\gamma \mapsto \omega(\dot\gamma).$$

Proof. We know that $\omega(\widehat{\xi}) = \xi$ by the definition of a connection on a principal bundle and so the result follows from

$$\widehat{\xi} \llcorner (\gamma(0), \dot\gamma(0)) = \mathrm{Tr}(\omega(\widehat{\xi})\omega(\dot\gamma)^T)$$

since we're using g_ω to identify $\dot\gamma(0)$ with a covector. $\square$

As a final remark before we discuss the reduced phase space, we notice that if we express our Hamiltonian H_Z as a function on $T^*P_0 \setminus 0$ via the isomorphism $T^*P_0 \setminus 0 \cong T^*_+P|_{P_0}$ it is given by

$$H_Z(\zeta) = \zeta(Z^\omega) + \sqrt{g_\omega(\zeta,\zeta)}\widehat{\nu}(Z^\omega)$$

We can calculate this more explicitly using the standard form for the metric g_ω with respect to $P \cong \mathbb{R}_t \times P_0$ via the flow of Z^ω. Indeed, here the future-directed unit normal to P_0 is simply the future-directed unit normal

$$\widehat{n} = N^{-1}(\partial_t - \vec{\beta})$$

to Σ, lifted horizontally to P. Thus, as $\widehat{\nu}$ is the future-directed unit conormal, we have

$$\widehat{\nu}(Z^\omega) = -g_\omega(\widehat{n}, Z^\omega) = N$$

Since

$$|\zeta(Z^\omega)| \leq \sqrt{g_\omega(\zeta,\zeta)}(N^2 - |\eta|_h^2)^{1/2}$$

and

$$N - (N^2 - |\eta|_h^2)^{1/2} = \sqrt{N^2} - (N^2 - |\eta|_h^2)^{1/2} > 0 \text{ pointwise everywhere}$$

we see that P_0 being compact implies H_Z is uniformly bounded away from zero. If one calculates the fiberwise Hessian of H_Z to be uniformly positive definite then occasionally one can guarantee the existence of periodic orbits, hence why we are about to spend so much time analyzing them.

1.3.1. The Reduced Phase Space

Fix a charge, i.e. a coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$. We now wish to form the symplectically reduced phase space of solutions with charge $\mathcal{O}$. The construction of this in Riemannian signature, and its relationship to Wong's equations can be found in [**15**] and it generalizes with almost no modifications to our setting.

Recall that our coadjoint orbit $\mathcal{O}$ is naturally a symplectic manifold. The symplectic form $\omega_{\mathcal{O}}$ can be defined as follows. Fix $\xi_0 \in \mathcal{O}$ and let G_{ξ_0} denote the stabilizer of ξ_0 under the coadjoint action. Then

$$G \to \mathcal{O}$$

$$g \mapsto \operatorname{Ad}_g^* \xi_0$$

induces an isomorphism

$$G/G_{\xi_0} \cong \mathcal{O}$$

which identifies

$$T_{\xi_0}\mathcal{O} \cong \mathfrak{g}/\mathfrak{g}_{\xi_0}$$

where $\mathfrak{g}_{\xi_0}$ is the Lie algebra of G_{ξ_0}. The other tangent spaces of $\mathcal{O}$ are also identified with $\mathfrak{g}/\mathfrak{g}_{\xi_0}$ by pushforward along the G-action. We then have:

$$\omega_{\mathcal{O}}(X, Y) = \langle \xi_0, [X, Y] \rangle.$$

We notice that this is well-defined on $\mathfrak{g}/\mathfrak{g}_{\xi_0}$ since

$$\mathfrak{g}_{\xi_0} = \{X \in \mathfrak{g} \ : \ \langle \xi_0, [X,Y] \rangle = 0 \ \text{ for all } Y \in \mathfrak{g}\}.$$

Let $\overline{\mathcal{O}}$ denote $\mathcal{O}$ but equipped with $-\omega_{\mathcal{O}}$ as its symplectic form instead of $\mathcal{O}$.

Lemma 1.3.12. *The extended moment map*

$$\mu_{\mathcal{O}} : \mathcal{N} \times \overline{\mathcal{O}} \to \mathfrak{g}^*$$

$$\mu_{\mathcal{O}}(\gamma, \xi) := \mu(\gamma) - \xi$$

is a submersion and G acts freely on $\mu_{\mathcal{O}}^{-1}(0)$.

Proof. The fact that G acts freely on $\mu_{\mathcal{O}}^{-1}(0)$ simply follows from G acting freely on $\mathcal{N} \cong T^*_+P|_{P_0}$ since P, P_0 are principal G-bundles. To see that $\mu_{\mathcal{O}}$ is a submersion, we notice that under the isomorphism $\mathcal{N} \cong T^*_+P|_{P_0}$ we have

$$\mu_{\mathcal{O}} : T^*_+P|_{P_0} \times \overline{\mathcal{O}} \to \mathfrak{g}^*$$

$$(\zeta, \xi) \mapsto \mathrm{Tr}(\omega(\zeta)^T \omega(-)) - \xi$$

and if we use our Ad-invariant inner product to identify $\mathfrak{g} \cong \mathfrak{g}^*$ then this maps

$$(\zeta, \xi) \mapsto \omega(\zeta) - \xi.$$

Forgetting ξ we can already see that $\zeta \mapsto \omega(\zeta)$ is a submersion (and therefore $\mu_{\mathcal{O}}$ is a submersion). Indeed, it suffices to prove that for every $\xi \in \mathfrak{g}$ there exists $\zeta \in T^*_+P|_{P_0}$ such that $\omega(\zeta) = \xi$. However, $\widehat{\xi}$ is tangent to P_0 with $g_\omega(\widehat{\xi}, \widehat{\xi}) = \mathrm{Tr}(\xi\xi^T)$ so $\zeta := \sqrt{\mathrm{Tr}(\xi\xi^T)}\widehat{n} + \widehat{\xi}$ is future-directed, has $\omega(\zeta) = \xi$ and $g_\omega(\zeta, \zeta) = 0$ as desired. $\square$

From the above proof we record as a remark the fact that $\mu_{\mathcal{O}}^{-1}(0)$ is precisely the space of pairs (γ, ξ) where $\gamma \in \mathcal{N}$ and $\xi \in \mathcal{O}$ satisfy

$$\mathrm{Tr}(\omega(\dot{\gamma})^T(-)) = \xi.$$

This $\mu_{\mathcal{O}}^{-1}(0)$ is precisely the space of solutions with charge $\mathcal{O}$, prior to quotienting by gauge transformations.

Definition 1.3.13. The **reduced phase space** is

$$\mathcal{N}_{\mathcal{O}} := \mu_{\mathcal{O}}^{-1}(0)/G$$

with symplectic form obtained from the one on $\mathcal{N} \times \overline{\mathcal{O}}$.

Lemma 1.3.14. *The Hamiltonian H_Z, extended to $\mathcal{N} \times \overline{\mathcal{O}}$ to be independent of $\overline{\mathcal{O}}$, is invariant under the G-action and therefore descends to a Hamiltonian $\widetilde{H}_Z$ on $\mathcal{N}_{\mathcal{O}}$ with flow $\widetilde{\Phi}^Z_s$.*

Proof. From the definition of H_Z we see that what we have to show is that $g_\omega(Z^\omega, \dot{\gamma} \cdot g) = g_\omega(Z^\omega, \dot{\gamma})$ for all $g \in G$ and $\gamma \in \mathcal{N}$. However:

$$g_\omega(Z^\omega, \dot{\gamma} \cdot g) = g_\omega(Z^\omega \cdot g^{-1}, \dot{\gamma}) = g_\omega(Z^\omega, \dot{\gamma})$$

since $Z^\omega = \partial_t$ is invariant under the G action. $\square$

The point of the previous construction is its manifestly gauge-invariant nature. Below we give an alternative characterization that might be more familiar to some readers, although we will not use it in our proof.

Fix $\xi_0 \in \mathcal{O}$ and recall from our proof that $\mu_{\mathcal{O}}$ is a submersion that μ is also a submersion, hence ξ_0 is automatically a regular value. Furthermore, while the full G-action on $\mathcal{N}$ doesn't preserve the submanifold $\mu^{-1}(\xi_0)$, it is preserved by the action of the stabilizer G_{ξ_0} of ξ_0. The action of G_{ξ_0} on $\mu^{-1}(\xi_0)$ is free since the action of G on $\mathcal{N}$ is free.

Definition 1.3.15. The **reduced phase space** (version II) is the quotient

$$\mu^{-1}(\xi_0)/G_{\xi_0}$$

with the symplectic form induced from that on $\mathcal{N}$.

Lemma 1.3.16. *[17] The map*

$$\mu^{-1}(\xi_0) \to \mathcal{N}_{\mathcal{O}}$$
$$\gamma \mapsto [(\gamma, \xi_0)]$$

induces a symplectomorphism $\mu^{-1}(\xi_0)/G_{\xi_0} \cong \mathcal{N}_{\mathcal{O}}$ intertwining the reductions of the Hamiltonian flow of H_Z to $\mu^{-1}(\xi_0)/G_{\xi_0}$ and $\mathcal{N}_{\mathcal{O}}$. Here $[(\gamma, \xi_0)]$ denotes the equivalence class of (γ, ξ_0) in the quotient.

Finally, let's note that since M is assumed to be spatially compact we expect the quantum system to have discrete spectrum and hence bound states. The leading order singularities in our distributional trace of the propagator will be therefore expressed as a sum over classical bound states: periodic orbits of null geodesics under $\widetilde{\Phi}_s^Z$. There are two aspects of these periodic orbits we will need to consider:

(1) the *(linearized) Poincaré first return map* of a periodic orbit, and

(2) the phase change due to a periodic orbit similar to the Aharonov-Bohm effect.

The first of these points relates to the classical dynamics of periodic orbits, while the second of these is only relevant for the quantum effects we will discuss later.

Following **[34]**, we fix an energy $E \in \mathbb{R}$ and restrict ourselves to the contact manifold given by the level surface

$$\widetilde{H}_Z^{-1}(E) \subseteq \mathcal{N}_{\mathcal{O}}.$$

This is invariant under the $\widetilde{\Phi}_s^Z$-flow and so we can define the set of **periods**:

$$\mathcal{P}_E := \{T \in \mathbb{R} \setminus \{0\} \ : \ \exists z \in \widetilde{H}_Z^{-1}(E) \ \text{ such that } \ \widetilde{\Phi}_T^Z(z) = z\}$$

and, for $T \in \mathcal{P}_E$, the set of **periodic points**:

$$\mathcal{P}_{E,T} := \{z \in \widetilde{H}_Z^{-1}(E) \ : \ \widetilde{\Phi}_T^Z(z) = z.\}$$

We say that $T > 0$ is the **minimum period** of z if and only if it is the smallest positive time for which $\widetilde{\Phi}_T^Z(z) = z$. The below result is a general fact concerning Hamiltonian dynamics and is a simple consequence of the implicit function theorem.

Lemma 1.3.17. *([25] Prop 8.5.3)*

Given a periodic point $z_0 \in \mathcal{P}_{E,T}$ where T is its minimum period there exists, in a sufficiently small neighborhood of z_0, a codimension 1 symplectic submanifold

$$z_0 \in S \subseteq \widetilde{H}_Z^{-1}(E)$$

*which is transverse to the flow $\widetilde{\Phi}_s^Z$. Furthermore, in a sufficiently small neighborhood of z_0 in S, the **first return time***

$$\mathcal{T}(z) := \min\{t > 0 \ : \ \widetilde{\Phi}_t^Z(z) \in S\}$$

is well-defined, smooth and satisfies $\mathcal{T}(z_0) = T$.

Definition 1.3.18. With z_0, S, T as above, we define the **linearized Poincaré first return map** to be

$$P_{z_0,S} := \frac{\partial}{\partial z}\Big|_{z=z_0} \widetilde{\Phi}^Z_{\mathcal{T}(z)}(z) : T_{z_0}S \to T_{z_0}S.$$

This is a linear symplectic map. For any other choice of local symplectic transversal S' there is a linear symplectic isomorphism

$$L : T_{z_0}S' \xrightarrow{\cong} T_{z_0}S$$

such that

$$P_{z_0,S'} = L^{-1} \circ P_{z_0,S} \circ L.$$

There is actually an alternate, perhaps simpler, description of these maps $P_{z_0,S}$. This alternate description is analogous to the more standard definition of the linearized Poincaré first return map for geodesic flow on Riemannian or Lorentzian manifolds, which is usually defined with the aid of Jacobi fields.

Definition 1.3.19. Given $z_0 \in \mathcal{P}_{E,T}$ with T the minimum period of z_0, we define the **Floquet operator** of z_0 to be:

$$V_{z_0}(T) := \frac{d}{dz}\Big|_{z=z_0} \widetilde{\Phi}^Z_T(z) : T_{z_0}\mathcal{N}_{\mathcal{O}} \to T_{z_0}\mathcal{N}_{\mathcal{O}}.$$

Lemma 1.3.20. *The subspace*

$$W_{z_0} := \mathrm{Span}\{\widetilde{Z}(z_0),\ \nabla\widetilde{H}(z_0)\}$$

is symplectic, as is the quotient $T_{z_0}\mathcal{N}_{\mathcal{O}}/W_{z_0}$, and W_{z_0} is preserved by the Floquet operator. The induced quotient map

$$V_{z_0}(T) : T_{z_0}\mathcal{N}_{\mathcal{O}}/W_{z_0} \to T_{z_0}\mathcal{N}_{\mathcal{O}}/W_{z_0}$$

is conjugate via a linear symplectomorphism to the linearized Poincaré first return map.

Let's discuss for some time the significance of these operators to us. For this, we will need the following assumption.

Definition 1.3.21. We say that E satisfies the **clean intersection hypothesis** if and only if E is a regular value for $\widetilde{H}_Z$ and the flow map

$$\mathbb{R} \times \widetilde{H}_Z^{-1}(E) \to \widetilde{H}_Z^{-1}(E) \times \widetilde{H}_Z^{-1}(E)$$

$$(t, \gamma) \mapsto (\gamma, \widetilde{\Phi}^Z_t(\gamma))$$

admits a clean fibered product over $\widetilde{H}^{-1}(E) \times \widetilde{H}^{-1}(E)$ with the diagonal map $\widetilde{H}^{-1}(E) \to \widetilde{H}^{-1}(E) \times \widetilde{H}^{-1}(E)$.

Let's discuss this hypothesis for a moment. The fibered product is given, as a set, by:

$$\mathfrak{Y}_E := \{(T,\gamma) \in \mathbb{R} \times \widetilde{H}_Z^{-1}(E) \ : \ \widetilde{\Phi}_T^Z(\gamma) = \gamma\}.$$

Notice that this contains $\{0\} \times \widetilde{H}_Z^{-1}(E)$ as a subset and the clean intersection hypothesis implies that $\mathfrak{Y}_E$ is a disjoint union of smooth submanifolds of $\mathbb{R} \times \widetilde{H}_Z^{-1}(E)$.

Lemma 1.3.22. *Under the clean intersection hypothesis, $\{0\} \times \widetilde{H}_Z^{-1}(E)$ is a clopen subset of $\mathfrak{Y}_E$ and every connected component $Y \subseteq \mathfrak{Y}_E$ has*

$$\dim(Y) \le \dim \widetilde{H}_Z^{-1}(E) = 2n + \dim \mathcal{O} - 1.$$

Proof. Let $Y \subseteq \mathfrak{Y}_E$ be any connected component. By the clean intersection hypothesis, for any $(T,\gamma) \in Y$ we must have

$$T_{(T,\gamma)}Y = \left\{ (\tau,\zeta) \in T_T\mathbb{R} \times T_\gamma \widetilde{H}_Z^{-1}(E) \ : \ \tau \frac{d}{dt}\Big|_{t=T} \widetilde{\Phi}_t^Z(\gamma) + D\widetilde{\Phi}_T^Z(\zeta) = \zeta \right\}$$

Since $\zeta \mapsto \zeta - D\widetilde{\Phi}_T^Z(\zeta)$ is linear the only way for the above constraint to be trivial (and not reduce the dimension) is if $\frac{d}{dt}|_{t=T}\widetilde{\Phi}_t^Z(\gamma) = 0$ and if $D\widetilde{\Phi}_T^Z = \mathrm{id}$. Indeed, if $\frac{d}{dt}|_{t=T}\widetilde{\Phi}_t^Z(\gamma) \neq 0$ and we didn't want the equation to constrain ζ then we would need to constrain τ to $\tau = 0$. But now since $\widetilde{\Phi}_T^Z(\gamma) = \gamma$ it follows that $\frac{d}{dt}|_{t=T}\widetilde{\Phi}_t^Z(\gamma) = 0$ implies $\frac{d}{dt}|_{t=0}\widetilde{\Phi}_t^Z(\gamma) = 0$ and so the gradient of the Hamiltonian $\nabla \widetilde{H}_Z$ vanishes at γ and so γ is an equilibrium point. However, we assumed that $\gamma \in \widetilde{H}_Z^{-1}(E)$ and that E was a regular value for $\widetilde{H}_Z$, which contradicts $\nabla \widetilde{H}_Z$ vanishing at γ.

Now, let Y be the smallest clopen subset containing $\{0\} \times \widetilde{H}_Z^{-1}(E)$. We have already shown that $\dim(Y') \leq \dim \widetilde{H}_Z^{-1}(E)$ for any connected component Y' and so we must have $\dim(Y) = \dim \widetilde{H}_Z^{-1}(E)$ since Y is a disjoint union of connected components. In particular, since the inclusion

$$\{0\} \times \widetilde{H}_Z^{-1}(E) \hookrightarrow Y$$

is an immersion it is automatically a submersion as well and hence a local diffeomorphism. Local diffeomorphisms are local homeomorphisms and are hence open maps. Thus the image $\{0\} \times \widetilde{H}_Z^{-1}(E)$ is open in Y, hence open in $\mathfrak{Y}_E$ since Y is open in $\mathfrak{Y}_E$. Since $\{0\} \times \widetilde{H}_Z^{-1}(E)$ is also closed in $\mathfrak{Y}_E$ it follows that it is clopen hence

$$\{0\} \times \widetilde{H}_Z^{-1}(E) = Y$$

as desired. $\square$

We should remark that there is no reason to expect $\widetilde{H}_Z^{-1}(E)$ to be connected even if M is connected since we have allowed disconnected structure groups such as $G = \mathrm{O}(d)$.

In our trace formula, the leading order singularities of the distributional trace will have symbols given by integrals over components of the above clean intersection. The linearized Poincaré map gives us a dynamical description of the volume density on these components. To describe how, let's first recall the invariant volume density on the energy hypersurface $\widetilde{H}_Z^{-1}(E)$.

Definition 1.3.23. Let Ω denote the volume form on $\mathcal{N}_{\mathcal{O}}$ induced by the symplectic form and equip $\mathcal{N}_{\mathcal{O}}$ with the Riemannian metric h_0 induced from the one on $\mathcal{N} \cong T^*_+P|_{P_0} \cong T^*P_0 \setminus 0$ and the Ad-invariant inner product on $\mathfrak{g}$. Using this metric we can define the gradient ∇H_Z and the $2(n+\ell)-1$-form on $\mathcal{N}_{\mathcal{O}}$:

$$|\nabla \widetilde{H}_Z|_{h_0}^{-2} \nabla \widetilde{H}_Z \llcorner \Omega.$$

Denote:

$$\nu_E := \text{ the pullback of the above form to } \widetilde{H}_Z^{-1}(E).$$

The ν_E is invariant under the Hamiltonian flow $\widetilde{\Phi}_t^Z$ and its absolute value $|\nu_E|$ defines an invariant measure on the energy hypersurface $\widetilde{H}_Z^{-1}(E)$.

Lemma 1.3.24. *Under the clean intersection hypothesis, the fibered product $\mathfrak{Y}_E$ comes equipped with a natural volume density. Consider then the case $\mathfrak{Y}_E$ is a union of $\{0\} \times \widetilde{H}_Z^{-1}(E)$ and finitely many disjoint isolated orbits:*

$$Y_1 := \{(T_1, \widetilde{\Phi}_t^Z(\gamma_1)) \ : \ t \in [0, T_1]\}, \ \ldots, \ Y_q := \{(T_q, \widetilde{\Phi}_t^Z(\gamma_q)) \ : \ t \in [0, T_q]\}$$

with $T_j \neq 0$ for all j. Then the Poincaré first return map of each γ_j is invertible and if Ω_γ is the symplectic volume form on $T^*_\gamma \mathcal{N}_{\mathcal{O}}$ then the induced volume density on $T^*_\gamma Y_j$ is given by:

$$|\det(I - P_\gamma)|^{-1/2} |\nu_E| \tag{1.4}$$

with P_γ the Poincaré first return map for one, hence any, choice of symplectic local transversal S.

Proof. Indeed this follows immediately from the expression for the tangent space of Y derived in the proof of 1.3.22, noticing that the constraint

$$\tau \frac{d}{dt}\Big|_{t=T} \widetilde{\Phi}^Z_t(\gamma) + D\widetilde{\Phi}^Z_T(\zeta) = \zeta$$

in the case of isolated periodic orbits is such that $\zeta \mapsto \zeta - D\widetilde{\Phi}^Z_T(\zeta)$ has a 1-dimensional kernel in $T_\gamma \widetilde{H}_Z^{-1}(E) = (\nabla \widetilde{H}_Z(\gamma))^\perp$ spanned the vector field $\widetilde{Z}$ corresponding to the reduced flow $\widetilde{\Phi}^Z_t$. Thus by 1.3.20 the Poincaré first return map is invertible and the induced volume form on TY_j is determined by the invariant volume form ν_E on $\widetilde{H}_Z^{-1}(E)$ and the Poincaré first return map acting on

$$T\mathcal{N}_{\mathcal{O}} / \operatorname{Span}\{\widetilde{Z}, \nabla \widetilde{H}_Z\} \cong T\widetilde{H}_Z^{-1}(E) / \operatorname{Span}\{\widetilde{Z}\}$$

yielding the formula 1.4. $\square$

Next, let's discuss the phase associated to a periodic orbit. For this we need the following basic result from representation theory.

Definition 1.3.25. The coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ is called **integral** if and only if the cohomology class $[\omega_{\mathcal{O}}]$ of its symplectic form $\omega_{\mathcal{O}}$ is in the image of $H^2(\mathcal{O};\mathbb{Z}) \to H^2(\mathcal{O};\mathbb{R}) \cong H^2_{dR}(\mathcal{O};\mathbb{R})$.

Lemma 1.3.26. *A coadjoint orbit $\mathcal{O} = G \cdot \xi_0$ is integral if and only if there exists a character*

$$\chi_{\xi_0} : G_{\xi_0} \to \mathrm{U}(1)$$

such that

$$(d\chi_{\xi_0})_I = 2\pi i\langle \xi_0, -\rangle : \mathfrak{g}_{\xi_0} \to i\mathbb{R}$$

where $I \in G$ is the identity matrix.

So, when our coadjoint orbit is integral we have a $\mathrm{U}(1)$-bundle defined by the character:

$$G \times_{\chi_{\xi_0}} \mathrm{U}(1) \to \mathcal{O}$$

where $G \times_{\chi_{\xi_0}} \mathrm{U}(1)$ is the quotient of $G \times \mathrm{U}(1)$ by the relation

$$(g, z) \sim (gh, \chi_{\xi_0}(h^{-1})z) \ \text{ for all } \ h \in G_{\xi_0}.$$

The right G-action on G yields a right G-action on the total space $G \times_{\chi_{\xi_0}} \mathrm{U}(1)$ since the stabilizer G_{ξ_0} is a normal subgroup. Through this, we identify every tangent space of the total space with the tangent space at the equivalence class $[I, 1] \in G \times_{\xi_0} \mathrm{U}(1)$ of $(I, 1) \in G \times \mathrm{U}(1)$.

Lemma 1.3.27. *We have a natural isomorphism*

$$T_{[I,1]}(G \times_{\chi_{\xi_0}} \mathrm{U}(1)) \cong \left\{ \left(Y + c\xi_0^{\#},\ \frac{i}{2\pi}c \right) \ :\ Y \in \mathfrak{g}_{\xi_0}^{\perp},\ c \in \mathbb{R} \right\}.$$

Furthermore, there is a principal $\mathrm{U}(1)$-connection α on $G \times_{\xi_0} \mathrm{U}(1)$ such that $d\alpha = \omega_{\mathcal{O}}$ and, under our above isomorphism, it is given by:

$$\alpha\left(Y + c\xi_0^{\#},\ \frac{i}{2\pi}c \right) = c = \langle \xi_0, Y + c\xi_0^{\#} \rangle.$$

This G-equivariant bundle with connection over $\mathcal{O}$ gives us a natural U(1)-bundle with connection over the reduced phase space $\mathcal{N}_{\mathcal{O}}$, which we describe now.

Definition 1.3.28. The U(1)-Bundle With Connection: Construction I
Recalling that $\mu_{\mathcal{O}} : \mathcal{N} \times \overline{\mathcal{O}} \to \mathfrak{g}^*$ we can consider the G-equivariant U(1)-bundle:

$$\Big(\mathcal{N} \times (G \times_{\chi_{\xi_0}} \mathrm{U}(1))\Big)\,|_{\mu_{\mathcal{O}}^{-1}(0)} \to \mu_{\mathcal{O}}^{-1}(0).$$

If α^0 denotes the Liouville 1-form on $\mathcal{N}$ and $i : \mu_{\mathcal{O}}^{-1}(0) \hookrightarrow \mathcal{N}$ the inclusion then we have a G-invariant 1-form on the total space of this bundle given by:

$$i^*(\alpha^0 - \alpha).$$

We then set:

$$Z_{\mathcal{O}} := \Big(\mathcal{N} \times (G \times_{\chi_{\xi_0}} \mathrm{U}(1))\Big)\,|_{\mu_{\mathcal{O}}^{-1}(0)}/G \to \mu_{\mathcal{O}}^{-1}(0)/G = \mathcal{N}_{\mathcal{O}}$$

with connection 1-form

$$\alpha_{\mathcal{O}} := \text{ the reduction of } i^*(\alpha^0 - \alpha) \text{ mod } G.$$

Definition 1.3.29. The U(1)-Bundle with Connection: Construction II
Here we instead extend our right G_{ξ_0}-action on $\mu^{-1}(\xi_0)$ so $\mu^{-1}(\xi_0)\times \mathrm{U}(1)$ via the character χ_{ξ_0}. We then set

$$Z_{\mathcal{O}} := \left(\mu^{-1}(\xi_0) \times \mathrm{U}(1)\right)/G_{\xi_0} \to \mu^{-1}(\xi_0)/G_{\xi_0} = \mathcal{N}_{\mathcal{O}}$$

with connection 1-form

$$\alpha_{\mathcal{O}} := \text{ the reduction of } i^*\alpha^0 + d\theta \text{ mod } G_{\xi_0}$$

where now $i : \mu^{-1}(\xi_0) \hookrightarrow \mathcal{N}$ is the inclusion.

Finally we arrive at the holonomies that describe the quantum phase translation that occurs upon traveling along a classical periodic orbit.

Definition 1.3.30. Let $\gamma : [0,T] \to \mathcal{N}_{\mathcal{O}}$ be a periodic orbit of the $\widetilde{\Phi}^Z_s$-flow (i.e. $\gamma(s) = \widetilde{\Phi}^Z_s(z_0)$ for some z_0 and $\gamma(0) = \gamma(T)$) and assume that T is the minimum period of γ. We denote:

$$\mathrm{Hol}_{\mathcal{O}}(\gamma) := \text{ the holonomy of } \alpha_{\mathcal{O}} \text{ about the loop } \gamma.$$

A key point is that while our construction of the U(1)-bundle with connection relied on a choice of character as well as a choice of $\xi_0 \in \mathcal{O}$, the element $\mathrm{Hol}_{\mathcal{O}}(\gamma) \in \mathrm{U}(1)$ is independent of these choices.

The following proposition is from **[17]** section 4. Their result applies here since it applies in the general context of symplectic reduction along an integral coadjoint orbit.

Proposition 1.3.31. *The map* $\mathrm{Hol}_{\mathcal{O}} : \mathfrak{Y}_E \to \mathrm{U}(1)$ *is locally constant. Furthermore, if we consider the symplectomorphism* $\mathcal{N}_{\mathcal{O}} \cong \mu^{-1}(\xi_0)/G_{\xi_0}$ *and suppose we had* $\gamma \in \mu^{-1}(\xi_0)$ *with* $H_Z(\gamma) = E$ *and* $T \in \mathbb{R}$, $g \in G_{\xi_0}$ *such that* $\Phi^Z_T(\gamma) = \gamma \cdot g$ *then if* $[\gamma] \in \mathcal{N}_{\mathcal{O}}$ *denotes the image in the quotient we have:*

$$\mathrm{Hol}_{\mathcal{O}}(T, [\gamma]) = \chi_{\xi_0}(g)e^{iTE}.$$

CHAPTER 2

Wave and Dirac-Type Equations on Stationary Spacetimes

The equations of interest to us will have as solutions sections of complex vector bundles over a globally hyperbolic spacetime M. We collect both general facts concerning normally hyperbolic and Dirac-type operators on such bundles, as well as some discussion of specials forms on standard stationary and standard static spacetimes.

Definition 2.0.1. Let $E \to M$ be a complex vector bundle and $P : \Gamma(M, E) \to \Gamma(M, E)$ a linear differential operator. Recall that the **principal symbol** of P is given by writing P locally (in terms of local coordinates on M and a local frame for E) as

$$P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$$

with the a_α's being locally defined linear endomorphisms of E, a_α not identically zero for some multiindex α with $|\alpha| = m$, and then setting

$$\sigma_P(\xi_x) := i^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \quad \text{for } \xi_x \in T_x^*M.$$

In this way σ_P is a section of the pullback bundle of $\mathrm{End}(E)$ to T^*M. In fact, for a differential operator of order m such as above, $\sigma_P \in \Gamma(M, \mathrm{Sym}^m TM \otimes \mathrm{End}(E))$.

Definition 2.0.2. Let $E \to M$ be a complex vector bundle on a spacetime (M, g). A second-order linear differential operator $P : \Gamma(M, E) \to \Gamma(M, E)$ is called **normally**

hyperbolic if and only if its principal symbol is given by $\sigma_P(\xi_x) = g_x^{-1}(\xi_x, \xi_x)\,\mathrm{id}_{E_x}$ for all $\xi_x \in T_x^*M$.

Normally hyperbolic operators are simply generalizations of the wave operator $\Box_g$ acting on functions. Instead of giving explicit examples, we simply state the next lemma since it demonstrates what every example will look like.

Lemma 2.0.3. *[2] Let $E \to M$ be a complex vector bundle on a spacetime (M, g) and P a normally hyperbolic operator on E. Then there exists a unique connection A on E and smooth endomorphism $\Upsilon : E \to E$ such that*

$$P = -\operatorname{Tr}_g(\nabla^A \circ \nabla^A) + \Upsilon$$

where for any section u of E we have

$$\operatorname{Tr}_g(\nabla^A \circ \nabla^A)u := |g|^{-1/2}\nabla^A_\mu \left(|g|^{1/2} g^{\mu\nu} \nabla^A_\nu u\right).$$

Conversely, every operator of this form is normally hyperbolic.

Example 2.0.4. We should note that if $E = \Lambda^* T^*_{\mathbb{C}} M$ is the bundle of complex-valued differential forms then the Hodge-wave-operator $dd^* + d^*d$ with respect to g can be seen to be normally hyperbolic either by explicitly computing its symbol, or by applying the Weitzenböck identity and the above lemma.

Lemma 2.0.5. *Let $E \to M$ be a complex vector bundle over a spacetime and P a differential operator on E. Then the **transpose** P^T of P is an operator on E^* given by*

$$\int_M (P^T u)(\phi) dV_g := \int_M u(P\phi) dV_g$$

for all $u \in \Gamma(M, E^*)$ and all $\phi \in \Gamma_c(M, E)$. P^T is also a differential operator.

For us, sesquilinear forms will always be conjugate-linear in the first variable and linear in the second.

Lemma 2.0.6. *Let $E \to M$ be a complex vector bundle over a spacetime and P a differential operator on E. Fix a non-degenerate sesquilinear form $\langle -, - \rangle$ on E. We call P **formally self-adjoint** with respect to $\langle -, - \rangle$ if and only if*

$$\int_M \langle P\phi_1, \phi_2 \rangle dV_g = \int_M \langle \phi_1, P\phi_2 \rangle dV_g$$

*for all sections $\phi_1, \phi_2 \in \Gamma(M, E)$ with at least one of them compactly supported. The $\langle -, - \rangle$**-formal-adjoint** of P is the operator P^* on E given by conjugating P^T by the $\mathbb{C}$-antilinear isomorphism $E \cong E^*$ induced by $\langle -, - \rangle$ and P is clearly formally self-adjoint with respect to $\langle -, - \rangle$ if and only if $P = P^*$.*

Example 2.0.7. We return again to differential forms $E = \Lambda^* T^*_{\mathbb{C}} M$ for this example. The metric g induces a natural Hermitian fiber metric g^{-p} on each $\Lambda^p T^*_{\mathbb{C}} M$ via

$$g^{-p}(\alpha, \beta) := \frac{1}{p!} g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p} \alpha_{\mu_1 \cdots \mu_p} \beta_{\nu_1 \cdots \nu_p}$$

The factor of $1/p!$ is to guarantee that if $e^0, ..., e^n$ is a local orthonormal coframe with $g^{-1}(e^0, e^0) = -1$ then the p-forms $e^{\mu_1} \wedge \cdots \wedge e^{\mu_p}$ with $\mu_1 < \cdots < \mu_p$ form a g^{-p}-orthonormal local coframe. From this we can also see that the metric g^{-p} has a mixed signature with $\binom{n}{p-1}$ negative eigenvalues. This is why we only required a non-degenerate sesquilinear form, and not for our form to be of either Lorentzian or Riemannian signature.

Declaring the subbundles $\Lambda^p T^*_{\mathbb{C}} M$ and $\Lambda^q T^*_{\mathbb{C}} M$ orthogonal for $p \neq q$ we obtain a non-degenerate sesquilinear form on all of $\Lambda^* T^*_{\mathbb{C}} M$ from the g^{-p}'s and $dd^* + d^*d$ is formally self-adjoint with respect to it.

Lemma 2.0.8. *Let $E \to M$ be a complex vector bundle over a spacetime, P a normally hyperbolic operator on E and $\langle -, - \rangle$ a non-degenerate sesquilinear form on E. Then P^* is normally hyperbolic and if A, Υ are the connection and endomorphism of E induced by P then $P = P^*$ with respect to $\langle -, - \rangle$ if and only if A is $\langle -, - \rangle$-compatible and Υ is pointwise $\langle -, - \rangle$-self-adjoint.*

Proof. Assuming A is $\langle -, - \rangle$-compatible and Υ is pointwise $\langle -, - \rangle$-self-adjoint it is straight-forward to show that P is formally self-adjoint so we only show the converse. Let $f : M \to \mathbb{R}$ be an arbitrary smooth function and notice that P being formally self-adjoint with respect to $\langle -, - \rangle$ implies that for all smooth sections ϕ_1, ϕ_2 with at least one of them compactly supported we have

$$\int_M \langle P(f\phi_1) - fP\phi_1, \phi_2 \rangle dV_g = \int_M \langle \phi_1, fP\phi_2 - P(f\phi_2) \rangle dV_g$$

hence if M_f is the 0'th order operator given by multiplication by f it follows that $[P, M_f]$ is formally skew-adjoint. However we can compute

$$[P, M_f]\phi = -2\nabla^A_{\nabla f}\phi + (\Box_g f)\phi$$

where $\Box_g$ is the wave operator on functions on the Lorentzian manifold (M, g). Thus for all ϕ_1, ϕ_2 we have

$$\int_M \left[\langle \phi_1, \nabla^A_{\nabla f}\phi_2\rangle + \langle \nabla^A_{\nabla f}\phi_1, \phi_2\rangle - (\Box_g f)\langle \phi_1, \phi_2\rangle\right] dV_g = 0.$$

Now, $\Box_g f$ is equal to the divergence of the gradient of f hence $\mathcal{L}_{\nabla f} dV_g = (\Box_g f) dV_g$ and so by Stokes' theorem we can rewrite our integral identity as

$$\int_M \left[\langle \nabla^A_{\nabla f}\phi_1, \phi_2\rangle + \langle \phi_1, \nabla^A_{\nabla f}\phi_2\rangle - \mathcal{L}_{\nabla f}\langle \phi_1, \phi_2\rangle\right] dV_g = 0$$

and since this holds for all ϕ_1, ϕ_2 sections with at least one of them compactly supported, and for all smooth real-valued f it follows that A is $\langle -, -, \rangle$-compatible. But then

$$P = -\operatorname{Tr}_g(\nabla^A \circ \nabla^A) + \Upsilon \text{ and } -\operatorname{Tr}_g(\nabla^A \circ \nabla^A) \text{ is formally self-adjoint}$$

thus Υ must be pointwise $\langle -, -\rangle$-self-adjoint. $\Box$

Of special interest to us are normally hyperbolic operators over standard stationary spacetimes.

Lemma 2.0.9. *Let $E \to M = \mathbb{R}_t \times \Sigma$ be a complex vector bundle over a standard stationary spacetime 1.1.11 and A a connection on E. Then E is isomorphic to the pullback* *of a bundle E_0 over Σ along the projection $\mathbb{R}_t \times \Sigma \to \Sigma$ and there is an automorphism of E identifying A with a t-dependent family of connections A_0 on E_0. In particular, $\partial_t \llcorner F_{A_0} = \partial_t A_0$ and $\nabla^{A_0}_{\partial_t} = \partial_t$. A connection A_0 of this form is said to be in **temporal gauge**.*

Proof. That E can be trivialized in the t-direction is a standard fact from topology so we assume from the beginning that $E = \mathbb{R}_t \times E_0$ as a bundle over $\mathbb{R}_t \times \Sigma$. When then have an operator ∂_t acting on sections of E such that the difference $\nabla^A_{\partial_t} - \partial_t$ is an endomorphism of E which we denote by B. Solving the first order ODE

$$g^{-1}\partial_t g = -B, \ g|_{t=0} = \mathrm{id} \ \text{ for a gauge transformation } g \in \Gamma(M, \mathrm{Aut}(E))$$

it follows that the gauge-transformed connection $\nabla^{g \cdot A}$ is of the form

$$\nabla^{g \cdot A} = dt \otimes \partial_t + dx^i \otimes \nabla^{A_0}_i$$

for A_0 a t-dependent family of connections on $E_0 \to \Sigma$. The desired expressions for $\partial_t \llcorner F_{g \cdot A}$ and $\nabla^{g \cdot A}_{\partial_t}$ then follow. $\square$

We will often make the simplifying assumption that our connections are in temporal gauge. Finally, in order for the frequency spectrum of solutions to $P\phi = 0$, P normally hyperbolic, to be defined we need to assume that $\ker P$ is invariant under some time-translation operator.

Lemma 2.0.10. *Let $E \to M$ be a complex vector bundle over a standard stationary spacetime $M = \mathbb{R}_t \times \Sigma$ and P a normally hyperbolic operator on E. Let A, Υ denote the connection and endomorphism corresponding to P. Then $[P, \nabla^A_{\partial_t}] = 0$ if and only if*

$\partial_t \llcorner F_A = 0$ and $\nabla^{A,\mathrm{End}}_{\partial_t}\Upsilon = 0$ where $\nabla^{A,\mathrm{End}}$ is the connection on $\mathrm{End}(E)$ induced by A. In Coulomb gauge, the condition $\nabla^{A,\mathrm{End}}_{\partial_t}\Upsilon = 0$ simply becomes $\partial_t \Upsilon = 0$.

Proof. Here we simply calculate for an arbitrary section ϕ of E:

$$\begin{aligned} 0 = [P, \nabla^A_0]\phi &= -g^{\mu\nu} F^A_{\mu 0} \nabla^A_\nu \phi - |g|^{-1/2} \nabla^A_\mu \left(|g|^{1/2} g^{\mu\nu} F^A_{\nu 0} \phi \right) - \left(\nabla^{A,\mathrm{End}}_0 \Upsilon \right) \phi \\ &= -2 g^{\mu\nu} F^A_{\mu 0} \nabla^A_\nu \phi - |g|^{-1/2} \left(\nabla^{A,\mathrm{End}}_\mu \left(|g|^{1/2} g^{\mu\nu} F^A_{\nu 0} \right) \right) \phi - \left(\nabla^{A,\mathrm{End}}_0 \Upsilon \right) \phi \end{aligned}$$

Looking at the first order term, it follows that we must have $F^A_{\mu 0} = 0$ for all μ, i.e. $\partial_t \llcorner F_A = 0$. Once we know this vanishes identically the 0'th order term simply becomes $(\nabla^{A,\mathrm{End}}_0 \Upsilon)\phi$ and therefore we must have $\nabla^{A,\mathrm{End}}_0 \Upsilon = 0$ since ϕ is arbitrary. □

A similar analysis to the above must now be performed for Dirac-type operators.

Definition 2.0.11. Let $E \to M$ be a complex vector bundle over a spacetime (M, g). A **Dirac-type operator** on E is a linear first order differential operator $D\!\!\!/$ on E with principal symbol satisfying

$$\sigma_{D\!\!\!/}(\xi)\sigma_{D\!\!\!/}(\eta) + \sigma_{D\!\!\!/}(\eta)\sigma_{D\!\!\!/}(\xi) = 2g^{-1}(\xi, \eta)\,\mathrm{id}_E$$

for all covectors ξ, η. Given a Dirac-type operator $D\!\!\!/$ on E we introduce the following notation for $v \in TM$:

$$v\!\!\!/ := g_{\mu\nu} v^\mu \sigma_{D\!\!\!/}(dx^\nu) \in \mathrm{End}(E).$$

Notice that we also have $v\!\!\!/ w\!\!\!/ + w\!\!\!/ v\!\!\!/ = 2g(v, w)\,\mathrm{id}_E$ for all vectors v, w.

In contrast to normally hyperbolic operators which have a unique corresponding connection and endomorphism, Dirac-type operators satisfy the following.

Lemma 2.0.12. *Let $\not{D}$ be a Dirac-type operator on a complex vector bundle E over a spacetime (M, g). Given any connection B on E there exists a unique endomorphism Θ_B of E such that*

$$\not{D} = -i\not{\nabla}^B + \Theta_B := -ig^{\mu\nu}\not{\partial}_\mu \nabla^B_\nu + \Theta_B.$$

Furthermore, $\not{D}^2$ is a normally hyperbolic operator on E and if A, Υ are the connection and endomorphism corresponding to $\not{D}^2$ then

$$g^{\mu\nu}\not{\partial}_\mu \left(\Gamma^\gamma_{\nu\alpha}\not{\partial}_\gamma - (\nabla^{A,\mathrm{End}}_\nu \not{\partial}_\alpha)\right) - i(\not{\partial}_\alpha \Theta_A + \Theta_A \not{\partial}_\alpha) = 0 \quad \textit{for all } \alpha$$

In fact, any connection A satisfying the above identity is necessarily the connection corresponding to the normally hyperbolic operator $\not{D}^2$.

Proof. Given a connection Θ_B, one obtains Θ_B by simply noticing that $\not{D}$ and $-i\not{\nabla}^B$ have the same principal symbol by construction. As they are both first order differential operators, their difference is therefore an endomorphism of E.

For A, Υ the connection and endomorphism corresponding to the normally hyperbolic operator $\not{D}^2$ we fix an arbitrary section ψ of E and compute:

$$\begin{aligned}
\not{D}^2\psi = & -g^{\mu\nu}\not{\partial}_\mu \nabla^A_\nu (g^{\alpha\beta}\not{\partial}_\alpha \nabla^A_\beta \psi) - ig^{\mu\nu}(\not{\partial}_\mu \Theta_A + \Theta_A \not{\partial}_\mu)\nabla^A_\nu \psi \\
& + (\Theta_A^2 - ig^{\mu\nu}\not{\partial}_\mu \nabla^{A,\mathrm{End}}_\nu \Theta_A)\psi \\
= & -\mathrm{Tr}_g(\nabla^A \circ \nabla^A)\psi - g^{\mu\nu}\Gamma^\beta_{\mu\nu}\nabla^A_\beta \psi - g^{\mu\nu}(\partial_\nu g^{\alpha\beta})\not{\partial}_\mu \not{\partial}_\alpha \nabla^A_\beta \psi \\
& - g^{\mu\nu}g^{\alpha\beta}\not{\partial}_\mu (\nabla^{A,\mathrm{End}}_\nu \not{\partial}_\alpha)\nabla^A_\beta \psi - ig^{\mu\nu}(\not{\partial}_\mu \Theta_A + \Theta_A \not{\partial}_\mu)\nabla^A_\nu \psi
\end{aligned}$$

$$+\left(\Theta_A^2+\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\partial\!\!\!/_\mu\partial\!\!\!/_\alpha F^A_{\nu\beta}-ig^{\mu\nu}\partial\!\!\!/_\mu\nabla^{A,\mathrm{End}}_\nu\Theta_A\right)\psi$$

$$\begin{aligned}=&-\mathrm{Tr}_g(\nabla^A\circ\nabla^A)\psi+\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}(\partial_\alpha g_{\mu\nu})\nabla^A_\beta\psi-\frac{1}{2}\left(g^{\mu\nu}\partial_\nu g^{\gamma\beta}-g^{\gamma\nu}\partial_\nu g^{\mu\beta}\right)\partial\!\!\!/_\mu\partial\!\!\!/_\gamma\nabla^A_\beta\psi\\&-g^{\mu\nu}g^{\alpha\beta}\partial\!\!\!/_\mu(\nabla^{A,\mathrm{End}}_\nu\partial\!\!\!/_\alpha)\nabla^A_\beta\psi-ig^{\mu\nu}(\partial\!\!\!/_\mu\Theta_A+\Theta_A\partial\!\!\!/_\mu)\nabla^A_\nu\psi\\&+\left(\Theta_A^2+\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\partial\!\!\!/_\mu\partial\!\!\!/_\alpha F^A_{\nu\beta}-ig^{\mu\nu}\partial\!\!\!/_\mu\nabla^{A,\mathrm{End}}_\nu\Theta_A\right)\psi\end{aligned}$$

however we can also compute:

$$\begin{aligned}g^{\mu\nu}g^{\alpha\beta}\Gamma^\gamma_{\nu\alpha}\partial\!\!\!/_\mu\partial\!\!\!/_\gamma&=\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\tau}(\partial_\alpha g_{\tau\nu})\partial\!\!\!/_\mu\partial\!\!\!/_\gamma+\frac{1}{2}\left(g^{\gamma\tau}\partial_\tau g^{\mu\beta}-g^{\mu\nu}\partial_\nu g^{\gamma\beta}\right)\partial\!\!\!/_\mu\partial\!\!\!/_\gamma\\&=\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha g_{\mu\nu}-\frac{1}{2}\left(g^{\mu\nu}\partial_\nu g^{\gamma\beta}-g^{\gamma\nu}\partial_\nu g^{\mu\beta}\right)\partial\!\!\!/_\mu\partial\!\!\!/_\gamma\end{aligned}$$

so we arrive at

$$\begin{aligned}D\!\!\!\!/^2\psi=&-\mathrm{Tr}_g(\nabla^A\circ\nabla^A)\psi+g^{\mu\nu}g^{\alpha\beta}\partial\!\!\!/_\mu\left(\Gamma^\gamma_{\nu\alpha}\partial\!\!\!/_\gamma-(\nabla^{A,\mathrm{End}}_\nu\partial\!\!\!/_\alpha)\right)\nabla^A_\beta\psi\\&-ig^{\alpha\beta}(\partial\!\!\!/_\alpha\Theta_A+\Theta_A\partial\!\!\!/_\alpha)\nabla^A_\beta\psi\\&+\left(\Theta_A^2+\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\partial\!\!\!/_\mu\partial\!\!\!/_\alpha F^A_{\nu\beta}-ig^{\mu\nu}\partial\!\!\!/_\mu\nabla^{A,\mathrm{End}}_\nu\Theta_A\right)\psi\end{aligned}$$

as desired. □

From the above result we can see that if we want our connection A corresponding to $D\!\!\!\!/^2$ to be compatible with the Clifford multiplication then we obtain fairly strict constraints on the potential term Θ_A.

Corollary 2.0.13. *Let $E\to M$ be a complex vector bundle and $D\!\!\!\!/$ a Dirac-type operator on E. A connection B on E is said to be* ***compatible*** *with Clifford multiplication if and* *only if for all sections ψ of E and vector fields v, w we have*

$$\nabla_v^B(\not{\psi}\psi) = \not{\nabla_v w}\psi + \not{\psi}\nabla_v^B\psi$$

Such a connection B coincides with the connection A determined by $\not{D}^2$ if and only if $\not{v}\Theta_B = -\Theta_B\not{v}$ for all vector fields v.

Proof. This follows from our earlier computation of $\not{D}^2$ in terms of B and the fact that Clifford compatibility is manifestly equivalent to

$$\nabla_\mu^{B,\mathrm{End}}\not{\partial}_\nu = \Gamma_{\mu\nu}^\gamma\not{\partial}_\gamma$$

□

Nevertheless, we can proceed with the analysis of $\not{D}$ analogously to our previous work with P.

Lemma 2.0.14. *Let $\not{D}$ be a Dirac-type operator on a complex vector bundle E over a spacetime M. Similar to the case of a normally hyperbolic operator, $\not{D}$ has a transpose $\not{D}^T$ acting on E^* and for any non-degenerate sesquilinear form $\langle -, - \rangle$ on E a corresponding formally adjoint operator $\not{D}^*$ on E. The formal adjoint $\not{D}^*$ of $\not{D}$ is again a Dirac-type operator.*

Proof. This follows from the adjoints of the Clifford multiplication endomorphisms $\sigma_{\not{D}}(\xi)^*$ with respect to $\langle -, - \rangle$ still satisfying the Clifford relation. □

In the below lemma we begin the trend of stating our results separately in terms of the connection A corresponding to the normally hyperbolic operator $\not{D}^2$, and any connection

B compatible with the sesquilinear form and Clifford multiplication. We do this since, while certain results require working with the normally hyperbolic operator $\not{D}^2$ it will often be more convenient to instead make a choice of connection compatible with Clifford multiplication.

Lemma 2.0.15. *Let $\not{D}$ be a Dirac-type operator on a complex vector bundle E over a spacetime M and let $\langle -, - \rangle$ be a non-degenerate sesquilinear form on E. Then $\not{D}$ is called* ***formally self-adjoint*** *with respect to $\langle -, - \rangle$ if and only if for all smooth sections ψ_1, ψ_2 of E with at least one of the compactly supported we have*

$$\int_M \langle \not{D}\psi_1, \psi_2 \rangle dV_g = \int_M \langle \psi_1, \not{D}\psi_2 \rangle dV_g$$

Then $\not{D}$ is formally self-adjoint with respect to $\langle -, - \rangle$ if and only if for A the connection on E corresponding to the normally hyperbolic operator $\not{D}^2$ we have

$$\Theta_A = \Theta_A^* + i|g|^{-1/2} g^{\mu\nu} \left(\Gamma^{\gamma}_{\mu\nu} \not{\partial}_{\gamma} - \nabla^{A,\mathrm{End}}_{\nu} \not{\partial}_{\mu} \right)$$

If we instead use a connection B with is both compatible with respect to $\langle -, - \rangle$ and Clifford multiplication then it instead follows that $\not{D}$ is $\langle -, - \rangle$-formally self adjoint if and only if both Clifford multiplication and Θ_B are $\langle -, - \rangle$-self-adjoint.

Proof. First of all, recall that for any section ψ and smooth real-valued function f we have

$$\not{D}(f\psi) = f\not{D}\psi - ig^{\mu\nu}\not{\partial}_{\mu}(\partial_{\mu} f)\psi = f\not{D}\psi - i\not{\nabla} f\psi$$

So if we assume that $\not{D}$ is formally self-adjoint with respect to $\langle -, - \rangle$ then for all real-valued smooth function f and all sections ψ_1, ψ_2 with at least one of the compactly supported we have

$$\int_M \langle -i\nabla\!\!\!/ f\psi_1, \psi_2\rangle dV_g = \int_M \langle \psi_1, f D\!\!\!/\psi_2 - D\!\!\!/(f\psi_2)\rangle dV_g = \int_M \langle \psi_1, i\nabla\!\!\!/ f\psi_2\rangle dV_g$$

so that $\nabla\!\!\!/ f$ is pointwise $\langle -,-\rangle$-self-adjoint and hence for any vector $v \in TM$ we have $v\!\!\!/$ is pointwise $\langle -,-\rangle$-self-adjoint. Furthermore, $D\!\!\!/$ being formally self-adjoint implies that the normally hyperbolic operator $D\!\!\!/^2$ is formally $\langle -,-\rangle$-self-adjoint hence from our earlier lemma it follows that A is $\langle -,-\rangle$-compatible. For any $\langle -,-\rangle$-compatible connection B we can write $D\!\!\!/ = -i\nabla\!\!\!/^B + \Theta_B$ and compute that

$$\begin{aligned} D\!\!\!/^*\psi &= -i|g|^{-1/2}\nabla^B_\nu\left(|g|^{1/2}g^{\mu\nu}\partial\!\!\!/_\mu\psi\right) + \Theta_B^*\psi \\ &= -i\nabla\!\!\!/^B\psi + \Theta_B^*\psi + i|g|^{-1/2}g^{\mu\nu}\left(\Gamma^\gamma_{\mu\nu}\partial\!\!\!/_\gamma - \nabla^{B,\mathrm{End}}_\nu\partial\!\!\!/_\mu\right)\psi \end{aligned}$$

In the case $B = A$ is our connection corresponding to $D\!\!\!/^2$ we obtain our desired result. Similarly, if B is instead any connection compatible with both $\langle -,-\rangle$ and Clifford multiplication then $\Theta_B = \Theta_B^*$ follows from $\Gamma^\gamma_{\mu\nu}\partial\!\!\!/_\gamma = \nabla^{B,\mathrm{End}}_\nu\partial\!\!\!/_\mu$. $\square$

Before proceeding we note the following existence result for connections B compatible with Clifford multiplication.

Theorem 2.0.16. *[21] [9] Let $E \to M$ be a complex vector bundle over a spacetime with Dirac-type operator $D\!\!\!/$ which is formally self adjoint with respect to some non-degenerate sesquilinear form $\langle -,-\rangle$. Then there exists a connection B compatible with both the Clifford multiplication and $\langle -,-\rangle$.*

Finally we are prepared to discuss the time translation operator on $\ker \not{D}$. As in the normally hyperbolic case, we must assume that $\not{D}$ commutes with $\nabla^A_{\partial_t}$. In principle one could work with $\nabla^B_{\partial_t}$ for some connection B compatible with Clifford multiplication instead, however one can no longer necessarily conclude $F^B_{\mu 0} = 0$ from simply assuming $[\not{D}, \nabla^A_0] = 0$, hence doing so adds additional complications.

Furthermore, the below lemma demonstrates that assuming the existence of a connection B compatible with Clifford multiplication for which $[\not{D}, \nabla^B_0] = 0$ imposes very strict requirements on the underlying stationary metric g.

Lemma 2.0.17. *Let $E \to M = \mathbb{R}_t \times \Sigma$ be a complex vector bundle over a standard stationary spacetime 1.1.11, $\not{D}$ a Dirac-type operator on E and A the connection corresponding to $\not{D}^2$. Then $[\not{D}, \nabla^A_{\partial_t}] = 0$ if and only if $\partial_t \llcorner F_A = 0$, $\nabla^{A,\mathrm{End}}_0 \Theta_A = 0$ and $\nabla^{A,\mathrm{End}}_0 \not{\partial}_\mu = 0$ for all μ. If we instead work with some connection B compatible with Clifford multiplication then the condition $[\not{D}, \nabla^B_0] = 0$ becomes equivalent to $g^{\mu\nu} \not{\partial}_\mu F^B_{\nu 0} = i \nabla^{B,\mathrm{End}}_0 \Theta_B$, $N^2 - |\eta|^2_h$ being constant, and $d\eta = 0$ where N, η, h are from the standard form 1.1.11 of g.*

Proof. Fixing an arbitrary connection B and a section ψ we can compute (using that $\partial_0 g_{\mu\nu} = 0$):

$$
\begin{aligned}
[\not{D}, \nabla^B_{\partial_t}]\psi &= -i g^{\mu\nu} \not{\partial}_\mu \nabla^B_\nu \nabla^B_0 \psi + i \nabla^B_0 (g^{\mu\nu} \not{\partial}_\mu \nabla^B_\nu \psi) - (\nabla^{B,\mathrm{End}}_0 \Theta_B)\psi \\
&= -i g^{\mu\nu} \not{\partial}_\mu (\nabla^B_\nu \nabla^B_0 - \nabla^B_0 \nabla^B_\nu)\psi + i g^{\mu\nu} (\nabla^{B,\mathrm{End}}_0 \not{\partial}_\mu) \nabla^B_\nu \psi - (\nabla^{B,\mathrm{End}}_0 \Theta_B)\psi \\
&= i g^{\mu\nu} (\nabla^{B,\mathrm{End}}_0 \not{\partial}_\mu) \nabla^B_\nu \psi - ((\nabla^{B,\mathrm{End}}_0 \Theta_B) + i g^{\mu\nu} \not{\partial}_\mu F^B_{\nu 0})\psi
\end{aligned}
$$

In the case where $B = A$ is the connection corresponding to $\not{D}^2$ we also have $[\not{D}^2, \nabla^A_{\partial_t}] = 0$ and so it follows that $F^A_{\mu 0} = 0$ for all μ. Thus our desired result follows. In the case where we assume that B is compatible with Clifford multiplication then the vanishing of the first order terms of $[\not{D}, \nabla^A_{\partial_t}]$ yields

$$0 = \nabla^{B,\mathrm{End}}_0 \not{\partial}_\mu = \Gamma^\gamma_{0\mu} \not{\partial}_\gamma$$

however $\Gamma^\gamma_{0\mu} = 0$ for all γ, μ if and only if

$$\partial_\mu g_{\nu 0} - \partial_\nu g_{\mu 0} = 0 \text{ for all } \mu, \nu$$

and so, looking separately at the cases where at least one of μ, ν are zero, and both are non-zero, we arrive at:

$$d(N^2 - |\eta|^2_h) = 0 \quad \text{and} \quad d\eta = 0.$$

$\square$

It should be noted that none of the naturally occurring stationary globally hyperbolic metrics from 1 satisfy the above condition save for the ultrastatic ones. Non-ultrastatic examples of such metrics are, in the case Σ is compact, simply twists of the ultrastatic metric

$$-dt^2 + h \quad \text{on} \quad \mathbb{R}_t \times \Sigma$$

by a closed 1-form η.

We close-out the introduction to this chapter by discussion one final aspect of Dirac-type operators: chirality.

Definition 2.0.18. Let E be a complex vector bundle over a spacetime M of dimension $\dim(M) = n + 1$ and suppose E is equipped with a Dirac-type operator $\not{D}$. For any local oriented orthonormal frame $e_0, ..., e_n$ for TM with $g(e_0, e_0) = -1$ we define

$$\omega_{\mathbb{C}} := i^{1+\frac{(n+1)n}{2}} \not{e}_0 \cdots \not{e}_n$$

As a local section of $\mathrm{End}(E)$ this is independent of our choice of local oriented orthonormal frame and hence $\omega_{\mathbb{C}}$ defines a global endomorphism of E.

Lemma 2.0.19. *We have $\omega_{\mathbb{C}}^2 = I$ and if E is equipped with a non-degenerate sesquilinear form $\langle -, - \rangle$ for which Clifford multiplication by elements in TM is self-adjoint, it follows that for any ψ_1, ψ_2 sections of E we have*

$$\langle \omega_{\mathbb{C}} \psi_1, \psi_2 \rangle = (-1)^n \langle \psi_1, \omega_{\mathbb{C}} \psi_2 \rangle$$

In particular if we denote by $E_\pm$ the ± 1-eigensubbundles of E corresponding to $\omega_{\mathbb{C}}$ then for $n + 1$ even and any $\psi \in E_\pm$ we have that $\langle \psi, - \rangle$ vanishes identically on $E_\pm$ and therefore, by non-degeneracy, $\psi \mapsto \langle \psi, - \rangle$ is a complex antilinear isomorphism $E_\pm \xrightarrow{\cong} E_\mp^*$. Furthermore, still for $n + 1$ even only, Clifford multiplication by any vector maps $E_\pm \to E_\mp$.

Proof. When squaring $\omega_{\mathbb{C}}$ one can use that $\not{e}_\mu \not{e}_\nu = -\not{e}_\nu \not{e}_\mu$ for $\mu \neq \nu$ to get

$$\omega_{\mathbb{C}}^2 = -\not{e}_0^2 \cdots \not{e}_n^2$$

and the result then follows from $\not{e}_j^2 = 1$ and $\not{e}_0^2 = -1$. The fact that $\langle \omega_{\mathbb{C}} \psi_1, \psi_2 \rangle$ is equal to $(-1)^n \langle \psi_1, \omega_{\mathbb{C}} \psi_2 \rangle$ follows from the same argument. The remaining results follow from applying the above and the definitions of $E_\pm$. Namely if ψ_1, ψ_2 are both in the same eigenbundle $E_\pm$ and $n+1$ is even then $\langle \psi_1, \psi_2 \rangle = -\langle \psi_1, \psi_2 \rangle$ by the above identity and hence this is zero. □

Lemma 2.0.20. *Let $E \to M$ be a complex vector bundle together with a Dirac-type operator $\not{D}$ and suppose B was a connection on E compatible with Clifford multiplication and satisfying $\Theta_B \omega_{\mathbb{C}} + \omega_{\mathbb{C}} \Theta_B = 0$. Then $\not{D}\omega_{\mathbb{C}} + \omega_{\mathbb{C}}\not{D} = 0$. Thus $\not{D}$ interchanges $E_\pm$.*

Proof. We notice that

$$\sum_\mu \not{e}_\mu \nabla^{A,\mathrm{End}}_{e_\mu} (\not{e}_0 \cdots \not{e}_n) = \sum_{\mu,\nu} \not{e}_\mu \not{e}_0 \cdots \not{e}_{\nu-1} \cancel{\nabla_{e_\mu} e_\nu} \not{e}_{\nu+1} \cdots \not{e}_n$$

Denoting $\nabla_{e_\mu} e_\nu =: \widetilde{\Gamma}^\gamma_{\mu\nu} e_\gamma$ it follows from the e_μ being orthonormal that $\widetilde{\Gamma}^\theta_{\mu\nu} = -\widetilde{\Gamma}^\nu_{\mu\theta}$. This together with $\not{e}_\mu \not{e}_\nu = -\not{e}_\nu \not{e} - \mu$ when $\mu \neq \nu$ allows us to conclude. □

2.1. Energy Estimates and Cauchy Data Isomorphisms

Here we summarize the standard known theory for normally hyperbolic and Dirac-type operators on globally hyperbolic spacetimes. As such, throughout we assume:

(1) M is a globally hyperbolic spacetime with complex vector bundle $E \to M$, non-degenerate sesquilinear form $\langle -, - \rangle$ and compatible connection A.
(2) $P = -\operatorname{Tr}_g(\nabla^A \circ \nabla^A) + \Upsilon$ for some $\langle -, - \rangle$-self-adjoint endomorphism Υ.
(3) For simplicity we also assume M is spatially compact. If this is not the case, simply add the assumption that all functions have supports which are compact when intersected with any Cauchy hypersurface and the results of this section will remain true.

There are then two important fundamental solutions for P which we will use to give integral kernels for the solutions to the Cauchy problem.

Theorem 2.1.1. *[2] There are fundamental solutions $E_{adv}, E_{ret} \in \mathcal{D}'(M \times M, E \boxtimes E^*)$ for P, respectively called the* ***advanced and retarded propagators*** *with the following properties:*

(1) $E_{adv/ret} : C_c^\infty(M, E) \to C^\infty(M, E)$, *i.e. they map compactly supported smooth sections to smooth sections.*

(2) *Given $\phi \in C_c^\infty(M, E)$, $E_{adv}\phi$ is the unique solution to $P(E_{adv}\phi) = \phi$ satisfying*

$$\mathrm{supp}(E_{adv}\phi) \subseteq J_-(\mathrm{supp}\,\phi).$$

Similarly $E_{ret}\phi$ is the unique solution to $P(E_{ret}\phi) = \phi$ satisfying

$$\mathrm{supp}(E_{ret}\phi) \subseteq J_+(\mathrm{supp}\,\phi).$$

(3) *Under our assumptions on P being $\langle -, - \rangle$-formally self-adjoint, it follows that $E_{adv/ret}$ is the adjoint of $E_{ret/adv}$ and so*

$$E_{adv/ret} : \mathcal{E}'(M, E) \to \mathcal{D}'(M, E).$$

(4) *The following complex is exact:*

$$0 \to C_c^\infty(M, E) \xrightarrow{P} C_c^\infty(M, E) \xrightarrow{E_{adv}-E_{ret}} C^\infty(M, E) \xrightarrow{P} C^\infty(M, E)$$

We introduce the notation $E_{caus} := E_{adv} - E_{ret}$ and call E the ***causal propagator****.*

If we already knew that the Cauchy problem for P was well posed on all Cauchy hypersurfaces $\Sigma \subseteq M$, one could give a pretty intuitive definition of $E_{adv/ret}$. Namely, given $\phi \in C_c^\infty(M, E)$, $E_{adv}\phi$ is obtained by solving the Cauchy problem $Pu = \phi$ for u where u has vanishing Cauchy data in the causal future of $\mathrm{supp}(\phi)$. Conversely, $E_{ret}\phi$ arises from solving the Cauchy problem with vanishing Cauchy data in the causal past of $\mathrm{supp}(\phi)$. Thus one can think of $E_{ret}\phi$ as the wave in the vacuum propagating forward in time from the source ϕ, and indeed it is E_{ret} that has physical significance in electrodynamics.

Since we will frequently be computing with integrals over Σ, let's briefly note that our explicit standard forms for g from 1.1 imply

$$dV_\Sigma = dV_h = N^{-1}(N^2 - |\eta|_h^2)^{1/2} dV_{\widetilde{h}} \quad \text{and} \quad dV_g = N dt dV_h$$

since indeed $g(\widehat{n}, -) = -N dt$. In order to present the solution to the Cauchy problem we first need the following lemma.

Lemma 2.1.2. *Let ϕ_1, ϕ_2 be smooth sections of E over M, $\Sigma \subseteq M$ a Cauchy hypersurface and $\widehat{n}$ the future directed unit normal to Σ. If one of ϕ_1, ϕ_2 has compact support in $J^+(\Sigma)$ then*

$$\int_M \left[\langle P\phi_1, \phi_2\rangle - \langle \phi_1, P\phi_2\rangle\right] dV_g = -\int_\Sigma \left[\langle \nabla^A_{\widehat{n}}\phi_1, \phi_2\rangle - \langle \phi_1, \nabla^A_{\widehat{n}}\phi_2\rangle\right] dV_\Sigma \tag{2.1}$$

Similarly if one of ϕ_1, ϕ_2 has compact support in $J^-(\Sigma)$ then we instead have

$$\int_M \left[\langle P\phi_1, \phi_2\rangle - \langle \phi_1, P\phi_2\rangle\right] dV_g = \int_\Sigma \left[\langle \nabla^A_{\widehat{n}}\phi_1, \phi_2\rangle - \langle \phi_1, \nabla^A_{\widehat{n}}\phi_2\rangle\right] dV_\Sigma$$

Proof. This is a consequence of

$$\langle P\phi_1, \phi_2\rangle - \langle \phi_1, P\phi_2\rangle = -\operatorname{div}_g\left(\left(\langle \nabla^A_\mu \phi_1, \phi_2\rangle - \langle \phi_1, \nabla^A_\mu \phi_2\rangle\right) g^{\mu\nu}\partial_\nu\right)$$

together with the divergence theorem. The difference in negative signs in front of the integrals arises when noting that $\widehat{n}$ is the unit inwards normal to $\partial J^+(\Sigma)$ while it is the unit outwards normal to $\partial J^-(\Sigma)$. Also one should take care to remember that $\widehat{n}$ is timelike so $g(\widehat{n}, \widehat{n}) = -1$. $\square$

Theorem 2.1.3. *Let $\Sigma \subseteq M$ be a Cauchy hypersurface and $u, v \in C^\infty(\Sigma, E)$. Then there exists a unique $\phi \in C^\infty(M, E)$ satisfying*

$$\begin{cases} P\phi & = 0 \\ \phi|_\Sigma & = u \\ \nabla^A_{\widehat{n}}\phi|_\Sigma & = v \end{cases}$$

and ϕ is given as follows. Define distributional sections of E on M via

$$\delta'_{\Sigma,u}(\psi) := -\int_\Sigma \langle \nabla^A_{\widehat{n}}\psi, u\rangle dV_\Sigma \quad \textit{and}$$
$$\delta_{\Sigma,v}(\psi) := \int_\Sigma \langle \psi, v\rangle dV_\Sigma.$$

Then

$$\phi = E_{caus}(\delta'_{\Sigma,u} + \delta_{\Sigma,v})$$

Proof. The distribution $\phi = E_{caus}(\delta'_{\Sigma,u} + \delta_{\Sigma,u})$ is defined on a compactly supported test section f by:

$$(E_{caus}(\delta'_{\Sigma,u} + \delta_{\Sigma,v}), f) := -(\delta'_{\Sigma,u} + \delta_{\Sigma,v})(E_{caus}f) = \int_\Sigma \left[\langle \nabla^A_{\widehat{n}} E_{caus}f, u\rangle - \langle E_{caus}f, v\rangle\right] dV_\Sigma$$

and for any such f, if we denote $\phi_\pm := E_{ret/adv}(\delta'_{\Sigma,u} + \delta_{\Sigma,v})$ then we can compute:

$$\begin{aligned}
(\delta'_{\Sigma,u} + \delta_{\Sigma,v})(f) &= (\delta'_{\Sigma,u} + \delta_{\Sigma,v})(E_{adv}Pf) \\
&= (E_{ret}(\delta'_{\Sigma,u} + \delta_{\Sigma,v}))(Pf) \\
&= \int_{J^+(\Sigma)} \langle Pf, \phi_+\rangle dV_g \\
&= \int_{J^+(\Sigma)} \langle Pf, \phi\rangle dV_g \quad \text{since } \operatorname{supp}(\phi_-) \subseteq J^-(\Sigma) \\
&= \int_{J^+(\Sigma)} \left[\langle Pf, \phi\rangle - \langle f, P\phi\rangle\right] dV_g \quad \text{since } P\phi = 0 \\
&= \int_\Sigma \left[\langle f, \nabla^A_{\widehat{n}}\phi\rangle - \langle \nabla^A_{\widehat{n}} f, \phi\rangle\right] dV_g \\
&= (\delta'_{\Sigma,\phi|_\Sigma} + \delta_{\Sigma,\nabla^A_{\widehat{n}}\phi|_\Sigma})(f)
\end{aligned}$$

and since this holds for all test sections f we obtain

$$\begin{cases} \phi|_\Sigma & = u \\ \nabla^A_{\widehat{n}}\phi|_\Sigma & = v \end{cases}$$

as desired. $\square$

As a corollary, we obtain the following isomorphism analogous to our symplectomorphism 1.2.12 identifying $\mathcal{N}$ with $T^*_+M|_\Sigma$.

Corollary 2.1.4. *Denoting*

$$\ker_{C^\infty}(P) := \{\phi \in C^\infty(M, E) \ : \ P\phi = 0\}$$

it follows that for any Cauchy hypersurface $\Sigma \subseteq M$ *we have the Cauchy data isomorphism*

$$\mathrm{CD}_\Sigma : \ker_{C^\infty}(P) \xrightarrow{\cong} C^\infty(\Sigma, E) \oplus C^\infty(\Sigma, E)$$
$$\phi \mapsto \big(\phi|_\Sigma, \ (\nabla^A_{\hat{n}}\phi)|_\Sigma\big)$$

which endows $\ker_{C^\infty}(P)$ *with the skew-hermitian form*

$$\sigma(\phi_1, \phi_2) := \int_\Sigma \left[\langle \nabla^A_{\hat{n}}\phi_1, \phi_2\rangle - \langle \phi_1 \nabla^A_{\hat{n}}\phi_2\rangle\right] dV_\Sigma$$

whose real part is a non-degenerate linear symplectic form. Furthermore, σ *satisfies*

$$\sigma(E_{caus}f_1, E_{caus}f_2) = \int_M \langle f_1, E_{caus}f_2\rangle dV_g = -\int_M \langle E_{caus}f_1, f_2\rangle dV_g$$

At this point we will assume our metric g has be placed in the standard form 1.1.9 that exists for all globally hyperbolic spacetimes. We recall that this means

$$M = \mathbb{R}_t \times \Sigma$$
$$g = -\alpha^2 dt^2 + k_t$$

for $\alpha : M \to \mathbb{R}_{>0}$ some smooth function and k_t a smooth family of t-dependent Riemannian metrics on Σ. Recall that each $\{t\} \times \Sigma$ is a Cauchy hypersurface and that if g was instead in the standard form 1.1.11 for stationary spacetimes, in can be placed in the above standard form 1.1.9 via the action of a t-dependent diffeomorphism of Σ.

We also make the further assumption:

$$\langle -, - \rangle \text{ is a positive definite hermitian fiber metric.}$$

We'll see that while our previous results on the advanced and retarded fundamental solutions to P can be applied to $\not{D}^2$ when working in the Dirac-type operator case, the above assumption prohibits us from using the following results in that context.

Definition 2.1.5. Under all of the above assumptions, we fix $t_1 < t_2$ and $s \in \mathbb{R}$. We define the finite energy spaces:

$$\mathrm{FE}^s([t_1,t_2] \times \Sigma, E) := C^0([t_1,t_2], W^{s,2}(\Sigma, E)) \cap C^1([t_1,t_2], W^{s-1,2}(\Sigma, E))$$

$$\mathrm{FE}^s([t_1,t_2] \times \Sigma, E, P) := \{\phi \in \mathrm{FE}^s([t_1,t_2] \times \Sigma, E) \ : \ P\phi \in L^2([t_1,t_2], W^{s-1,2}(\Sigma, E))\}$$

$$\mathrm{FE}^s(\ker P) := \{\phi \in \mathrm{FE}^s([t_1,t_2] \times \Sigma, E, P) \ : \ P\phi = 0\}$$

Note that we used the assumption that $\langle -, - \rangle$ was positive definite together with the connection A corresponding to P to define the Sobolev spaces $W^{s,2}(\Sigma, E)$ of sections.

The reason for using the standard form 1.1.9 for the metric here is that the wave operator $-\operatorname{Tr}(\nabla^A \circ \nabla^A)$ takes on a form in which there are no terms consisting of derivatives in both spatial and time directions. More precisely we have

$$-\operatorname{Tr}_g(\nabla^A \circ \nabla^A)\phi = \alpha^{-1}|k_t|^{-1/2}\nabla^A_0\left(\alpha^{-1}|k_t|^{1/2}\nabla^A_0\phi\right) - \alpha^{-1}|k_t|^{-1/2}\nabla^A_i\left(\alpha^{-1}|k_t|^{1/2}k_t^{ij}\nabla^A_j\phi\right)$$

thus all terms involving time derivatives have no space derivatives. Furthermore the future-directed unit normal to each $\{t\} \times \Sigma$ is particularly simple when g is written in this standard form: $\widehat{n} = \alpha^{-1}\partial_t$. Using this together with Gronwall's inequality and many applications of Cauchy-Schwarz one can obtain the following energy estimates.

Theorem 2.1.6. *[**5**] For any $t_1 < t_2$ and $s \in \mathbb{R}$ there exists a constant $C > 0$ depending on t_1, t_2, s and on the metric g in the region $[t_1, t_2] \times \Sigma$ such that for all $\phi \in \mathrm{FE}^s([t_1, t_2] \times \Sigma, E, P)$ we have the energy estimate*

$$\begin{aligned}\|\phi(t_2)\|_{W^{s,2}(\Sigma)} + \|(\nabla^A_{\widehat{\nu}}\phi)(t_2)\|_{W^{s-1,2}(\Sigma)} \leq \left(\|\phi(t_1)\|_{W^{s,2}(\Sigma)} + \|(\nabla^A_{\widehat{n}}\phi)(t_1)\|_{W^{s-1,2}(\Sigma)}\right) e^{C(t_2-t_1)} \\ + \int_{t_1}^{t_2} e^{C(t_2-t)}\|(P\phi)(t)\|_{W^{s-1,2}(\Sigma)} dt\end{aligned}$$

We will need two corollaries of the above from [**5**].

Corollary 2.1.7. *Via the energy estimates, elements of $\mathrm{FE}^s(\ker P)$ extend uniquely to continuous functions $\mathbb{R} \to W^{s,2}(\Sigma, E)$ which are C^1 as maps into $W^{s-1,2}(\Sigma, E)$, hence why we didn't include t_1, t_2 in the notation for $\mathrm{FE}^s(\ker P)$. When $s = 1$, $\mathrm{FE}^1(\ker P)$ is independent (up to bounded isomorphism with bounded inverse) of the choice of foliation by Cauchy hypersurfaces.*

Recall that when converting a metric in standard stationary form 1.1.11 to the above standard form 1.1.9, the resulting isometry acts as a diffeomorphism on each $\{t\} \times \Sigma$ individually but preserves the foliation as a whole.

Corollary 2.1.8. *For each $s \in \mathbb{R}$ our Cauchy data isomorphism CD_Σ extends uniquely to a Cauchy data isomorphism*

$$\mathrm{CD}_\Sigma : \mathrm{FE}^s(\ker P) \xrightarrow{\cong} W^{s,2}(\Sigma, E) \oplus W^{s-1,2}(\Sigma, E)$$

Our next step is to perform the same analysis for Dirac-type operators. While this will begin quite analogously to the above case of normally hyperbolic operators, we'll see that there are some fundamental differences in the assumptions we will make when studying finite energy solutions. For now, we make completely analogous assumptions to those from the start of this section:

(1) M is a globally hyperbolic spacetime with complex vector bundle $E \to M$ and non-degenerate sesquilinear form $\langle -, - \rangle$.
(2) $\not{D}$ is a $\langle -, - \rangle$-formally-self-adjoint Dirac-type operator on E.
(3) For simplicity we continue to assume M is spatially compact. Again, the contents in the rest of this section remain true if one adds in assumptions of compact spacelike support everywhere instead.

Just as in the normally hyperbolic case, there are two fundamental solutions for $\not{D}$ relevant to solving the Cauchy problem.

Theorem 2.1.9. *[26] [21] Let $E_{adv/ret}$ be the advanced and retarded fundamental solutions for the normally hyperbolic operator $\not{D}^2$ and E_{caus} its causal propagator. Then*

$$G_{adv/ret} := \not{D} E_{adv/ret} \in \mathcal{D}'(M \times M, E \boxtimes E^*)$$

are fundamental solutions for $\not{D}$ *and if we set*

$$G_{caus} := \not{D} E_{caus}$$

then they satisfy the following properties:

(1) $G_{adv/ret} : C_c^\infty(M,E) \to C^\infty(M,E)$,

(2) *Given* $\psi \in C_c^\infty(M,E)$, $G_{adv}\psi$ *is the unique solution to* $\not{D} G_{adv}\psi = \psi$ *satisfying*

$$\operatorname{supp}(G_{adv}\psi) \subseteq J_-(\operatorname{supp}\psi).$$

Similarly $G_{ret}\psi$ *is the unique solution to* $\not{D} G_{ret}\psi = \psi$ *satisfying*

$$\operatorname{supp}(G_{ret}\psi) \subseteq J_+(\operatorname{supp}\psi).$$

(3) *Under our assumptions that* $\not{D}$ *is formally self-adjoint, it follows that* $G_{adv/ret}$ *is the adjoint of* $G_{ret/adv}$ *and so*

$$G_{adv/ret} : \mathcal{E}'(M,E) \to \mathcal{D}'(M,E).$$

(4) *The following complex is exact:*

$$0 \to C_c^\infty(M,E) \xrightarrow{\not{D}} C_c^\infty(M,E) \xrightarrow{G_{caus}} C^\infty(M,E) \xrightarrow{\not{D}} C^\infty(M,E).$$

The next result is slightly different than in the globally hyperbolic case (aside from the order of the operator being different) since we will later want to use $\langle \not{n}(-), - \rangle$ not $\langle -, - \rangle$ when integrating over Σ in our Cauchy data isomorphism.

Lemma 2.1.10. *Let $\Sigma \subseteq M$ be a Cauchy hypersurface with unit forward normal $\widehat{n}$ and $\psi_1, \psi_2 \in C^\infty(M, E)$ with at least one of ψ_1, ψ_2 having compact support in $J^+(\Sigma)$. Then*

$$\int_M \left[\langle \not{D}\psi_1, \psi_2\rangle - \langle \psi_1, \not{D}\psi_2\rangle\right] dV_g = i \int_\Sigma \langle \not{\widehat{n}}\psi_1, \psi_2\rangle dV_\Sigma$$

If instead we had at least one of ψ_1, ψ_2 having compact support in $J^-(\Sigma)$ then we instead obtain

$$\int_M \left[\langle \not{D}\psi_1, \psi_2\rangle - \langle \psi_1, \not{D}\psi_2\rangle\right] dV_g = -i \int_\Sigma \langle \not{\widehat{n}}\psi_1, \psi_2\rangle dV_\Sigma$$

Proof. As before, this follows from an application of the divergence theorem using that

$$\mathrm{div}_g \left(i g^{\mu\nu}\langle \not{\partial}_\mu \psi_1, \psi_2\rangle \partial_\nu\right) = \langle \not{D}\psi_1, \psi_2\rangle - \langle \psi_1, \not{D}\psi_2\rangle$$

Again, the signs in front of the integrals come from $\widehat{n}$ being inward-pointing for $J^+(\Sigma)$ and outwards-pointing for $J^-(\Sigma)$, and from $g(\widehat{n}, \widehat{n}) = -1$. $\square$

Theorem 2.1.11. *Let $\Sigma \subseteq M$ be a Cauchy hypersurface and $s \in C^\infty(\Sigma, E)$. Then there exists a unique $\psi \in C^\infty(M, E)$ such that*

$$\begin{cases} \not{D}\psi & = 0 \\ \psi|_\Sigma & = s \end{cases}$$

and ψ is given by $\psi = G_{caus}(\not{\delta}_{\Sigma,s})$ where $\not{\delta}_{\Sigma,s} \in \mathcal{E}'(M, E)$ is the distributional section given by

$$\not{\delta}_{\Sigma,s}(f) := i \int_\Sigma \langle \not{\widehat{n}} f, s\rangle dV_\Sigma$$

Proof. By construction we clearly have $\not{D}\psi = 0$ so let's check what its restriction to Σ is. We write $\psi_\pm := G_{ret/adv}(\not{\delta}_{\Sigma,s})$ and for f a smooth compactly supported test section we compute

$$\begin{aligned}
\not{\delta}_{\Sigma,s}(f) &= \not{\delta}_{\Sigma,s}(G_{adv}\not{D}f) \\
&= \int_M \langle \not{D}f, \psi_+\rangle dV_g \\
&= \int_{J^+(\Sigma)} \langle \not{D}f, \psi_+\rangle dV_g \\
&= \int_{J^+(\Sigma)} \langle \not{D}f, \psi\rangle dV_g \;\; \text{since } \operatorname{supp}(\psi_-) \subseteq J^-(\Sigma) \\
&= \int_{J^+(\Sigma)} \left[\langle \not{D}f, \psi\rangle - \langle f, \not{D}\psi\rangle\right] dV_g \;\; \text{since } \not{D}\psi = 0 \\
&= i\int_\Sigma \langle \not{n}f, \psi\rangle dV_\Sigma \\
&= \not{\delta}_{\Sigma,\psi|_\Sigma}
\end{aligned}$$

as desired. □

Corollary 2.1.12. *Again, we denote by* $\ker_{C^\infty}(\not{D})$ *the collection of smooth sections* $\psi \in C^\infty(M,E)$ *such that* $\not{D}\psi = 0$*. For each Cauchy hypersurface* $\Sigma \subseteq M$ *the restriction map defines a Cauchy data isomorphism*

$$\mathrm{CD}_\Sigma : \ker_{C^\infty}(\not{D}) \xrightarrow{\cong} C^\infty(\Sigma, E)$$

$$\psi \mapsto \psi|_\Sigma$$

At this point we again fix a foliation by Cauchy hypersurfaces. However, unlike the normally hyperbolic case our operator $\not{D}$ is first order and so there are no mixed space-time derivatives to worry about. Thus our below results will apply to any smooth foliation

$$M = \mathbb{R}_t \times \Sigma$$

by Cauchy hypersurfaces, regardless of the form of the metric. However, as we saw earlier with the example of $\mathbb{C}$-valued differential forms, if we want $\not{D}$ to be $\langle -, - \rangle$-formally self-adjoint we cannot require $\langle -, - \rangle$ to be positive definite as we did in the normally hyperbolic case. Nevertheless, the energy estimates do still hold if one drops the assumption that $\not{D}$ or P is formally self adjoint and simply equips E with an auxiliary positive definite hermitian fiber metric [**5**], [**3**]. We simply chose to assume this metric could be taken to be $\langle -, - \rangle$ in the normally hyperbolic case since this is what occurs in practice. So from this point we:

(1) fix any smooth foliation $M = \mathbb{R}_t \times \Sigma$ by Cauchy hypersurfaces with ∇t past-directed timelike, and
(2) fix an auxiliary positive definite Hermitian fiber metric for E which we use to define our Sobolev spaces (and we make no assumption concerning self-adjointness with respect to this auxiliary fiber metric).

Definition 2.1.13. Given $t_1 < t_2$ and $s \in \mathbb{R}$ we define the finite energy spaces:

$$\mathrm{FE}^s_D([t_1, t_2] \times \Sigma, E) := C^0([t_1, t_2], W^{s,2}(\Sigma, E))$$

$$\mathrm{FE}^s_D([t_1, t_2] \times \Sigma, E, \not{D}) := \{\psi \in \mathrm{FE}^s_D([t_1, t_2] \times \Sigma, E) \ : \ \not{D}\psi \in L^2([t_1, t_2], W^{s-1,2}(\Sigma, E))\}$$

$$\mathrm{FE}^s_D(\ker \not{D}) := \{\psi \in \mathrm{FE}^s_D([t_1, t_2] \times \Sigma, E) \ : \ \not{D}\psi = 0\}$$

Similar energy estimates, now from chapter 4 of [**36**], allow one to extend our Cauchy data isomorphism CD_Σ (for the Dirac equation) to the finite energy spaces.

Theorem 2.1.14. *[3] For every $s \in \mathbb{R}$, elements of $\mathrm{FE}^s_D(\ker \not{D})$ extend uniquely to continuous functions $\mathbb{R} \to W^{s,2}(\Sigma, E)$ and for $s = 0$, $\mathrm{FE}^0_D(\ker \not{D})$ is independent (up to bounded isomorphism with bounded inverse) of the choice of foliation by Cauchy hypersurfaces. Furthermore, CD_Σ extends uniquely to a Cauchy data isomorphism*

$$\mathrm{CD}_\Sigma : \mathrm{FE}^s_D(\ker \not{D}) \xrightarrow{\cong} W^{s,2}(\Sigma, E)$$

for each $s \in \mathbb{R}$ and every Cauchy hypersurface Σ.

2.2. The Stress Tensor, Dirac Charge and Time Translation

In this section we specialize the results of the previous section to the stationary case and begin our analysis of the frequencies of solutions to normally hyperbolic and Dirac-type equations on stationary spacetimes. Therefore we assume the following throughout:

(1) (M, g) is a globally hyperbolic spatially compact spacetime together with complex vector bundle $E \to M$, non-degenerate sesquilinear form $\langle -, - \rangle$ and compatible connection A.

(2) (M, g) is stationary and in the standard form 1.1.11 with complete timelike Killing vector field $Z := \partial_t$.

(3) The bundle E, connection A and sesquilinear form $\langle -, - \rangle$ have all been placed in temporal gauge and $Z \llcorner F_A = 0$. Thus E is the pullback of a bundle over Σ, $\langle -, - \rangle$ is t-independent and A has no dt-component (and thus $\partial_t A = \partial_t \llcorner F_A = 0$).

Examples of the above setup are obtained by simply choosing any bundle with fixed fiber metric and compatible connection over Σ and then pulling them back to M along the projection $\mathbb{R}_t \times \Sigma \to \Sigma$.

In the first part of this section, we fix a normally hyperbolic operator P on E with A as its corresponding connection. Thus

$$P = -\operatorname{Tr}_g(\nabla^A \circ \nabla^A) + \Upsilon$$

and include the following additional assumption concerning P:

(4) $\nabla^{A,\mathrm{End}}_{\partial_t} \Upsilon = \partial_t \Upsilon = 0$ and P is $\langle -, - \rangle$-formally-self-adjoint.

From the beginning of 2 we know that these assumptions all together imply that P commutes with $\nabla^A_Z = \nabla^A_{\partial_t}$ and we arrive at one of the main goals of this thesis:

we wish to understand the spectral theory of $D_Z := -i \nabla^A_Z$ on $\ker_{C^\infty}(P)$.

Unfortunately D_Z is not typically self-adjoint with respect to any of the inner products obtained from the spaces $\mathrm{FE}^s(\ker P)$ and so, as done in **[34]** for the scalar case, we seek an inner product based on the stress-energy tensor.

Definition 2.2.1. As the equation $P\phi = 0$ arises as the Euler-Lagrange equations for compactly supported variations of

$$\int_M \left[\langle \nabla^A \phi, \nabla^A \phi \rangle + \langle \Upsilon \phi, \phi \rangle\right] dV_g$$

we have an associated stress-energy tensor

$$T_{\mu\nu} = T(\phi)_{\mu\nu} := \Re\langle \nabla^A_\mu \phi, \nabla^A_\nu \phi \rangle - \frac{1}{2}\left(\langle \nabla^A \phi, \nabla^A \phi \rangle + \langle \Upsilon \phi, \phi \rangle\right) g_{\mu\nu}$$

This is a $\mathbb{C}$-valued symmetric 2-tensor on M.

Lemma 2.2.2. *Let ϕ be a smooth solution to $P\phi = 0$ on M. Then*

$$\operatorname{div}_g(T(\phi)) = -\frac{1}{2}\langle (\nabla^{A,\mathrm{End}} \Upsilon)\phi, \phi \rangle$$

Proof. Writing

$$\Re\langle \nabla^A_\mu \phi, \nabla^A_\nu \phi \rangle = \frac{1}{2}\left(\langle \nabla^A_\mu \phi, \nabla^A_\nu \phi \rangle + \langle \nabla^A_\nu \phi, \nabla^A_\mu \phi \rangle\right)$$

we simply compute $\operatorname{div}_g(T)_\nu = |g|^{-1/2}\partial_\mu(|g|^{1/2} g^{\mu\omega} T_{\omega\nu})$ using that A is $\langle -, - \rangle$-compatible. $\square$

Since Υ is assumed to be time-independent and Z is assumed to be Killing we obtain from this the following conserved quantity.

Lemma 2.2.3. *If ϕ is a smooth solution to $P\phi = 0$ then*

$$Q(\phi) := \int_{\{t\}\times\Sigma} T(\phi)(Z, \widehat{n}) dV_\Sigma$$

is independent of our choice of t. The quantity obtained by applying the polarization identity to Q is then a sesquilinear form on $\ker_{C^\infty}(P)$ and it is given by

$$\langle \phi_1, \phi_2 \rangle_Q = \frac{1}{2} \int_\Sigma \langle \nabla^A_Z \phi_1, \nabla^A_Z \phi_2 \rangle dV_\Sigma + \frac{1}{2} \int_\Sigma \left[\widetilde{h}^{ij} \langle \nabla^A_i \phi_1, \nabla^A_j \phi_2 \rangle + \langle \Upsilon \phi_1, \phi_2 \rangle \right] N dV_\Sigma$$

Proof. By the divergence theorem it suffices to calculate the divergence of $T(\phi)(Z, -)$. However since Z is Killing the symmetric part of ∇Z vanishes and so

$$\mathrm{div}_g(Z \llcorner T(\phi)) = (\mathrm{div}_g T)(Z) + T(\nabla Z) = 0 + T(\mathrm{Sym}(\nabla Z)) = 0$$

as desired. $\square$

We now arrive at the key lemma allowing us to analyze the spectrum of D_Z.

Lemma 2.2.4. *Since $D_Z = -i\nabla^A_Z$ commutes with P we can let e^{isD_Z} denote the operator on $\ker_{C^\infty}(P)$ given by composing $\phi \in \ker_{C^\infty}(P)$ with the flow of Z (so $\phi(t, x)$ gets mapped to $\phi(t + s, x)$). Then D_Z and e^{isD_Z} are respectively symmetric and unitary on $\ker_{C^\infty}(P)$ with respect to $\langle -, - \rangle_Q$. Furthermore we have*

$$\langle \phi_1, \phi_2 \rangle_Q = \frac{i}{2} \sigma(\phi_1, D_Z \phi_2)$$

Proof. The fact that e^{isD_Z} is unitary in the sense that

$$\langle e^{isD_Z} \phi_1, \phi_2 \rangle_Q = \langle \phi_1, e^{-isD_Z} \phi_2 \rangle_Q$$

for all $\phi_1, \phi_2 \in \ker_{C^\infty}(P)$ is precisely the statement of our previous lemma since the flow of Z is an isometry and $A, \langle -, - \rangle$ are t-independent. Differentiating this relation in s at

$s = 0$ then yields

$$\langle D_Z\phi_1, \phi_2\rangle_Q = \langle \phi_1, D_Z\phi_2\rangle$$

for all ϕ_1, ϕ_2 as desired. For the relation between $\langle -, -\rangle_Q$ and our skew-hermitian form σ we notice that for $\phi \in \ker_{C^\infty}$, we have the following identities of differential forms where $*_\Sigma, *$ are respectively the Hodge-$*$ operators on Σ and M and we use that ∇^A_Z commutes with $\nabla^A_{\widehat{n}}$ due to $Z \llcorner F_A = 0$ and $\widehat{n}$ being t-independent.

$$\begin{aligned}\left[\langle \nabla^A_{\widehat{n}}\phi, \nabla^A_Z\phi\rangle - \langle \phi, \nabla^A_{\widehat{n}}\nabla^A_Z\phi\rangle\right] *_\Sigma 1 &= *\left(\langle \nabla^A_{(-)}\phi, \nabla^A_Z\phi\rangle - \langle \phi, \nabla^A_Z\nabla^A_{(-)}\phi\rangle\right) \\ &= 2T(\phi)(Z, \widehat{n}) *_\Sigma 1 - 2d(Z \llcorner *(\Re\langle \nabla^A_{(-)}\phi, \phi\rangle))\end{aligned}$$

Integrating over Σ and applying Stokes yields our desired result. $\square$

Unfortunately we arrive at the main problem with $\langle -, -\rangle_Q$: it need not be non-degenerate, nor need it be positive definite, even when our fiber metric $\langle -, -\rangle$ is. The below proposition is our saving grace.

Proposition 2.2.5. *Any $\phi \in \ker_{C^\infty}(P)$ for which $\langle \phi, -\rangle_Q \equiv 0$ on $\ker_{C^\infty}(P)$ satisfies $D_Z\phi = 0$ and the set of all such ϕ lies in the kernel of a second-order elliptic operator on Σ. In particular, since Σ is compact, the set of such degenerate ϕ's is finite dimensional.*

Proof. Let $\phi \in \ker_{C^\infty}(P)$ be such that $\langle \phi, -\rangle_Q \equiv 0$ on $\ker_{C^\infty}(P)$. By well-posedness of the Cauchy problem and since $\widehat{n} = N^{-1}(Z - \vec{\beta})$, the set of sections over Σ of the form $(\nabla^A_Z\widetilde{\phi})|_\Sigma$ with $\widetilde{\phi} \in \ker_{C^\infty}(P)$ for which $\widetilde{\phi}|_\Sigma \equiv 0$ is all of $C^\infty(\Sigma, E)$. Thus using non-degeneracy of $\langle -, -\rangle$ and $\langle \phi, \widetilde{\phi}\rangle_Q = 0$ for all such sections we obtain

$$0 = \nabla^A_Z\phi = iD_Z\phi$$

Thus for all $\widetilde{\phi} \in \ker_{C^\infty}$ we have

$$0 = \langle \phi, \widetilde{\phi} \rangle_Q = \frac{1}{2} \int_\Sigma \left[\widetilde{h}^{ij} \langle \nabla_i^A \phi, \nabla_j^A \widetilde{\phi} \rangle + \langle \Upsilon \phi, \widetilde{\phi} \rangle \right] N dV_\Sigma$$

Thus

$$(\nabla_j^A)^{*,\widetilde{h}} \left(N \widetilde{h}^{ij} \nabla_i^A \phi \right) + N \Upsilon \phi = 0$$

i.e. ϕ is in the kernel of an elliptic operator on Σ, as desired. □

Without assuming that the fiber metric $\langle -, - \rangle$ is positive definite, there is not much more that can be done at this point. Before assuming this, we state the below proposition which expresses the equation $P\phi = 0$ in terms of a quadratic operator pencil as was done in [**34**]. In order to proceed further without assuming $\langle -, - \rangle$ is positive definite, one may have to develop a theory analogous to [**32**] where the resolvent of an operator is replaced with the resolvent of an operator pencil as below.

Proposition 2.2.6. *Define*

$$\begin{aligned} D_Z &:= -i \nabla_0^A, \\ e^f &:= (N^2 - |\eta|_h^2)^{-1/4} N^{(3-n)/2} \\ \widehat{X} &:= \frac{1}{2} \left(\nabla_{\vec{\beta}}^A - \left(\nabla_{\vec{\beta}}^A \right)^{*, N^{-2}\widetilde{h}} \right) \ \textit{and} \\ \Delta_{\widetilde{h}}^A &:= |N^{-2} \widetilde{h}|^{-1/2} \nabla_i^A \left(|N^{-2}\widetilde{h}|^{1/2} N^2 \widetilde{h}^{ij} \nabla_j^A (-) \right) \end{aligned}$$

Then for all $\phi \in C^\infty(M, E)$ we have

$$P\phi = N^{-2} e^f \left(-D_Z^2 - 2i \widehat{X} D_Z - \Delta_{\widetilde{h}}^A + e^{2f} (\Delta_{N^{-2}\widetilde{h}} e^{-f}) + N^2 \Upsilon \right) e^{-f} \phi.$$

Proof. We begin by recalling the relation

$$N^{-1}|g|^{1/2} = |h|^{1/2} = N^{-1}(N^2 - |\eta|_h^2)^{1/2}|\widetilde{h}|^{1/2}$$

between the determinants that follows from the explicit expressions for g, g^{-1} from the start of 1.2. Using $F^A_{0\mu} = 0$ we then compute

$$\begin{aligned}
|h|^{1/2}\operatorname{Tr}_g(\nabla^A \circ \nabla^A)u &= \nabla^A_\mu \left(|h|^{1/2} g^{\mu\nu}\nabla^A_\nu u\right) \\
&= -\nabla^A_0\left(|h|^{1/2}N^{-2}\nabla^A_0 u\right) + \nabla^A_0\left(|h|^{1/2}N^{-2}\beta^j\nabla^A_j u\right) \\
&\quad + \nabla^A_j\left(|h|^{1/2}N^{-2}\beta^j\nabla^A_0 u\right) + \nabla^A_i\left(|h|^{1/2}N^{-2}\widetilde{h}^{ij}\nabla^A_j u\right) \\
&= -|h|^{1/2}N^{-2}\left(\nabla^A_0\right)^2 u + |N^{-2}\widetilde{h}|^{1/2}N^{n-3}(N^2 - |\eta|_h^2)^{1/2}\nabla^A_{\vec{\beta}}\nabla^A_0 u \\
&\quad + \nabla^A_j\left(N^{-2}N^{n-1}(N^2 - |\eta|_h^2)^{1/2}|N^{-2}\widetilde{h}|^{1/2}\beta^j\nabla^A_0 u\right) \\
&\quad + \nabla^A_i\left(N^{-2}N^{n-1}(N^2 - |\eta|_h^2)^{1/2}|N^{-2}\widetilde{h}|^{1/2}N^2\widetilde{h}^{ij}\nabla^A_j u\right) \\
&= -|h|^{1/2}N^{-2}(\nabla^A_0)^2 u + |h|^{1/2}N^{-2}(2\nabla^A_{\vec{\beta}} + \operatorname{div}_{N^{-2}\widetilde{h}}(\vec{\beta}))\nabla^A_0 u \\
&\quad + \mathcal{L}_{\vec{\beta}}(N^{n-3}(N^2 - |\eta|_h^2)^{1/2})|N^{-2}\widetilde{h}|^{1/2}\nabla^A_0 u \\
&\quad + N^{-2}|h|^{1/2}\operatorname{Tr}_{N^{-2}\widetilde{h}}(\nabla^A \circ \nabla^A)u \\
&\quad + |N^{-2}\widetilde{h}|^{1/2}N^2\widetilde{h}^{-1}(d(N^{n-3}(N^2 - |\eta|_h^2)^{1/2}), \nabla^A u)
\end{aligned}$$

Using the notation $\Delta^A_{\widetilde{h}} = \operatorname{Tr}_{N^{-2}\widetilde{h}}(\nabla^A \circ \nabla^A)$ (despite the fact that we allow A to have a dt-component) and our above definition of f we can simplify this to get

$$N^2\operatorname{Tr}_g(\nabla^A \circ \nabla^A)u = -(\nabla^A_0)^2 u + (\nabla^A_{\vec{\beta}} - (\nabla^A_{\vec{\beta}})^{*,N^{-2}\widetilde{h}})\nabla^A_0 u + e^{2f}(\mathcal{L}_{\vec{\beta}}e^{-2f})\nabla^A_0 u$$

$$
\begin{aligned}
&+ \Delta^A_{\widetilde{h}} u + e^{2f} N^2 \widetilde{h}^{-1}(de^{-2f}, \nabla^A u) \\
&= -e^f (\nabla^A_0)^2 (e^{-f} u) + e^f \left(\nabla^A_{\vec{\beta}} - (\nabla^A_{\vec{\beta}})^{*,N^{-2}\widetilde{h}} \right) \nabla^A_0 (e^{-f} u) \\
&+ e^f (\Delta^A_{\widetilde{h}} - e^{2f} f (\Delta_{N^{-2}\widetilde{h}} e^{-f}))(e^{-f} u)
\end{aligned}
$$

Recalling now that $D_Z = -i\nabla^A_0$ and denoting

$$
\widehat{X} := \frac{1}{2} \left(\nabla^A_{\vec{\beta}} - (\nabla^A_{\vec{\beta}})^{*,N^{-2}\widetilde{h}} \right)
$$

we arrive at

$$
\mathrm{Tr}_g(\nabla^A \circ \nabla^A) u = N^{-2} e^f \left(D_Z^2 + 2i\widehat{X} D_Z + \Delta^A_{\widetilde{h}} - e^{2f} (\Delta_{N^{-2}\widetilde{h}} e^{-f}) \right) e^{-f} u
$$

$\square$

As mentioned before, in order to proceed we make the following assumption:

the sesquilinear fiber metric $\langle -, - \rangle$ is assumed to be positive definite hermitian.

Definition 2.2.7. We denote by $\ker P$ the space $\mathrm{FE}^1(\ker P) \supseteq \ker_{C^\infty}(P)$ equipped with the topology of $\mathrm{FE}^1(\ker P)$ and the (continuous) sesquilinear form $\langle -, - \rangle_Q$.

Lemma 2.2.8. *Let D_Z act on $\ker P$ with domain*

$$
\mathrm{dom}(D_Z) := \mathrm{FE}^2(\ker P) \subseteq \ker P
$$

Then D_Z is a closed densely defined operator on $\ker P$ and is symmetric with respect to the Hermitian form $\langle -, - \rangle_Q$.

Proof. Once we show D_Z is closed and densely defined on $\ker P$ with domain $\mathrm{FE}^2(\ker P)$ it follows from our previous computations and density that it is $\langle -,-\rangle_Q$-symmetric. However, we can employ the modified Cauchy data isomorphism

$$\mathrm{CD}_\Sigma : \mathrm{FE}^2(\ker P) \to W^{2,2}(\Sigma, E) \oplus W^{1,2}(\Sigma, E)$$

where the normal derivative $\nabla^A_{\widehat{n}}\phi$ is replaced by $\nabla^A_{\partial_t}\phi$ in order to express D_Z as a 2×2 block matrix via the operator pencil 2.2.6 description of $\ker P$:

$$D_Z = iM_{e^f}\begin{pmatrix} 0 & -1 \\ -\Delta^A_{N^{-2}\widetilde{h}} + W & -2\widehat{X} \end{pmatrix} M_{e^{-f}}$$

where W is the potential term

$$W = e^{2f}(\Delta_{N^{-2}\widetilde{h}}e^{-f}) + N^2\Upsilon$$

Thus D_Z is conjugate to a closed operator by the bounded invertible multiplication operators $M_{e^f}, M_{e^{-f}}$, as desired. $\square$

Since we are now assuming that our fiber metric is positive definite, it follows that if Υ is non-negative definite then so is $\langle -,-\rangle_Q$ and, since we assume M is connected, if Υ non-negative definite on all of M and is positive definite at even just one point then $\langle -,-\rangle_Q$ will be a positive definite Hermitian inner product on $\ker P$. However, we have good reason to allow for negative potential terms Υ. The scalar wave equation obtained from applying a conformal change to g in the Hilbert-Einstein action is:

$$-\Box_g\phi + \frac{n-1}{4n}\operatorname{Scal}_g\phi = 0$$

Specifically, considering the conformal metrics $\widetilde{g} := e^{2f}g$ where $\phi := e^{(n-2)f/2}$ one can compute the Hilbert-Einstein action:

$$\int_M \mathrm{Scal}_{\widetilde{g}}\, dV_{\widetilde{g}} = \int_M \phi \left(-\Box_g \phi + \frac{n-1}{4n} \mathrm{Scal}_g \, \phi \right) dV_g$$

and so we can obtain interesting wave equations with negative potential terms from metrics with scalar curvature that is negative (even if it is only negative in some regions).

The general case where $\langle -, - \rangle_Q$ need not be positive definite, or even non-degenerate, will be handled as in [**34**] with the general theory of Pontryagin and Krein spaces [**23**], [**12**]. These are "Hilbert spaces" for which the inner product is permitted to have finite dimensional negative-definite and/or degenerate subspaces. The proof of the next lemma is verbatim to the scalar case done in [**34**].

Lemma 2.2.9. $\ker Q$ *is finite dimensional, consists of smooth sections and satisfies* $\ker Q \subseteq \ker D_Z$*. Because of this,* $\ker P / \ker Q$ *is a Krein space when equipped with the induced Hermitian form from* $\langle -, - \rangle_Q$ *and the induced closed densely defined operator from* D_Z *is Krein-self-adjoint.*

At this point it will be useful to introduce the bundle $\overline{E}$. As a real vector bundle, we simply set $\overline{E} := E$, although we define multiplication by $i \in \mathbb{C}$ on $\overline{E}$ to be multiplication by $-i$ on E. For notational purposes, if $\phi \in E$ (or if ϕ is a section of E) then we denote the corresponding element/section of $\overline{E}$ by $\overline{\phi}$. The map $\phi \mapsto \overline{\phi}$ is a $\mathbb{C}$-antilinear isomorphism $E \to \overline{E}$.

Furthermore, we can introduce a Hermitian form on $\overline{E}$ via the one on E. Namely, we set

$$\langle\overline{\phi}_1, \overline{\phi}_2\rangle := \langle\phi_2, \phi_1\rangle = \overline{\langle\phi_1, \phi_2\rangle}$$

The connection A on E induces a connection on $\overline{E}$ with covariant derivative given simply by

$$\nabla^A_X \overline{\phi} := \overline{\nabla^A_X \phi} \quad \text{for all } X$$

and this connection is compatible with the Hermitian fiber metric on $\overline{E}$. Together with our original connection, we now have a $\langle -, -\rangle$-compatible connection on $E \oplus \overline{E}$ that commutes with $\phi \mapsto \overline{\phi}$.

Our operator P also extends to $\overline{E}$ and $E \oplus \overline{E}$ in a way that commutes with $\phi \mapsto \overline{\phi}$ by defining $\Upsilon\overline{\phi} := \overline{\Upsilon\phi}$. As such, our previous discussion all applies verbatim to our new operator P on $E \oplus \overline{E}$ and we get a space

$$\ker_{\mathbb{C}} P := \mathrm{FE}^1(\ker P) \quad \text{on } E \oplus \overline{E}.$$

Definition 2.2.10. We extend D_Z to $E \oplus \overline{E}$ via $D_Z := -i\nabla^A_Z$. Note that $D_Z\overline{\phi} = -\overline{D_Z\phi}$.

We now present a detailed study of the spectrum of D_Z on $\ker P$. Towards this end, for $\lambda \in \mathbb{C}$ we denote

$$W_\lambda := \{\phi \in \ker_{\mathbb{C}} P \ : \ \exists m \in \mathbb{Z}_{\geq 1} \text{ such that } (D_Z - \lambda)^m\phi = 0\}$$

Implicit in the above definition is that ϕ must also live in the domain of $(D_Z - \lambda)^m$. Furthermore, the value of $m \in \mathbb{Z}_{\geq 1}$ is allowed to depend on ϕ. We say that W_λ **has**

no non-trivial Jordan blocks to mean that $D_Z\phi = \lambda\phi$ for all $\phi \in W_\lambda$. Since $\ker_{\mathbb{C}} P$ consists of sections of the bundle $E \oplus \overline{E}$ which has complex conjugates, we have

$$\overline{W_\lambda} = W_{-\overline{\lambda}} \text{ for all } \lambda \in \mathbb{C}. \tag{2.2}$$

Furthermore it's easy to see that for $\lambda \neq \lambda'$ we have $W_\lambda \cap W_{\lambda'} = \{0\}$. From these observations we can see that W_λ has no non-trivial Jordan blocks if and only if $W_{-\overline{\lambda}}$ has no non-trivial Jordan blocks.

Lemma 2.2.11. *For $\lambda = 0$, $\Re\sigma$ restricts to a non-degenerate linear symplectic form on W_0. For $\lambda \neq 0$, $\Re\sigma$ restricts to a non-degenerate linear symplectic form on $W_\lambda \oplus W_{-\overline{\lambda}}$ and for any $\phi_1 \in W_\lambda$, any $\lambda' \neq -\overline{\lambda}$, and any $\phi_2 \in W_{\lambda'}$ we have $\Re\sigma(\phi_1, \phi_2) = 0$.*

Proof. First, let $\lambda \neq 0$, $\phi_1 \in W_\lambda$ and $\lambda' \neq -\overline{\lambda}$. Choose $m \geq 1$ such that $(D_Z - \lambda)^m\phi_1 = 0$. The restriction of $(D_Z + \overline{\lambda})^m$ to $W_{\lambda'}$ acts as multiplication by $(\lambda' + \overline{\lambda})^m$ which is non-zero and hence invertible. So every element of $W_{\lambda'}$ can be written as $(D_Z + \overline{\lambda})^m\phi_2$ for some $\phi_2 \in W_{\lambda'}$ and so

$$\Re\sigma\left(\phi_1, (D_Z + \overline{\lambda})^m\phi_2\right) = \Re\sigma\left((D_Z - \lambda)^m\phi_1, \phi_2\right) = 0$$

hence $\sigma(\phi_1, -)$ is identically vanishing on $W_{\lambda'}$. From the general theory of Krein and Pontryagin spaces [**34**], [**12**], [**23**] we know that the Direct sum of all W_λ's is dense in $\ker_{\mathbb{C}}(P)$ and since $\Re\sigma$ is non-degenerate it follows that when $\lambda \neq 0$ its restriction to $W_\lambda \oplus W_{-\overline{\lambda}}$ is non-degenerate symplectic and when $\lambda = 0$ it simply restricts to a non-degenerate symplectic form on W_0. □

The proof of the following lemma is again verbatim to [**34**].

Lemma 2.2.12. *We have* $\ker Q \subseteq W_0$ *and for every* $\lambda \in \mathbb{C}$, W_λ *is finite-dimensional and consists of smooth sections.*

We can now decompose $\ker_{\mathbb{C}} P$ into two parts: one in which the spectrum of D_Z is well behaved and a finite dimensional part in which we may have a non-trivial Jordan canonical form for D_Z.

Definition 2.2.13. Let W denote the closed span in $\ker_{\mathbb{C}} P$ of all W_λ's such that $\lambda \in \mathbb{R}$ and W_λ has no non-trivial Jordan blocks.

Lemma 2.2.14. *[34] For all* $\lambda \in \mathbb{R} \setminus \{0\}$, *the eigenvalues of* Q *on* W_λ *are all non-zero and there exists a decomposition into* D_Z*-invariant subspaces*

$$W_\lambda = W_{\lambda,1} \oplus \cdots \oplus W_{\lambda,k_\lambda}$$

such that Q *is sign-definite on each* $W_{\lambda,j}$ *and* σ *is symplectic (i.e. non-degenerate) on each* $W_{\lambda,j} \oplus \overline{W_{\lambda,j}}$. *We can choose this decomposition for each* $\lambda \in \mathbb{R} \setminus \{0\}$ *such that* $\overline{W_{\lambda,j}} = W_{-\lambda,j}$ *and* $k_\lambda = k_{-\lambda}$.

Theorem 2.2.15. *[34] The set of* $\lambda \in \mathbb{C}$ *such that* λ *is either in* $\mathbb{C} \setminus \mathbb{R}$ *or* $\lambda \in \mathbb{R}$ *but* W_λ *has a non-trivial Jordan block is finite. Letting* $\lambda_1, ..., \lambda_N$ *be this finite list of* λ*'s we denote*

$$\mathcal{V} := \bigoplus_{k=1}^{N} (W_{\lambda_k} \oplus \overline{W_{\lambda_k}}).$$

Then

$$\ker_{\mathbb{C}} P = W \oplus \mathcal{V}, \quad W^{\perp Q} = \ker Q \oplus \mathcal{V}, \quad \textit{and} \quad \mathcal{V}^{\perp Q} = W.$$

As such the spectrum of D_Z on $\ker_{\mathbb{C}} P$ is discrete, consists of finitely many generalized eigenvalues $\lambda_1, ..., \lambda_N$ and an infinite set of real eigenvalues accumulating only at $\pm\infty$. Furthermore the spectrum is invariant under $\lambda \leftrightarrow -\overline{\lambda}$

We now consider the case of Dirac-type operators. Our original three assumptions from the start of 2.2 are still assumed to hold, but now we fix a Dirac-type operator

$$\not{D} = -i\not{\nabla}^A + \Theta_A$$

on E and make the assumption:

(4) $\not{D}$ is $\langle -, - \rangle$-formally-self-adjoint, A is the connection corresponding to $\not{D}^2$ and $[\not{D}, \nabla^A_{\partial_t}] = 0$.

Notice how we assume both $[\not{D}, \nabla^A_{\partial_t}] = 0$ and $\partial_t \llcorner F_A = 0$ here, whereas in the normally hyperbolic case it sufficed to assume that the potential was t-independent along with $\partial_t \llcorner F_A = 0$. This is due to the results of our calculations in the introduction to 2.

As mentioned earlier, we cannot typically assume that $\langle -, - \rangle$ is positive definite in the case of a Dirac-type operator since it is not positive definite for the special cases of interest (differential forms, Spin^c-bundles etc.). As is done in **[21]**, **[3]**, **[4]** we will make the assumption that

(5) $\langle \widehat{n}(-), - \rangle$ is a positive-definite Hermitian fiber metric.

This is satisfied for twists of spinor bundles on M, assuming M admits a Spin or Spin^c structure, but is not satisfied for the bundle of $\mathbb{C}$-valued differential forms.

One major advantage of this assumption is that it avoids many of the unfortunate difficulties of the normally hyperbolic case due to this fiber metric automatically yielding a conserved quantity.

Definition 2.2.16. For $\psi \in \ker_{C^\infty}(\not{D})$ we denote by

$$j(\psi) := \langle \not{\partial}_\mu \psi, \psi \rangle dx^\mu$$

the globally-defined 1-form called the **Dirac charge**. In the introduction to 2 we already saw that $d^* j(\psi) = 0$ and so for $\psi_1, \psi_2 \in \ker_{C^\infty}(\not{D})$ we have that

$$\langle \psi_1, \psi_2 \rangle_J := \int_{\{t\}\times\Sigma} \langle \widehat{n}\psi_1, \psi_2 \rangle dV_\Sigma$$

is independent of t and is positive-definite Hermitian on $\ker_{C^\infty}(\not{D})$.

We can now simply state the below theorem whose proof, unlike the normally hyperbolic case, does not have any of the technical difficulties involving Jordan blocks.

Theorem 2.2.17. *[21] We denote* $\ker \not{D} := \mathrm{FE}^0_D(\ker \not{D})$ *equipped with the structure of a Hilbert space with inner product* $\langle -, - \rangle_J$*. If* $D_Z := -i\nabla^A_Z$ *is given the domain*

$$\mathrm{dom}(D_Z) := \mathrm{FE}^1_D(\ker \not{D}) \subseteq \ker \not{D}$$

then D_Z *is a closed densely defined self adjoint operator with respect to* $\langle -, - \rangle_J$*. Its spectrum* $\mathrm{Spec}(D_Z)$ *is discrete, contained in* $\mathbb{R}$*, consists of eigenvalues and accumulates at only* $\pm\infty$*.*

CHAPTER 3

Main Results

3.1. Representation Theory of $\ker \Box_\omega$

Here we return to the case of Kaluza-Klein spacetimes, applying the results of the previous chapter to the case of the total space P of a principal $G \subseteq \mathrm{SO}(k)$-bundle over a standard stationary spacetime M. Concretely, $M = \mathbb{R}_t \times \Sigma$ is in standard form 1.1.11, P is the pullback of a principal G-bundle $\pi : P_0 \to \Sigma$ over Σ and ω is the pullback of a connection on P_0. Then

$$P = \mathbb{R}_t \times P_0$$

with Kaluza-Klein metric

$$g_\omega = \pi^* g - \mathrm{Tr}(\omega(-)\omega(-)) = -(N^2 - |\eta|_h^2)dt^2 + dt \otimes \eta + \eta \otimes dt + h - \mathrm{Tr}(\omega(-)\omega(-))$$

where in the last equality we have omitted the pullbacks along $\pi : P_0 \to \Sigma$. We will be interested in normally hyperbolic operators on the trivial complex line bundle over P. Namely we let

$$\Box_\omega := -\Box_{g_\omega} + V \quad \text{acting on} \quad C^\infty(P, \mathbb{C})$$

where our potential V is the pullback of a smooth real-valued function on Σ. Recalling that $Z^\omega = \partial_t$ is the horizontal lift of the complete timelike Killing vector field on M (and we have placed everything in temporal gauge already) we have operators

$$D_Z := -i\mathcal{L}_{Z^\omega} \quad \text{and} \quad D_\xi := -i\mathcal{L}_{\widehat{\xi}} \quad \text{for} \quad \xi \in \mathfrak{g}$$

Lemma 3.1.1. *We have* $[\Box_\omega, D_Z] = 0 = [\Box_\omega, D_\xi]$ *for all* $\xi \in \mathfrak{g}$.

Proof. We have $[\Box_\omega, D_Z] = 0$ since $\Box_\omega$ satisfies all of the assumptions from our previous chapter 2. For D_ξ we note that $\widehat{\xi}$ is Killing for the Kaluza-Klein metric since the right G-action on P is by isometries and furthermore since V is the pullback of a function on Σ it is constant on the fibers of P hence $\mathcal{L}_\xi V = 0$ for all $\xi \in \mathfrak{g}$. $\Box$

As such, we have our space $\ker \Box_\omega$ from before together with the stress-energy quadratic form

$$Q_\omega(\phi) = \int_{P_0} T(\phi)(Z^\omega, \widehat{n}) dV_{P_0}$$

with respect to which D_Z is symmetric.

Lemma 3.1.2. Q_ω *is invariant under the action of* G.

Proof. We write $\phi \cdot g$ for the function $x \mapsto \phi(xg^{-1})$ and also continue to use the notation $\zeta \cdot g$ for the induced right action of G on covectors ζ. Since G acts by isometries we have $|d(\phi \cdot g)|^2_{g_\omega} = |d\phi|^2_{g_\omega} \cdot g$ and therefore

$$T(\phi \cdot g)(Z^\omega, \widehat{n}) = T(\phi)(g^{-1} \cdot Z^\omega, g^{-1} \cdot \widehat{n}) \cdot g.$$

But $Z^\omega = \partial_t$ and $\widehat{n} = N^{-1}(\partial_t - \beta)$ are both invariant under the G-action so

$$T(\phi \cdot g)(Z^\omega, \widehat{n}) = T(\phi)(Z^\omega, \widehat{n}) \cdot g.$$

Finally, the volume form dV_{P_0} is invariant under the G-action since it is an action by isometries hence we can perform the change of variables $x \mapsto xg^{-1}$ in the integral to get

$$\int_{P_0} T(\phi \cdot g)(Z^\omega, \widehat{n}) dV_{P_0} = \int_{P_0} T(\phi)(Z^\omega, \widehat{n}) \cdot g dV_{P_0} = \int_{P_0} T(\phi)(Z^\omega, \widehat{n}) dV_{P_0}$$

as desired. □

At this point we are prepared to study how the spectral theory of D_Z on $\ker \Box_\omega$ interacts with the G-action. Recall from 2.2 that $\ker Q_\omega$ is finite-dimensional, consists of smooth functions, is contained in $\ker D_Z$ and if $\widetilde{Q}_\omega$ is the Hermitian form on the quotient $\ker \Box_\omega / \ker Q_\omega$ induced by Q_ω then

$$\left(\ker \Box_\omega / \ker Q_\omega,\ \widetilde{Q}_\omega\right) \quad \text{is a Pontryagin space.}$$

Since the subspace of smooth solutions in $\ker \Box_\omega$ is invariant under the G-action (it is a smooth action by assumption) it also follows that the domain of D_Z contains a dense G-invariant subspace.

Lemma 3.1.3. *[34], [23], [12] Let $\widetilde{Q}_\omega$ denote the induced quadratic form on the quotient* $\ker \Box_\omega / \ker Q_\omega$. *Then there exists a maximal negative definite subspace*

$$\widetilde{V}^- \subseteq (\ker \Box_\omega / \ker Q_\omega,\ \widetilde{Q}_\omega)$$

which is invariant under D_Z and e^{-itD_Z}. Furthermore, it is finite-dimensional with dimension an invariant of the Krein space and D_Z themselves. Finally, $\widetilde{V}^-$ is invariant under the action of G.

PROOF. The only part of this not proven in the above-cited papers is the G-invariance. Indeed, suppose for contradiction that there was some $g \in G$ and $v \in \widetilde{V}^-$ with $g \cdot v \notin \widetilde{V}^-$. Consider the subspace $\widetilde{W}^- := g \cdot \widetilde{V}^-$. Then, as $g \cdot v \notin \widetilde{V}^-$ we have that the subspace $\widetilde{W}^- + \widetilde{V}^-$ properly contains $\widetilde{V}^-$. Furthermore, it is invariant under both D_Z and e^{-itD_Z} since D_Z commutes with the G-action. Finally, $\widetilde{Q}_\omega$ is negative-definite on $\widetilde{W}^-$ since it is negative-definite on $\widetilde{V}^-$ and invariant under the G-action, hence $\widetilde{Q}_\omega$ is negative definite on $\widetilde{W}^- + \widetilde{V}^-$, contradicting maximality. $\square$

Since $\widetilde{Q}_\omega$ is non-degenerate and invariant under both the G-action and e^{-isD_Z} we obtain the following immediate corollary.

Corollary 3.1.4. *The subspace*

$$\widetilde{V}^+ := (\widetilde{V}^-)^{\perp \widetilde{Q}_\omega}$$

is a Hilbert space with inner product $\widetilde{Q}_\omega$, and is equipped with a unitary representation of $\mathbb{R} \times G$ given by the restriction of e^{-isD_Z} and the G-action from above.

We can now begin the process of showing that Q_ω is positive definite on isotypic subspaces for irreducible representations with sufficiently large dominant integral weights.

Lemma 3.1.5. *Let V^- be the preimage of $\widetilde{V}^-$ in $\ker \Box_\omega$ under the quotient map $\ker \Box_\omega \to \ker \Box_\omega / \ker Q_\omega$. Then V^- is finite dimensional and contains $\ker Q_\omega$.*

PROOF. Indeed the quotient map restricts to a map $V^- \to \widetilde{V}^-$ with kernel $\ker Q_\omega$. Choosing a splitting of this linear surjection gives us an isomorphism of vector spaces $V^- \cong \widetilde{V}^- \oplus \ker Q_\omega$ and since $\widetilde{V}^- \oplus \ker Q_\omega$ so is V^-. $\square$

Definition 3.1.6. Fix $\mathcal{O} \subseteq \mathfrak{g}^*$ an integral coadjoint orbit. For each $m \in \mathbb{Z}_{\geq 1}$ we let κ_m denote the irreducible representation corresponding to the integral coadjoint orbit $m\mathcal{O} \subseteq \mathfrak{g}^*$.

Proposition 3.1.7. *There exists an $m_0 \in \mathbb{Z}_{\geq 1}$ depending only on $\mathcal{O}$, D_Z and the Krein space $(\ker \Box_\omega / \ker Q_\omega, \widetilde{Q}_\omega)$ such that for any $m \geq m_0$ and any $\phi \in \ker \Box_\omega$ which generates a cyclic G-representation $V_\phi \subseteq \ker \Box_\omega$ isomorphic to κ_m we have*

$$V_\phi \cap V^- = \{0\}.$$

Thus for each $m \geq m_0$ we have a closed subspace

$$\mathcal{H}_m := \overline{\mathrm{Span}_{\mathbb{C}}\{\phi \in \ker \Box_\omega \ : \ V_\phi \cong \kappa_m\}}$$

on which Q_ω restricts to a positive definite Hilbert space inner product. Furthermore, our representation of $\mathbb{R} \times G$ arising as the product of the G-action and e^{-isD_Z} leaves $\mathcal{H}_m$ invariant and is unitary.

Proof. Let $\phi \ker \Box_\omega$ generate a cyclic G-representation V_ϕ isomorphic to κ_m. Suppose that $V_\phi \cap V^- \neq \{0\}$ and so there existed a non-zero $\psi \in V_\phi \cap V^-$. Since V^- is a G-invariant subspace we have $V_\psi \subseteq V^-$ where V_ψ is the cyclic G-representation generated by ψ. Furthermore, $0 \neq V_\psi \subseteq V_\phi$ and since V_ϕ is irreducible it follows that $V_\psi = V_\phi$. So it follows that:

$$\text{if } V_\phi \cong \kappa_m \ \text{ and } \ V_\phi \cap V^- \neq \{0\} \ \text{ then } \ V_\phi \subseteq V^-.$$

Since V^- is finite dimensional this can happen for at most finitely many irreducible cyclic invariant subspaces and hence for at most finitely many m. In fact, since the dimension of V^- is an invariant of D_Z and the Krein space $\ker \Box_\omega / \ker Q_\omega$ it follows that for m_0 large enough (with dependence as in the statement of the proposition) and all $m \geq m_0$ we have:

$$\text{if } \phi \in \ker \Box_\omega \text{ with } V_\phi \cong \kappa_m \text{ then } V_\phi \cap V^- = \{0\}.$$

In particular, for $m \geq m_0$ and $\mathcal{H}_m$ defined as in the statement of the proposition, Q_ω is positive definite on $\mathcal{H}_m$.

To show that our $\mathbb{R} \times G$ action leaves $\mathcal{H}_m$ invariant and is unitary it suffices to show that it leaves $\mathrm{Span}_{\mathbb{C}}\{\phi \in \ker \Box_\omega \ : \ V_\phi \cong \kappa_m\}$ invariant and is unitary here, since it will then extend to $\mathcal{H}_m$ by uniform continuity. Since Q_ω is invariant under the full $\mathbb{R} \times G$-action, unitarity is immediate. All that remains is to check invariance. However, since κ_m is irreducible it follows that for any ϕ with $V_\phi \cong \kappa_m$ and any $g \in G$ we have $0 \neq V_{\phi \cdot g} \subseteq V_\phi$ hence $V_{\phi \cdot g} = V_\phi$ thus we have invariance, as desired. $\Box$

It is worth noting that, as remarked in the previous chapter and in [**34**], if $V \geq 0$ and there exists some $x \in \Sigma_0$ for which $V(x) > 0$ then Q_ω is positive definite. This is especially true for the massive Klein-Gordon equation where V is a positive constant. In [**35**] the special case of our results where $G = \mathrm{U}(1)$ and (P, ω) was a trivial bundle was considered. In this case it was shown that when projected down to M our parameter $m \in \mathbb{Z}_{\geq 1}$ above actually corresponds to mass. We will demonstrate an analogue of this later in this section.

Another important remark is that not every $\phi \in \mathcal{H}_m$ has $V_\phi \cong \kappa_m$. This is most easily seen in the Euclidean-signature case where M is a single point. Then $P = G$ and our

Hilbert space is $L^2(G)$ which, by the Peter-Weyl theorem, contains every irreducible representation of G as a cyclic subspace. However, as was shown in [**14**], since G is compact Hausdorff and second-countable, the entire representation $L^2(G)$ is itself a cyclic representation.

Combining our previous facts, for $m \geq m_0$ we can decompose:

$$\mathcal{H}_m = \bigoplus_{\ell \in \mathbb{Z}}^{L^2} \mathcal{H}_{m,\ell}$$

with $\mathcal{H}_{m,\ell}$ the $\lambda_{m,\ell}$-eigenspace for D_Z on $\mathcal{H}_m$, organized so that $\lambda_{m,\ell} \leq \lambda_{m,\ell+1}$ for all $\ell \in \mathbb{Z}$. If $\lambda_{m,\ell} = \lambda_{m,\ell+1}$ then $\mathcal{H}_{m,\ell} = \mathcal{H}_{m,\ell+1}$ and otherwise these spaces are orthogonal (this is the sense in which the above is indeed an L^2-direct sum). We can then further decompose:

$$\mathcal{H}_{m,\ell} = \bigoplus_{j=1}^{\mu(m,\ell)} \kappa_m$$

and it is worth noticing that $\mu(m, \ell)$ is indeed always finite since $\mathcal{H}_{m,\ell}$ itself is finite dimensional (being an eigenspace for D_Z).

Since we will be studying asymptotics as $m \to \infty$, there's no harm in replacing κ with κ_{m_0} so that we may assume $m_0 = 1$. As such, we want to study the time evolution of quantum states in the subspace

$$\mathcal{H} := \bigoplus_{m \geq 1}^{L^2} \mathcal{H}_m \subseteq \ker \Box_\omega.$$

However, we still haven't fully specified a direction in which to take our large quantum numbers limit. Indeed, for fixed m the eigenvalues $\lambda_{m,\ell}$ very well might accumulate at $\pm\infty$ as ℓ tends to $\pm\infty$. Thus for each $E \in \mathbb{R}$ we could consider eigenvalues satisfying

$$\lambda_{m,\ell} \sim mE$$

and different choices of E might very well yield different $m \to \infty$ asymptotics. Classically this is reflected in the fact that symplectic reduction along $\mathcal{O}$ generally leads to phase spaces which are not conical. As such, our problem is broken into two steps:

(1) For m fixed, "count" eigenvalues satisfying $\lambda_{m,\ell} \sim mE$.

(2) Understand the asymptotics of the above count as $m \to \infty$.

The first step is fairly straight-forward. It is highly unlikely for us to have any eigenvalues satisfying $\lambda_{m,\ell} = mE$ exactly and so we instead sum over all $\ell \in \mathbb{Z}$, weighting eigenvalues near mE the most. By stationary phase, this is described for large frequencies by the distribution:

$$\varphi \mapsto \mathrm{Tr}\left(\int_{-\infty}^{\infty} \varphi(t) e^{-it(D_Z - mE)}|_{\mathcal{H}_m} dt\right) = \sum_{\ell\in\mathbb{Z}} \widehat{\varphi}(\lambda_{m,\ell} - mE) =: \mu(E, m, \varphi).$$

We use the letter μ to denote this distribution since it can be viewed as a multiplicity for the representation on $\mathcal{H}$ of $\mathbb{R} \times G$ associated to the coadjoint orbit $\{E\} \times \mathcal{O} \subseteq \mathbb{R} \oplus \mathfrak{g}^*$. The point is that (modulo factors of 2π), $\widehat{\varphi}$ approaches δ_0 as $\varphi \to 1$ and so in this limit the right hand side approaches the literal multiplicity of mE as an eigenvalue on $\mathcal{H}_m$. However, this is only a moral since the above limit does not converge. Instead we first notice that $\mu(E, m, -)$ defines a linear functional on the collection of all $\varphi \in \mathcal{S}(\mathbb{R})$ with compactly supported Fourier transform. Our goal now is to apply an analogue of a result from [**21**] which generalizes the Weyl law of [**34**] to vector bundles in order to prove that $\mu(E, m, -)$ is actually tempered and hence $\mu(E, m, \varphi)$ is defined for any $\varphi \in \mathcal{S}(\mathbb{R})$.

We begin by recalling the well-known fact that for any unitary representation V of G there is an isomorphism

$$C^\infty(P, V)^G \cong \Gamma(M, P \times_G V)$$

between V-valued G-equivariant smooth functions on P and smooth sections of the associated vector bundle $P \times_G V$ over M. Furthermore, the Hermitian inner product on V defines a Hermitian fiber metric on $P \times_G V$. We will need a less well-known, but related construction.

Definition 3.1.8. We fix an $m \geq m_0$ so that Q_ω is positive definite on $\mathcal{H}_m \subseteq \ker \Box_\omega$ and denote by $\kappa_m : G \to \mathrm{U}(V_m)$ our irreducible representation corresponding to $m\mathcal{O}$. We also let $d_m := \dim_{\mathbb{C}} V_m$ and fix an orthonormal basis $\vec{e}_1, ..., \vec{e}_{d_m}$ for V_m, writing $\langle -, - \rangle_m$ for our Hermitian inner product on V_m.

Lemma 3.1.9. *Let $\vec{\psi} \in C^\infty(P, V_m)^G$ and $\vec{v} \in V_m$ both be non-zero. Define a function*

$$\phi : P \to \mathbb{C}$$

$$\phi(p) := \langle \vec{\psi}(p), \vec{v} \rangle_m.$$

Then $V_\phi \cong V_m$ as G-representations. Furthermore, if $\Box_\omega$ is extended to act on V_m-valued smooth functions it follows that $\Box_\omega \vec{\psi} = 0$ if and only if $\Box_\omega \phi = 0$.

Proof. Since $\vec{v} \in V_m$ is non-zero and V_m is irreducible, it is a cyclic vector and so for each $j = 1, ..., d_m$ there are finitely many group elements $g^i_j \in G$ such that $\sum_i g^i_j \vec{v} = \vec{e}_j$. Thus

$$\sum_i \phi(p(g^i_j)^{-1}) = \sum_i \langle \vec{\psi}(p), g^i_j \vec{v} \rangle_m = \langle \psi(p), \vec{e}_j \rangle_m.$$

So the functions $\langle \vec{\psi}(-), \vec{e}_j \rangle_m$ are in V_ϕ for all $j = 1, ..., d_m$. Furthermore every function $p \mapsto \psi(pg^{-1}) = \langle \vec{\psi}(p), g\vec{v} \rangle_m$ is in the span of the functions $\langle \vec{\psi}(-), \vec{e}_j \rangle_m$ hence

$$V_\phi = \mathrm{Span}_{\mathbb{C}} \left\{ \langle \vec{\psi}(-), \vec{e}_1 \rangle_m, ..., \langle \vec{\psi}(-), \vec{e}_{d_m} \rangle_m \right\}.$$

The set of functions $\langle \vec{\psi}(-), \vec{e}_j \rangle_m$ are linearly independent since if $a^j \in \mathbb{C}$ are such that $\langle \vec{\psi}(p), a^j \vec{e}_j \rangle_m = 0$ for all $p \in P$ then since $\vec{\psi} \neq 0$ there exists a $p \in P$ with $0 \neq \vec{\psi}(p) \in V_m$. Since V_m is irreducible there exists elements $g_k \in G$ such that $\sum_k g\vec{\psi}(p) = a^j \vec{e}_j$ and so

$$0 = \sum_k \langle \vec{\psi}(pg_k^{-1}), a^j \vec{e}_k \rangle = \sum_j |a^j|^2$$

hence $a^j = 0$ for all j as desired. Therefore the map

$$\vec{e}_j \leftrightarrow \langle \vec{\psi}(-), \vec{e}_j \rangle_m$$

induces an isomorphism of G-representations $V_m \cong V_\phi$.

If $\Box_\omega \vec{\psi} = 0$ then by definition ($\vec{v}$ and $\langle -, - \rangle_m$ are constant) $\Box_\omega \phi = 0$. Conversely, G-invariance of $\Box_\omega$ implies that if $\Box_\omega \phi = 0$ then $\Box_\omega f = 0$ for all $f \in V_\phi$ and hence $\Box_\omega \langle \vec{\psi}(-), \vec{e}_j \rangle_m = 0$ for all j. Therefore $\Box_\omega \vec{\psi} = 0$ as desired. $\Box$

Usually one doesn't look at the full wave operator $\Box_\omega$ applied to $\vec{\psi} \in C^\infty(P,V)^G$ but only at the "horizontal" wave operator. To relate these two wave operators, we fix a root system for $\mathfrak{g}$ compatible with our Ad-invariant inner product and let:

$$\rho := \quad \text{the sum of all positive roots}$$

and

$$\Lambda_0 := \quad \text{the dominant integral weight for } \kappa_{m_0}.$$

Lemma 3.1.10. *The wave operator $\Box_\omega$ on $C^\infty(P)$ splits as a sum of vertical and horizontal parts:*

$$\Box_\omega = \Box_H - \Delta_G$$

where $\Box_H$ is the horizontal wave operator (plus the potential) and Δ_G is the Laplacian on the fibers. These operators commute and if $\phi \in \mathcal{H}_m$ has $V_\phi \cong V_m$ then Δ_G acts on V_m as multiplication by a constant. Hence Δ_G acts by multiplication by a constant on all of $\mathcal{H}_m$ and this constant is given by:

$$\Delta_G|_{\mathcal{H}_m} = \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle.$$

Proof. The existence of the splitting and the fact that $[\Box_H, \Delta_G] = 0$ follows from **[17]** section 6. Since Δ_ω and Δ_H both commute with the G-action it follows that Δ_G does as well hence Δ_G does indeed preserve V_ϕ. In fact, by the explicit form of Δ_G we see that its action on V_ϕ is precisely the action of the quadratic Casimir and hence is given by multiplication by $\langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle$. $\Box$

In fact, we see that Δ_G preserves our space

$$\mathcal{H} = \bigoplus_{m \geq m_0}^{L^2} \mathcal{H}_m$$

and on this space $\mathcal{H}_m$ is precisely the $\langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle$-eigenspace of Δ_G.

Definition 3.1.11. We denote by $\Box_m$ the operator

$$\Box_m := \Box_H - \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle.$$

Lemma 3.1.12. *Denote by*

$$\mathrm{Gr}_m(P) := \{V \subseteq \ker \Box_\omega \cap C^\infty(P)\ :\ V \textit{ is } G\textit{-invariant and } V \cong V_m\}$$

the collection of all invariant subspaces of $\ker \Box_\omega$ *which are isomorphic to* V_m *as* G*-representations. Then for each* $V \in \mathrm{Gr}_m(P)$ *we have* $V \subseteq \mathcal{H}_m$*. Furthermore if* $\Phi : V_m \to V$ *is any isomorphism of* G*-representations then*

$$\vec{\psi}(p) := \sum_{j=1}^{d_m} \Phi(\vec{e}_j)(p)\vec{e}_j$$

is a G*-equivariant* V_m*-valued function with*

$$\Box_m \vec{\psi} = \Box_H \vec{\psi} - \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle \vec{\psi} = 0. \tag{3.1}$$

Finally, the definition of $\vec{\psi}$ *is independent of our choice of orthonormal basis* $\vec{e}_j$*.*

Proof. Since each $\Phi(\vec{e}_j)$ generates a cyclic representation isomorphic to V_m it automatically follows that $V \subseteq \mathcal{H}_m$ and $\vec{\psi}$ satisfies 3.1. So all that remains to be checked is $\vec{\psi}$'s equivariance and basis-independence. However since Φ is an isomorphism of G-representations we can compute:

$$\vec{\psi}(pg^{-1}) = \sum_{j=1}^{d_m} \Phi(\vec{e}_j)(pg^{-1})\vec{e}_j = \sum_{j=1}^{d_m} \Phi(g\vec{e}_j)(p)\vec{e}_j.$$

But if we write $g\vec{e}_j = g^i_j\vec{e}_i$ then we arrive at:

$$\vec{\psi}(pg^{-1}) = \sum_{j=1}^{d_m} g^i_j\Phi(\vec{e}_i)(p)\vec{e}_j = \sum_{i=1}^{d_m} \Phi(\vec{e}_i)\ g\vec{e}_i$$

proving equivariance. Similarly, if $\vec{f}_j \in V_m$ is another orthonormal basis then there exists a unitary matrix A satisfying $\vec{e}_j = A^i_j\vec{f}_i$ hence

$$\vec{\psi}(p) = \sum_{j=1}^{d_m} A^i_jA^k_j\Phi(\vec{e}_i)(p)\vec{e}_k = \sum_{i=1}^{d_m} \Phi(\vec{e}_i)(p)\vec{e}_i$$

as desired. $\square$

Since V_m is irreducible, Schur's lemma tells us that any two isomorphisms $V_m \cong V$ of G-representations differ by a multiplicative non-zero constant complex number. As such, we obtain the following corollary.

Corollary 3.1.13. *There is a natural isomorphism*

$$\mathrm{Gr}_m(P) \to \{\vec{\psi} \in C^\infty(P, V_m)^G \ :\ \Box_m\vec{\psi} = 0\}/\mathbb{C}^\times$$
$$V \mapsto \sum_{j=1}^{d_m} \Phi(\vec{e}_j)(-)\vec{e}_j \ \mathit{mod}\ \mathbb{C}^\times$$

where in the above expression $\vec{e}_j$ is any choice of orthonormal basis for V_m and Φ is any choice of isomorphism of G-representations $V_m \cong V$.

Proof. This is simply a combination of 3.1.9 and 3.1.12, taking care to remark that the two constructions from these two lemmas are inverse to one-another (taking $\vec{v} = \vec{e}_1$ in 3.1.9). $\square$

Our final step is to compare elements of $C^\infty(P, V_m)^G$ with sections of the associated vector bundle.

Definition 3.1.14. We define a map $\Psi : C^\infty(P, V_m)^G \to \Gamma(M, P \times_G V_m)$ as follows. Given $\vec{\psi} \in C^\infty(P, V_m)^G$ and $x \in M$ we choose an arbitrary $p \in P$ in the fiber over x and define

$$\Psi(\vec{\psi})(x) := \text{ the equivalence class of } (p, \vec{\psi}(p)) \text{ in the fiber } (P \times_G V_m)_x.$$

We recall from [**8**] Chapter 3, for example, that Ψ is an isomorphism. Furthermore there is an induced covariant derivative ∇^m on $P \times_G V_m$ which corresponds under Ψ to the horizontal exterior derivative on P with respect to ω, and there is a Hermitian fiber metric $\langle -, - \rangle_m$ on $P \times_G V$ corresponding to the constant Hermitian inner product $\langle -, - \rangle_m$ on V_m.

Now, let's let $V \in \mathrm{Gr}_m(P)$ and choose an isomorphism $\Phi : V_m \to V$ which is unitary where V is given the Q_m-inner product. Writing

$$\vec{\psi}(p) := \sum_{j=1}^{d_m} \Phi(\vec{e}_j)(p)\vec{e}_j$$

it follows that the expression

$$Q_\omega(\vec{\psi}) := \sum_{j=1}^{d_m} Q_\omega(\Phi(\vec{e}_j))$$

is independent of our choice of orthonormal basis $\vec{e}_j$ or unitary isomorphism Φ. We also have the following explicit formula from [**34**] where we briefly break notational convention and use Greek $\mu, \nu, ...$ for indices of coordinates tangent to $\Sigma_0 \subseteq M$ and Roman $a, b, ...$ indices for coordinates tangent to the fibers of P_0:

$$Q(\Phi(\vec{e}_j)) = \int_{P_0} N^{-1}\Big(|\partial_t \Phi(\vec{e}_j)|^2 + (N^2 h^{\mu\nu} - \beta^\mu \beta^\nu)(\partial_\mu \Phi(\vec{e}_j))(\partial_\nu \overline{\Phi(\vec{e}_j)}) \\ + \operatorname{Tr}\left(\omega(d\Phi(\vec{e}_j))\omega(d\overline{\Phi(\vec{e}_j)})^T\right) + |\Phi(\vec{e}_j)|^2 V\Big) dV_{P_0}$$

By equivariance it follows that if $\xi_1, ..., \xi_d$ is an orthonormal basis for $\mathfrak{g}$ then

$$\omega(d\Phi(\vec{e}_j)) = \sum_a (\mathcal{L}_{\widehat{\xi}_a} \Phi(\vec{e}_j))\widehat{\xi}_a = \sum_a \Phi(\xi_a \cdot \vec{e}_j)\widehat{\xi}_a$$

and so

$$\operatorname{Tr}\left(\omega(d\Phi(\vec{e}_j))\omega(d\overline{\Phi(\vec{e}_j)})^T\right) = \sum_a |\Phi(\xi_a \cdot \vec{e}_j)|^2 = \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle |\Phi(\vec{e}_j)|^2.$$

Thus we obtain

$$Q(\Phi(\vec{e}_j)) = \int_{P_0} N^{-1}\Big(|\partial_t \Phi(\vec{e}_j)|^2 + (N^2 h^{\mu\nu} - \beta^\mu \beta^\nu)(\partial_\mu \Phi(\vec{e}_j))(\partial_\nu \overline{\Phi(\vec{e}_j)}) \\ + |\Phi(\vec{e}_j)|^2 \left(V + \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle\right)\Big) dV_{P_0}$$

Furthermore, from this explicit expression we see that the sum

$$
\begin{aligned}
(3.2)\qquad \sum_{j=1}^{d_m} N^{-1}\Big(& |\partial_t \Phi(\vec{e}_j)|^2 + (N^2 h^{\mu\nu} - \beta^\mu\beta^\nu)(\partial_\mu \Phi(\vec{e}_j))(\partial_\nu \overline{\Phi(\vec{e}_j)}) \\
& + |\Phi(\vec{e}_j)|^2 (V + \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle)\Big)
\end{aligned}
$$

is invariant under the G action.

Definition 3.1.15. Given a section $s \in \Gamma(M, P \times_G V_m)$ we define the **bundle stress-energy tensor** $T_m(s)$ to be the symmetric 2-tensor on M given by:

$$T_m(s)_{ij} := \langle \nabla_i^m s, \nabla_j^m s\rangle - \frac{1}{2}\left(|\nabla^m s|^2 + |s|^2\left(V + \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle\right)\right) g_{ij}$$

where we recall that ∇^m is the covariant derivative on $P \times_G V_m$ induced by the connection ω.

Since $\langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle$ is a constant and the connection ∇^m is compatible with the fiber metric it follows exactly as in the scalar case that if we abuse notation and also use $\Box_m$ to denote

$$\Box_m = (\nabla^m)^*\nabla^m + V + \langle m\Lambda_0,\ m\Lambda_0 + \rho\rangle$$

acting on sections of $P \times_G V_m$ then

$$\mathrm{div}_M(T_m(s)) = -\langle \Box_m s, \nabla^m s\rangle - \frac{1}{2}|s|^2 dV$$

$$\mathrm{div}_M(T_m(s)(Z)) = (\mathrm{div}_M\, T_m(s))(Z) = 0 \ \text{ if } \Box_m s = 0$$

where we note that despite the raised and lowered m's appearing, we are not summing over them: they merely denote the representation of G we are considering.

Applying 2 to each of these bundles $P \times_G V_m$ and each of these normally hyperbolic operators $\Box_m$ we obtain our finite energy spaces $\ker \Box_m$ together with their stress-energy quadratic forms Q_m and closed densely defined operators $D_{m,Z} := -i\nabla^m_Z$ (recalling again that ∇^m unfortunately denotes the connection induced on $P \times_G V_m$ by ω, and not the m'th covariant derivative). Combining all of our results in this section and especially using 3.2 we arrive at the following result.

Proposition 3.1.16. *Let $V \in \mathrm{Gr}_m(P)$ and $\Phi : V_m \to V$ a unitary isomorphism so that we can define*

$$\vec{\psi}(p) := \sum_{j=1}^{d_m} \Phi(\vec{e}_j)(p)\vec{e}_j.$$

Then $\Psi(\vec{\psi}) \in \ker \Box_m$ and

$$Q_m(\Psi(\vec{\psi})) := \int_{\Sigma_0} T_m(\Psi(\vec{\psi}))(Z, \widehat{n})dV_{\Sigma_0} = \mathrm{Vol}(G)Q_\omega(\vec{\psi})$$

where $\mathrm{Vol}(G)$ is taken with respect to the volume form induced by our Ad-invariant inner product on $\mathfrak{g}$. Furthermore, since $m \geq m_0$ by assumption it follows that Q_m is positive definite on the finite energy space $\ker \Box_m$.

The following theorem then follows from a simple modification of the arguments from [**34**] to the vector bundle case via the formalism in 2. The modifications to the relevant FIO compositions from [**34**] are done in [**21**].

Theorem 3.1.17. *[21] The operator $D_{m,Z}$ is self-adjoint on $(\ker \Box_m, Q_m)$ with $\sigma(D_{m,Z}) \subseteq \mathbb{R}$ discrete and accumulating at $\pm\infty$ with polynomial growth.*

We can now conclude this section with a description of the spectral theory of D_Z on a fixed isotypic subspace $\mathcal{H}_m \subseteq \ker \Box_\omega$, back on the total space of our principal bundle.

Corollary 3.1.18. *The spectrum of D_Z on $\mathcal{H}_m$ is real, discrete and accumulates at $\pm\infty$ with polynomial growth. Furthermore, the multiplicity of $\lambda \in \sigma(D_Z)$ is equal to $d_m = \dim(V_m)$ times the multiplicity of $\lambda \in \sigma(D_{m,Z})$. Furthermore the distribution $\mu(E, m, -)$ given by*

$$\mu(E, m, \varphi) := \sum_{\ell \in \mathbb{Z}} \widehat{\varphi}(\lambda_{m,\ell} - mE)$$

is a tempered distribution on $\mathbb{R}$. Here we recall that $\cdots \leq \lambda_{m,\ell} \leq \lambda_{m,\ell+1} \leq \cdots$ are the eigenvalues of D_Z on $\mathcal{H}_m$.

3.2. The Trace Formula on Kaluza-Klein Spacetimes

This section answers the following question:

What are the $m \to \infty$ asymptotics of the frequency spectrum of $D_Z - mE$, $E \in \mathbb{R}$ fixed, on $\mathcal{H}_m$?

This is a relativistic analogue of the question studied in [**17**], and is a generalization to non-trivial principal bundles with arbitrary compact structure groups of the results in [**35**]. As mentioned earlier in section 3.1, we make precise the notion of the distribution of the frequency spectrum about the value $mE \in \mathbb{R}$ by the tempered distribution $\mu(E, m, -)$ on $\mathbb{R}$. Recall:

$$\mu(E, m, \varphi) := \sum_{\ell \in \mathbb{Z}} \widehat{\varphi}(\lambda_{m,\ell} - mE)$$

Its $m \to \infty$ asymptotics will be studied by assembling these distributions for a fixed test function $\varphi \in \mathcal{S}(\mathbb{R})$ into a periodic generating function

$$\Upsilon(\varphi)(\theta) := \sum_{m=1}^{\infty} \mu(E, m, \varphi) e^{im\theta} \in \mathcal{D}'(S^1) = \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$$

While we have already shown $\mu(E, m, -)$ is tempered, $\Upsilon(\varphi)$ will only actually be shown to be a distribution later in this section.

To state our main results we need to make a dynamical assumption akin to the "clean intersection hypotheses" that appear in [**35**], [**16**] and [**17**]. This was already discussed briefly in 1.3 but we review some of the basic definitions here.

- $\mathcal{N}$ is the symplectic manifold of affinely parametrized inextendible future-directed null geodesics on P modulo the action of translation in the affine parameter,
- $\mathcal{N}_{\mathcal{O}}$ is the symplectic reduction of $\mathcal{N}$ along the coadjoint orbit $\mathcal{O}$ as in [**17**],
- $\widetilde{H}_Z : \mathcal{N}_{\mathcal{O}} \to \mathbb{R}$ and $\widetilde{\Phi}_t^Z$ are respectively the Hamiltonian and Hamiltonian flow corresponding to the reduction of the flow on $\mathcal{N}$ induced by the flow of $Z = \partial_t$ on P.

The **clean intersection hypothesis** then states that $E > 0$ is a regular value for $\widetilde{H}_Z$ and the fibered product $\mathfrak{Y}_E$ of the flow map

$$\mathbb{R} \times \widetilde{H}_Z^{-1}(E) \to \widetilde{H}_Z^{-1}(E)$$

with the diagonal map $\widetilde{H}_Z^{-1}(E) \to \widetilde{H}_Z^{-1}(E) \times \widetilde{H}_Z^{-1}(E)$ is a clean fibered product (it is a smooth manifold with tangent spaces given by the not-necessarily-transverse fibered products of the respective tangent spaces of the factors). We now state our main theorems. These are completely analogous to the main theorems in **[17]** and are direct relativistic generalizations of these.

Theorem 3.2.1. *The wave front set of* $\Upsilon(\varphi) \in \mathcal{D}'(S^1)$ *is contained in:*

$$\mathfrak{D}_{\varphi,E} := \Big\{(\omega, r) \in S^1 \times \mathbb{R}_{>0} \ : \ \exists (T,\gamma)) \in \mathfrak{Y}_E \text{ with } T \in \operatorname{supp}\widehat{\varphi} \\ \text{such that } \omega = \operatorname{Hol}_{\mathcal{O}}([0,T] \ni t \mapsto \widetilde{\Phi}^Z_t(\gamma))\Big\}$$

where this holonomy is taken with respect to a natural $\mathrm{U}(1)$*-bundle with connection over* $\mathcal{N}_{\mathcal{O}}$ *defined in 1.3.28.*

Theorem 3.2.2. *Let* $n+1 = \dim(M)$*. Under the clean intersection hypothesis,* $\mathfrak{D}_{\varphi,E}$ *is a union of the positive parts of finitely many fibers of* T^*S^1*, and* $\Upsilon(\varphi) \in I^{n+\ell-1+\frac{1}{4}}(S^1; \mathfrak{D}_{\varphi,E})$ *where* $2\ell := \dim \mathcal{O}$*. Furthermore, we obtain an asymptotic expansion as* $m \to \infty$*:*

$$\mu(E, m, \varphi) \sim \sum_{k=0}^{\infty} m^{n+\ell-1-k} a_k(\varphi, m)$$

with each $a_k(\varphi, m)$ *a distribution in* φ*, bounded in* m *for* k, φ *fixed, and*

$$a_0(\varphi, m) = C_{n,d}\widehat{\varphi}(0) \operatorname{Vol}\left(\widetilde{H}_Z^{-1}(E)\right)$$

where by Vol *we mean to take the invariant measure on the energy hypersurface.*

The above theorem comes from the analysis of the big singularity in $\Upsilon(\varphi)$ at $\theta = 0$. The theorem below handles analyzes a special case of the singularities at $\theta \neq 0$.

Theorem 3.2.3. *Suppose that, in addition to the clean intersection hypothesis, we assumed that* $0 \notin \operatorname{supp}(\widehat{\varphi})$ *and there existed only finitely many non-degenerate periodic orbits* $(T_1, \gamma_1), \cdots, (T_q, \gamma_q) \in \mathfrak{Y}_E$ *with each* $T_j \neq 0$. *Then we actually obtain a better asymptotic expansion as* $m \to \infty$:

$$\mu(E, m, \varphi) \sim \sum_{k=0}^{\infty} m^{-k} b_k(\varphi, m)$$

and $b_0(\varphi, m)$ *is of the form:*

$$b_0(\varphi, m) = C_{n,d} \sum_{j=0}^{q} \operatorname{Hol}_{\mathcal{O}}(T_j, \gamma_j)^m \frac{T_j^{\#}}{2\pi} \widehat{\varphi}(T_j) \frac{e^{i\pi\mathfrak{m}_j/4}}{|\det(I - P_j)|^{1/2}}.$$

Where $T_j^{\#}$ *is the minimum positive value of* T *such that* $\widetilde{\Phi}_T^Z(\gamma_j) = \gamma_j$, P_j *is the linearized Poincaré first return map of* γ_j, *and* $\mathfrak{m}_j$ *is the Conley-Zehnder index.*

Before proving these theorems, we apply the results of 3.1 to obtain a corollary concerning the frequency distribution of D_Z for vector bundles. To simplify notation, we denote

$$\mathcal{V}_m := P \times_G V_m$$

and recall that $\square_m$ is the induced normally hyperbolic operator on sections of this bundle and $D_{m,Z}$ is the induced time-translation operator on $\ker \square_m$.

Corollary 3.2.4. *For* m *sufficiently large, define the tempered distribution* $\mu(E, \mathcal{V}_m, -)$ *by*

$$\mu(E, \mathcal{V}_m, \varphi) := \sum_{\lambda \in \operatorname{Spec}(D_{m,Z})} \widehat{\varphi}(\lambda - mE)$$

Then under the clean intersection hypothesis we have an asymptotic expansion

$$\mu(E, \mathcal{V}_m, \varphi) \sim \frac{1}{d_m} \sum_{k=0}^{\infty} m^{n+\ell-1-k} c_k(\varphi, m).$$

In general, one can compute the values of ℓ and d_m in terms of the dominant integral element Λ_0. Indeed, if R^+ is the set of positive roots then

$$\ell = \frac{1}{2} \dim \mathcal{O} = \quad \text{the number of positive roots not orthogonal to } \Lambda_0$$

and as a consequence of the Weyl character formula we have

$$d_m = \frac{\prod_{\alpha \in R^+} \langle \alpha,\ m\Lambda_0 + \frac{1}{2}\rho \rangle}{\prod_{\alpha \in R^+} \langle \alpha, \frac{1}{2}\rho \rangle}.$$

In particular we see that d_m is a polynomial of degree ℓ and so our leading order asymptotics for $\mu(E, \mathcal{V}_m, \varphi)$ as $m \to \infty$ are m^{n-1}. This is in agreement with [**35**] where $\ell = 0$ and $d_m = 1$ for all m. When $G = \mathrm{SU}(2)$ and $\mathcal{O}$ corresponds to the vector representation then $\ell = 1$ and $d_m = m + 1$.

We now begin the proofs of these theorems. As alluded to in the statements, the $m \to \infty$ asymptotics of $\mu(E, m, \varphi)$ depend significantly on whether or not $0 \in \operatorname{supp} \widehat{\varphi}$. For now we illustrate the method from Section 7 of [**16**] where φ is fixed and arbitrary. This method takes advantage of the periodicity and "positive frequency" property of our distributions to express them in terms of linear combinations of the basic homogeneous periodic distributions

$$\sum_{m=1}^{\infty} m^k z^{-m} e^{im\theta}$$

with $z \in S^1$ and $k \in \mathbb{Z}_{\geq 0}$ determining the location of the singularity and the homogeneity respectively. A key advantage of these techniques from [**16**] is that it circumvents the need for general Tauberian theorems.

From now on we replace $\mathcal{O}$ with $m_0\mathcal{O}$ so that we may assume $m_0 = 1$.

Definition 3.2.5. We define the **generating function** of the multiplicities $\mu(E, m, \varphi)$ to be the periodic distribution in the real variable θ:

$$\Upsilon(\varphi)(\theta) := \sum_{m=1}^{\infty} \mu(E, m, \varphi) e^{im\theta}$$

defined for any function $f(\theta)$ which is the Fourier transform of a compactly supported function on $\mathbb{R}$.

Distributions of the form $\sum_{m=1}^{\infty} a_m e^{im\theta}$ with a_m real are called **Hardy distributions**. These are precisely the distributions on the sphere S^1 whose negative Fourier coefficients all vanish and so they have nice descriptions in terms of boundary values of holomorphic functions on the unit disk via the Paley-Weiner theorem. The asymptotics of the Fourier coefficients of such distributions, especially when a_m is a homogeneous function of m, have been studied in books such as [**10**] Sections 12 and 13, and applied to spectral asymptotics in [**16**], [**17**], [**35**] for example.

Later in this section we will write $\Upsilon(\varphi)$ as a composition of Fourier integral operators and through this we will show that it is actually in $\mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$. For now we illustrate how the asymptotics of the Fourier coefficients of a general Lagrangian distribution Υ on S^1 can be related to its principal symbol.

Definition 3.2.6. Let $\Upsilon \in \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$. An element $s_0 \in \operatorname{singsupp}(\Upsilon)$ is called **classical of degree** k if and only if when interpreting Υ as a $2\pi\mathbb{Z}$-periodic distribution on $\mathbb{R}$ we have:

(1) s_0 is an isolated singularity, and
(2) for any $\rho \in C_c^\infty(\mathbb{R})$ with $\rho \equiv 1$ on a neighborhood of s_0 and $\operatorname{singsupp}(\Upsilon) \cap \operatorname{supp}(\rho) = \{s_0\}$ we have asymptotic expansions:

$$\widehat{\rho\Upsilon}(\xi) \sim e^{-is_0\xi} \sum_{\ell=0}^{\infty} c_\ell^+ \xi^{k-\ell} \quad \text{as} \quad \xi \to +\infty \quad \text{and}$$

$$\widehat{\rho\Upsilon}(\xi) \sim e^{-is_0\xi} \sum_{\ell=0}^{\infty} c_\ell^- \xi^{k-\ell} \quad \text{as} \quad \xi \to -\infty.$$

Lemma 3.2.7. *Let $s_0 \in \mathbb{R}$ be a classical singularity of Υ of degree k, and let $\rho \in C_c^\infty(\mathbb{R})$ have $\rho \equiv 1$ on a neighborhood of s_0 and* $\operatorname{singsupp}(\Upsilon(\varphi)) \cap \operatorname{supp}(\rho) = \{s_0\}$. *Then*

$$\rho\Upsilon(\varphi) \in I^{k+1/4}(\mathbb{R}, \Lambda)$$

*where $\Lambda = \{(s_0, \xi) \in T^*\mathbb{R} \setminus 0 \ : \ \xi \neq 0\}$. If $c_\ell^- = 0$ for all ℓ (in which case the singularity is called **positive**) then instead $\Lambda = \{(s_0, \xi) \in T^*\mathbb{R} \setminus 0 \ : \ \xi > 0\}$.*

Proof. We can write the distribution $\rho\Upsilon$ as

$$\langle \rho\Upsilon, \psi \rangle = \int_{\mathbb{R}} e^{i(s-s_0)\xi} (e^{is_0\xi} \widehat{\rho\Upsilon}(\xi)) \psi(s) ds d\xi$$

and so it suffices to check whether the function $e^{is_0\xi}\widehat{\rho\Upsilon}(\xi)$ lives in the correct symbol class. Since $\rho\Upsilon \in \mathcal{E}'(\mathbb{R})$ its Fourier transform is a smooth function and our asymptotics precisely tell us that it lives in the symbol class

$$S^k(\mathbb{R}_s \times \mathbb{R}_\xi) \ \ \text{(it is independent of } s\text{)}.$$

Since $\dim(\mathbb{R}_s) = 1 = \dim(\mathbb{R}_\xi)$ this is the correct order for a symbol to define an FIO of order

$$k - (1 - 2 \cdot 1)/4 = k + 1/4.$$

As for the Lagrangian, one simply notices first that the ξ-critical points of the phase are precisely the set of (s, ξ) with $s = s_0$, meanwhile the support of $e^{is_0\xi}\widehat{\rho\Upsilon}(\xi)$ is everywhere in the non-positive singularity case and is a positive ray in the case of a positive singularity.

$\square$

Lemma 3.2.8. *Suppose Υ had only finitely many singularities $z_1, ..., z_q \in S^1$ and that for $s_1, ..., s_q \in [0, 2\pi]$ with $e^{-is_1} = z_1, ..., e^{-is_q} = z_q$ the singularities $s_1, ..., s_q$ were all classical with respective degrees $k_1, ..., k_q$. For some $\rho_j \in C_c^\infty(\mathbb{R})$ smooth cutoffs with $\rho_j \equiv 1$ on a neighborhood of s_j and* $\operatorname{singsupp}(\Upsilon) \cap \operatorname{supp}(\rho_j) = \{s_j\}$, *and for $c_\ell^{\pm,j}$ the coefficients of our asymptotic expansions for $\widehat{\rho_j\Upsilon}$:*

$$\widehat{\rho_j\Upsilon}(\xi) \sim \sum_{\ell=0}^{\infty} c_\ell^{\pm,j}\xi^{k_j-\ell} \quad \text{as} \quad \xi \to \pm\infty$$

we have:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ims}\Upsilon(\theta)ds \sim \sum_{\ell=0}^{\infty}\sum_{j=1}^{q} c_\ell^{+,j}\omega_j^{-m}m^{k_j-\ell} \quad \text{as} \quad m \to \infty.$$

Proof. Choose our cutoffs ρ_j to be non-negative with disjoint supports and such that there exists $\eta \in C_c^\infty(\mathbb{R})$ with $0 \le \eta \le 1$ such that

$$\rho_1 + \cdots + \rho_q + \eta \equiv 1 \quad \text{on} \quad [0, 2\pi] \quad \text{and} \quad \operatorname{singsupp}(\Upsilon) \cap \operatorname{supp}(\eta) = \emptyset.$$

Then, taking Fourier transforms we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ims}\Upsilon(s)ds = \frac{1}{2\pi}\sum_{j=1}^{q}\int_0^{2\pi} e^{-ims}\rho_j(s)\Upsilon(s)ds + \frac{1}{2\pi}\int_0^{2\pi} e^{-ims}\eta(s)\Upsilon(s)ds.$$

Since $\eta\Upsilon \in C_c^\infty(\mathbb{R})$ we have that the last term is going to 0 rapidly as $m \to \infty$. For the remaining terms we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ims}\rho_j(s)\Upsilon(s)ds = \widehat{\rho_j\Upsilon}(m) \sim e^{-is_j m}\sum_{\ell=0}^{\infty} c_\ell^{+,j} m^{k_j-\ell} \quad \text{as} \quad m \to \infty.$$

Summing these asymptotics together then yields our desired result. $\square$

So we see that in order to obtain the leading order asymptotics of $\mu(E, m, \varphi)$ as $m \to \infty$ it suffices to demonstrate that the singularities of $\Upsilon(\varphi)$ are classical and to compute both its order as an FIO, and the leading terms $c_0^{+,j}$ in the asymptotic expansions of its Fourier transform. Let's now check how to obtain $c_0^{+,j}$ from the principal symbol.

Lemma 3.2.9. *Let s_0 be a classical singularity of degree k of Υ, let $\rho \in C_c^\infty(\mathbb{R})$ be a cutoff as in the previous lemma and let $a(s,\xi)$ be any principal symbol for $\rho\Upsilon$. i.e.*

$$a(s,\xi) - e^{is_0\xi}\widehat{\rho\Upsilon}(\xi) \in S^{k-1}(\mathbb{R}_s \times \mathbb{R}_\xi).$$

Then

$$c_0^{\pm} = \lim_{\xi \to \pm\infty} a(s,\xi)\xi^{-k}.$$

Proof. Indeed, if $a(s,\xi)$ is any principal symbol for $\rho\Upsilon$ then, by definition

$$|a(s,\xi) - e^{is_0\xi}\widehat{\rho\Upsilon}(\xi)| \lesssim (1+|\xi|)^{k-1}$$

and so dividing by ξ^k and taking limits yields our desired result. $\square$

So, our goal has now been reduced to writing $\Upsilon(\varphi)$ as a composition of well-understood FIOs and computing the order and principal symbol of the composition in terms of its constituents. Let's begin by introducing the relevant operators from **[17]** and **[34]**. Recall the distributions $E_{adv/ret}$ and E_{caus} from 2. Here they are corresponding to the normally hyperbolic operator $\Box_\omega$. Definitions of the Hörmander spaces I^s of Lagrangian distributions we use below can be found in **[20]**.

Lemma 3.2.10. *[34]*

We have $E_{caus} \in I^{-3/2}(P \times P; C_1')$ with the canonical relation C_1 given by

$$C_1 = \{(\zeta_1;\ \zeta_2) \in T_0^*P \times T_0^*P\ :\ \exists s \in \mathbb{R} \text{ such that } \zeta_2 = G_{-s}(\zeta_1)\}.$$

Parametrizing the left copy of T_0^*P in C_1 by $T_0^*P|_{P_0} \times \mathbb{R} \cong \mathcal{N} \times \mathbb{R}_{s'}$ via the geodesic flow and then the $\zeta_2 = G_{-s}(\zeta_1)$ by the parameter s, the principal symbol of E_{caus} is given by the half-density

$$|d_{C_1}|^{1/2} := -\frac{1}{2}|\Omega_\mathcal{N}|^{1/2} \otimes |ds'|^{1/2} \otimes |ds|^{1/2}$$

where $\Omega_\mathcal{N}$ is the Liouville volume form on $\mathcal{N}$ induced by the symplectic form.

Before we get to composing FIO's, let's recall how this works [**20**]. Suppose we had smooth manifolds X, Y, Z of respective dimensions n_X, n_Y, n_Z respectively and $C_1 \subseteq (T^*Z \setminus 0) \times (T^*Y \setminus 0)$, $C_2 \subseteq (T^*Y \setminus 0) \times (T^*X \setminus 0) \setminus 0$ canonical relations. We write C_j' for the result of multiplying the left fiber variables by -1 so that the result is a Lagrangian submanifold. Given

$$A_1 \in I^{d_1}(Z \times Y; C_1') \quad \text{and} \quad A_2 \in I^{d_2}(Y \times X; C_2')$$

we interpret A_1 and A_2 as operators

$$A_1 : C_c^\infty(Y) \to \mathcal{D}'(Z) \quad \text{and} \quad A_2 : C_c^\infty(X) \to \mathcal{D}'(Y).$$

One can then often form the composition

$$A_1 \circ A_2 \in I^{d_1+d_2+\frac{e}{2}}(Z \times X; (C_1 \circ C_2)')$$

where e and $(C_1 \circ C_2)'$ are defined as follows. Since C_1 and C_2 are Lagrangian they have dimensions:

$$\dim(C_1) = n_X + n_Y \quad \text{and} \quad \dim(C_2) = n_Y + n_Z.$$

The product $C_1 \times C_2$ lives in $T^*Z \times (T^*Y)^{\times 2} \times T^*X$ and has dimension

$$\dim(C_1 \times C_2) = n_X + 2n_Y + n_Z.$$

Meanwhile we also have a diagonal submanifold

$$D := (T^*Z \setminus 0) \times \operatorname{diag}(T^*Y \setminus 0) \times (T^*X \setminus 0)$$

of dimension $\dim(D) = 2n_Z + 2n_Y + 2n_X$. Since the total space $T^*Z \times (T^*Y)^{\times 2} \times T^*X$ has dimension $2n_Z + 4n_Y + 2n_X$ it follows that if D and $C_1 \times C_2$ intersected transversely then the intersection would have dimension

$$\dim(D \cap (C_1 \times C_2)) = n_Z + n_X$$

and if π_X, π_Z are respectively the projection maps from $T^*Z \times (T^*Y)^{\times 2} \times T^*X$ to T^*X and T^*Z then the restriction

$$\pi_Z \times \pi_X|_{D \cap (C_1 \times C_2)} : D \cap (C_1 \times C_2) \to C_1 \circ C_2 := (\pi_Z \times \pi_X)(D \cap (C_1 \times C_2))$$

is a local diffeomorphism and $C_1 \circ C_2$ is a Lagrangian submanifold of $(T^*Z \setminus 0) \times (T^*X \setminus 0)$. In this case, as long as everything is properly supported, we can take $e = 0$ and we have

$$A_1 \circ A_2 \in I^{d_1+d_2}(Z \times X; (C_1 \circ C_2)').$$

We call this a **transverse composition** of FIO's. Furthermore, in this case if a_1, a_2 are the principal symbols of A_1, A_2 then:

the principal symbol of $A_1 \circ A_2$ is given by the restriction of $a_1 \times a_2$ to $(C_1 \circ C_2)'$.

However, one can still form the composition $A_1 \circ A_2$ if the intersection of D and $C_1 \times C_2$ in $T^*Z \times (T^*Y)^{\times 2} \times T^*X$ is merely **clean**. In this case the intersection is still a smooth manifold, its tangent spaces are given by the intersections of the tangent spaces of D and

$C_1 \times C_2$, $C_1 \circ C_2$ is still defined in the same way, but now the projection map

$$\pi_Z \times \pi_X|_{D\cap(C_1\times C_2)} : D \cap (C_1 \times C_2) \to C_1 \circ C_2$$

is merely required to be a submersion. Since we're assuming everything is properly supported it follows that the fibers are compact manifolds. We define:

$$e := \text{ the dimension of the fibers of } \pi_Z \times \pi_X|_{D\cap(C_1\times C_2)}.$$

This is called the **excess**. Then from Proposition 25.1.5' in [**20**] we have

$$A_1 \circ A_2 \in I^{d_1+d_2+\frac{e}{2}}(Z \times X; (C_1 \circ C_2)')$$

where if a_1, a_2 are the principal symbols of A_1, A_2 respectively then the principal symbol of $A_1 \circ A_2$ at a point $z \in (C_1 \circ C_2)'$ is given by

$$\int_{F_z} a_1 \times a_2 \quad \text{where} \quad F_z \quad \text{is the fiber over } z.$$

We will call this a **clean composition** of FIOs. It should be noted that the above results can be occasionally tweaked to apply when some of the hypotheses (such as C_j being a canonical relation) aren't exactly satisfied as long as one is careful to ensure that the wavefront sets line up correctly in order for the desired products to be defined.

Finally we should say that in our below computations we omit the Maslov index factors until the very end.

We are now ready to apply the above FIO calculus in order to better understand our generating function $\Upsilon(\varphi)$. Let's recall our notation from earlier:

- d is the dimension of G,
- $n+1$ is the dimension of M with n the dimension of Σ,
- $n+1+d$ is the dimension of P,
- T_0^*P is a cone subbundle of $T^*P \setminus 0$ and has dimension $2(n+1+d)-1$.
- The restriction $T_0^*P|_{P_0}$ is symplectomorphic to $T^*P_0 \setminus 0$ (but not in a $\mathbb{R}_{>0}$-equivariant way) and both have dimension $2(n+d)$.
- $\dim \mathcal{O} =: 2\ell$ so $\mathcal{N}_{\mathcal{O}}$ has dimension $2(n+\ell)$.

The below result is also from **[34]** and again we state it for the reader's convenience.

Lemma 3.2.11. *[34]*

Let $E_t(x,y) := e^{-it(D_Z)_x} E_{caus}(x,y)$. *Then*

$$E_t(x,y) \in I^{-7/4}(P \times P \times \mathbb{R}; C_2')$$

where C_2 *is the canonical relation:*

$$\begin{aligned} C_2 := \{(\zeta_1;\ \zeta_2;\ t, \tau) \in (T_0^*P)^{\times 2} \times (T^*\mathbb{R} \setminus 0) \\ :\ \tau + \langle Z^\omega, \zeta_1 \rangle = 0,\ \exists s \text{ such that } \zeta_2 = (G_{-s} \circ \Phi_t^Z)(\zeta_1)\}. \end{aligned}$$

Parametrizing $C_2 \cong C \times \mathbb{R}_t$ *as the flowout of* C *under the* Z*-flow the principal symbol of* $E_t(x,y)$ *is given by:*

$$\mp \frac{i}{2}(2\pi)^{3/4} |d_{C_1}|^{1/2} \otimes |dt|^{1/2} \quad on \quad C_\pm$$

where $C_\pm$ *is the subset of* C_1 *where both covectors are in* $T^*_\pm P$*.*

In the next lemma we begin combining results from [**34**] and [**17**].

Lemma 3.2.12. *The right* G*-action gives us an action map* $C^\infty(P) \to C^\infty(P \times G)$ *which is an FIO*

$$F \in I^{-d/4}(P \times P \times G; \Gamma_0')$$

with Γ_0' *the moment Lagrangian, whose canonical relation is:*

$$\Gamma_0 := \{(\zeta;\ \zeta \cdot g;\ g, \eta) \in (T^*P \setminus 0)^{\times 2} \times (T^*G \setminus 0)\ :\ \mu(\zeta) = \eta\}.$$

The composition $E_t \circ F$*, denoted by* $E_t(x, yg)$*, arises from a transverse intersection of canonical relations and is therefore an FIO:*

$$E_t(x, yg) \in I^{-(d+7)/4}(P \times P \times G \times \mathbb{R}; \Gamma')$$

with canonical relation

$$\begin{aligned} \Gamma := \{(\zeta_1;\ \zeta_2;\ g, \eta;\ t, \tau) \in (T_0^*P)^{\times 2} \times (T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0) \\ : \tau + \langle Z^\omega, \zeta_1 \rangle = 0,\ \mu(\zeta_2 g^{-1}) = \eta,\ \exists s \text{ such that } \ \zeta_2 = (G_{-s} \circ \Phi_t^Z)(\zeta_1) g\} \end{aligned}$$

Parametrizing Γ *by* $C_2 \times G \times \mathbb{R} \cong C_1 \times \mathbb{R}_{t_1} \times G \times \mathbb{R}_{t_2}$ *the principal symbol of* $E_t(x, yg)$ *is given by:*

$$\mp \frac{i}{2} (2\pi)^{(d+3)/4} |d_{C_1}|^{1/2} \otimes |dt_1|^{1/2} \otimes |dg|^{1/2} \otimes |dt_2|^{1/2} \quad \text{on} \quad C_\pm$$

where $|dg|$ *is the volume measure on* G *induced by our* Ad*-invariant inner product on the tangent spaces.*

PROOF. The expression for the moment Lagrangian and the fact that $\Gamma_0 \in I^{-d/4}(P \times P \times G; \Gamma_0')$ is proven in [**17**]. By construction we have

$$\Gamma = C_2 \circ \Gamma_0$$

and the composition is clean so the orders of the FIOs simply add up. □

In [**34**], the distributional trace of e^{-itD_Z} was expressed in terms of E_t and so $E_t(x, yg)$ will play a similar role for our equivariant trace.

Lemma 3.2.13. *Write $\widehat{n}_x, \widehat{n}_y$ for the Lie derivatives along the unit normal $\widehat{n}$ in the variables x and y respectively. Then*

$$\mathcal{F} := \widehat{n}_x E_t(x, yg) - \widehat{n}_y E_t(x, yg) \in I^{-(d+3)/4}(P \times P \times G \times \mathbb{R}; \Gamma)$$

with Γ given in 3.2.12. Under the same parametrization of Γ as in 3.2.12, the principal symbol of $\mathcal{F}$ is given by

$$\pm\frac{1}{2}(2\pi)^{(d+3)/4}\langle\widehat{n}, -\rangle|d_{C_1}|^{1/2} \otimes |dt_1|^{1/2} \otimes |dg|^{1/2} \otimes |dt_2|^{1/2} \quad \text{on } C_\pm$$

where $\langle\widehat{n}, -\rangle$ is the function on Γ given by pairing the first cotangent vector with $\widehat{n}$.

PROOF. Differentiation is a differential operator, hence pseudodifferential operator, and so its Lagrangian is just the diagonal. Therefore differentiating an FIO does not affect the Lagrangian and merely increases the order by 1. □

For the next couple of lemmas we hold off on computing the principal symbols since it will be easier to directly compute the principal symbol of the wave trace after all of these compositions.

Lemma 3.2.14. *Let* diag : $P \to P \times P$ *denote the diagonal map so that pulling back along* diag *is an FIO*

$$\operatorname{diag}^* \in I^{(n+1+d)/4}(P \times P \times P; C_3')$$

with Lagrangian C_3' *where*

$$C_3 = \{(p, \zeta_2 - \zeta_1;\ p, \zeta_1;\ p, \zeta_2) \in (T^*P \setminus 0)^{\times 3}\}.$$

Then the composition $\operatorname{diag}^* \mathcal{F}$ *arises from a transverse intersection and is therefore an FIO*

$$\operatorname{diag}^* \mathcal{F} \in I^{(n-2)/4}(P \times G \times \mathbb{R}; \Gamma_1)$$

where

$$\begin{aligned}\Gamma_1 := \{(\zeta_2 - \zeta_1;\ g, \eta;\ t, \tau) \in T^*P \times (T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0) \\ :\ \tau + \langle Z^\omega, \zeta_1 \rangle = 0,\ \mu(\zeta_2 g^{-1}) = \eta,\ \exists s \textit{ such that } \zeta_2 = (G_{-s} \circ \Phi_t^Z)(\zeta_1) g \\ \textit{and } \zeta_1, \zeta_2 \in T_0^* P \textit{ live over the same point in } P\}\end{aligned}$$

Proof. The expression for Γ_1 above is precisely the definition of $C_3 \circ \Gamma$ so let's check that this is indeed a transverse composition. Notice that in the definition of Γ_1, one ζ_1 and t are chosen, s and g are uniquely determined by the requirement that $(G_{-s} \circ \Phi_t^Z)(\zeta_1) g$ must live over the same point in P as ζ_1. Furthermore, the constraint that there must exist s, g so that $(G_{-s} \circ \Phi_t^Z)(\zeta_1)g$ lives over the same point in P adds n independent constraints on ζ_1 since they must also live over the same point in Σ. This is unless $t = 0$.

So, given $0 \neq t, \zeta_1$ satisfying our n independent constraints: s, g and therefore ζ_2 and η are completely determined. τ is directly determined by ζ_1. Hence we see that there are exactly

$$\dim(T_0^* P) + 1 - n = 2(n + 1 + d) - 1 + 1 - n = (n + 1 + d) + d + 1$$

independent directions in both the composition $C_3 \circ \Gamma$ and in the fiber over the point corresponding to $0 \neq t, \zeta_1$.

In the case $t = 0$ we necessarily have $s = 0$ and $g = 1 \in G$, however the $t = 0$ local is a proper submanifold of Γ_1 and the tangent space to Γ_1 at $t = 0$ has $d + 1$ tangent directions arising from how s, g vary as we move off the $t = 0$ local. Within the $t = 0$ local we then have $\zeta_2 = \zeta_1$ and τ, η are determined by $\zeta_1 = \zeta_2$. While, in this case, we do have $\dim(T_0^* P) = 2(n + 1 + d) - 1$ choices for ζ_1, it is the quantity $\zeta_2 - \zeta_1$ that appears in Γ_1 and so the fiber coordinates of the first component of Γ_1 always vanish. Thus in both Γ_1 and in the fiber over the point corresponding to $(0, \zeta_1)$ we have

$$\dim(P) + d + 1 = (n + 1 + d) + d + 1$$

tangent directions. Hence indeed we have a transverse intersection and the order of $\operatorname{diag}^* \mathcal{F}$ is the sum of the orders of diag^* and $\mathcal{F}$. $\square$

Lemma 3.2.15. *Let $\iota : P_0 \hookrightarrow P$ denote the inclusion. Pulling back along ι is an FIO*

$$\iota^* \in I^{1/4}(P_0 \times P; C_4')$$

with Lagrangian C_4' defined by

$$C_4 = \{(x, \zeta_1;\ x, \zeta_2) \in T^*P_0 \times T^*P|_{P_0}\ :\ \zeta_2|_{TP_0} = \zeta_1\}.$$

As in [34], the canonical relation of $\operatorname{diag}^* \mathcal{F}$ *is disjoint from the conormal bundle N^*P_0 and $C_4 \circ \Gamma_1$ arises from a tranverse intersection so the composition $\iota^* \operatorname{diag}^* \mathcal{F}$ can be formed as if it were a transverse composition of FIOs and*

$$\iota^* \operatorname{diag}^* \mathcal{F} \in I^{(n-1)/4}(P_0 \times G \times \mathbb{R}; \Gamma_2)$$

where

$$\begin{aligned}\Gamma_2 := \{((\zeta_2 - \zeta_1)|_{TP_0};\ g, \eta;\ t, \tau) \in T^*P_0 \times (T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0)\\ : \zeta_1, \zeta_2 \in T_0^*P|_{P_0} \textit{ lie over the same point in } P_0,\ \tau + \langle Z^\omega, \zeta_1 \rangle = 0,\\ \mu(\zeta_2 g^{-1}) = \eta,\ \exists s \textit{ such that } \zeta_2 = (G_{-s} \circ \Phi_t^Z)(\zeta_1)g\}.\end{aligned}$$

Proof. The proof that the composition $i^* \operatorname{diag}^* \mathcal{F}$ can be formed is exactly the same as in Lemma 8.3 and the discussion preceding it in [**34**], and our Γ_2 is precisely defined to be $C_4 \circ \Gamma_1$. Transversality again follows from noticing that the intersection is clean and then dimension-counting, however we should remark that in order to get exactly $n+2d+1$ degrees of freedom one uses the fact that the restriction of covectors in $T_0^*P|_{P_0}$ to TP_0

yields the isomorphism $T_0^*P|_{P_0} \cong T^*P \setminus 0$ and hence the only way the fiber variable of the first component is zero is if $\zeta_1 = \zeta_2$. $\square$

So, we've arrived at the following object:

$$\iota^* \operatorname{diag}^* \mathcal{F} = (\widehat{n}_x E_t(x, yg) - \widehat{n}_y E_t(x, yg))\,|_{x=y\in P_0} \in I^{(n-1)/4}(P_0 \times G \times \mathbb{R}; \Gamma_2).$$

The importance of this object arises from the following slight generalization of Theorem 4.1 from [**34**].

Proposition 3.2.16. *Let $\Pi_* : C^\infty(P_0 \times G \times \mathbb{R}) \to C^\infty(G \times \mathbb{R})$ be the operator given by integration over P_0. Then since the base has dimension $d+1$ and the fibers have dimension $n+d$ we have*

$$\Pi_* \in I^{\frac{d+1}{2} - \frac{n+d}{4}}(G \times \mathbb{R} \times P_0 \times G \times \mathbb{R}; C_5')$$

where

$$\begin{aligned} C_5 = \{(g, \eta;\ t, \tau;\ x, 0;\ g, \eta;\ t, \tau) \\ \in (T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0) \times T^*P_0 \times (T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0)\ :\ x \in P_0\}. \end{aligned}$$

Furthermore, since $m_0 = 1$ and Q_ω is positive definite on $\mathcal{H} = \bigoplus_{m\geq 1}^{L^2} \mathcal{H}_m$, if we set

$$\mathcal{V} := \mathcal{H}^{\perp Q_\omega} \quad \textit{then} \quad \ker \Box_\omega = \mathcal{V} \oplus \mathcal{H}$$

then we have

$$\mathcal{K}(g, t) := \Pi_* \iota^* \operatorname{diag}^* \mathcal{F} = \int_{P_0} (\widehat{n}_x E_t(x, yg) - \widehat{n}_y E_t(x, yg))\,\Big|_{x=y} dV_{P_0}(x) \tag{3.3}$$

$$= \mathrm{Tr}_{\mathcal{V}}(e^{-itD_Z} \circ F) + \sum_{m=1}^{\infty} \sum_{\ell \in \mathbb{Z}} \mu(m, \ell)\, \mathrm{Tr}(\kappa_m(g)) e^{-it\lambda_{m,\ell}}. \tag{3.4}$$

Proof. The basic facts concerning push-forward distributions such as Π_* can be found in section 7.1 of [**35**] and we omit the proofs here as they are well known.

Let's now derive the above explicit expression 3.4 for $\mathcal{K}(g, t)$. Indeed, by the computation in Theorem 4.1 of [**34**], $\mathcal{K}(g, t)$ is the equivariant trace of the operator $e^{-itD_Z} \circ F$ on $\ker \Box_\omega$. Recalling that $\mu(m, \ell)$ is simply the multiplicity of κ_m in the $\lambda_{m,\ell}$-eigenspace and that F acts by κ_m on this eigenspace by definition of $\mathcal{H}_m$ we obtain our above expression 3.4 for $\mathcal{K}(g, t)$, as desired. $\square$

We will now build a distribution on $P_0 \times G \times \mathbb{R} \times S^1$ which we will then compose with $\iota^* \operatorname{diag}^* \mathcal{F}$ to produce $\Upsilon(\varphi)$. A key motivating fact in the below definition is the orthogonality of the functions $g \mapsto \mathrm{Tr}(\kappa_m(g))$ for different m's. This is a well-known fact from abstract harmonic analysis (see Section 5.3 of [**13**], for example), however one should take care not to confuse the two distinct notions of "character" of a representation.

Lemma 3.2.17. *[**16, 17**]*

The operator $\mathcal{L}_{\mathcal{O}} : C^\infty(G) \to \mathcal{D}'(S^1)$ with Schwartz kernel given by the distribution

$$\mathcal{L}(e^{i\theta}, g) := \sum_{m=1}^{\infty} \mathrm{Tr}(\kappa_m(g)) e^{im\theta}$$

is in

$$\mathcal{L}_{\mathcal{O}} \in I^{(1-d)/4}(S^1 \times G; \Lambda_{\mathcal{O}})$$

with Lagrangian

$$\Lambda_{\mathcal{O}} = \{(z, r; g, r\xi) \in (T^*S^1 \setminus 0) \times (T^*G \setminus 0) \mid \xi \in \mathcal{O},\ g \in G_\xi,\ z = \chi_\xi(g)\}$$

where $\chi_\xi : G_\xi \to U(1)$ *is the character associated to* $\xi \in \mathcal{O}$*.*

Our final to-do before we have, at least morally, obtained a description of $\Upsilon(\varphi)$ as a composition of FIOs is to localize about the ray $\lambda_{m,\ell} \sim mE$ via φ. Towards this end, we define an operator

$$T_{\varphi,E} : C_c^\infty(S^1 \times \mathbb{R}) \to \mathcal{D}'(S^1)$$

by declaring its Schwartz kernel to be given by the oscillatory integral

$$T_{\varphi,E}(\theta'; \theta, t) := (2\pi)^{-2} \widehat{\varphi}(t) \int_{-\infty}^{\infty} ds\ e^{is(\theta' - \theta - tE)}.$$

Lemma 3.2.18. *[17]*
$T_{\varphi,E} \in I^{-1/4}(S^1 \times S^1 \times \mathbb{R}; \Lambda_E)$ *where*

$$\Lambda_E := \{(ze^{itE}, r;\ z, r;\ t, rE) \in (T^*S^1 \setminus 0) \times (T^*S^1 \setminus 0) \times (T^*\mathbb{R} \setminus 0) \mid r \in \mathbb{R},\ z \in S^1\}.$$

Lemma 3.2.19. *We can form the composition*

$$T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}}) \in I^{-d/4}(S^1 \times G \times \mathbb{R}; \Theta'_{\varphi,E})$$

where

$$\Theta'_{\varphi,E} = \{(\chi_\xi(g)e^{itE}, r; g, r\xi; t, Er) \in T^*S^1 \times T^*G \times T^*\mathbb{R}\ :\ \xi \in \mathcal{O},\ g \in G_\xi\}.$$

Furthermore:

$$(T_{\varphi,E} \circ \mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})\mathcal{K} = \Upsilon(\varphi).$$

Proof. The fact that this composition $T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})$ can be formed, has the above order, and the above canonical relation $\Theta'_{\varphi,E}$ is proven in [**17**]. So, we just need to demonstrate that we do indeed obtain $\Upsilon(\varphi)$ when applying it to $\mathcal{K}$. Recalling the formula 3.4 for $\mathcal{K}(g,t)$ we note that by [**34**] the trace over $\mathcal{V}$ still decomposes as a sum over (possibly generalized) eigenvalues of D_Z counted with multiplicity, only now not all are real and some may be zero modes. Furthermore, the G-dependence in the trace over $\mathcal{V}$ is still in the form of the $\mathrm{Tr}(\kappa(g))$ for κ the representation generated by that specific (generalized) eigenvector. Indeed, while the Hilbert space inner products from the Cauchy data isomorphism are not D_Z-invariant they are still G-invariant and so $\mathcal{V}$ is completely decomposable since it is a unitary G-representation. Since characters are orthogonal with respect to the Haar measure on G, we obtain:

$$(\mathcal{L}_{m_0\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})\mathcal{K} = \sum_{m=1}^{\infty} \sum_{\ell \in \mathbb{Z}} e^{-it\lambda_{m,\ell}} e^{im\theta}$$

as a distribution on $S^1 \times \mathbb{R}$. Finally, applying $T_{\varphi,E}$ we immediately obtain:

$$(T_{\varphi,E} \circ (\mathcal{L}_{m_0\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}}))\mathcal{K} = \sum_{m=1}^{\infty} \sum_{\ell \in \mathbb{Z}} \widehat{\varphi}(\lambda_{m,\ell} - mE) e^{im\theta}$$

as desired. $\square$

Our next step is to understand the composition $(T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}}))\mathcal{K}$ as an actual Lagrangian distribution. As it turns out, it is more clear if one first computes $(T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})) \circ \Pi_*$.

Lemma 3.2.20. *The composition*

$$(T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})) \circ \Pi_* \in I^{\frac{1}{2}-\frac{n}{4}}(S^1 \times P_0 \times G \times \mathbb{R}; C_6')$$

is a transverse composition of FIOs with Lagrangian determined by

$$C_6 = \{(\chi_\eta(g)e^{itE}, r;\ x, 0;\ g, r\eta;\ t, rE) \in (T^*S^1 \setminus 0) \times T^*P_0 \times (T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0)$$
$$:\ \eta \in \mathcal{O},\ g \in G_\eta\}.$$

Proof. This is immediate since this composition does not affect the T^*P_0-variables and in the $(T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0)$-variables the Lagrangian for Π_* is just the diagonal. $\square$

Theorem 3.2.21. *The clean intersection hypothesis implies that the composition of* $T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}}) \circ \Pi_*$ *and* $\iota^* \operatorname{diag}^* \mathcal{F}$ *is a clean composition of FIOs with excess*

$$e = 2(n+\ell) - 2 \quad \textit{and therefore order} \quad \left(\frac{1}{2} - \frac{n}{4}\right) + \frac{n-1}{4} + \frac{e}{2} = n + \ell - 1 + \frac{1}{4}$$

Thus

$$\Upsilon(\varphi) \in I^{n+\ell-1+\frac{1}{4}}(S^1; \mathfrak{C}_E')$$

where

$$\mathfrak{C}_E' = \{(\chi_\eta(g)e^{itE}, r) \in T^*S^1 \setminus 0\ :\ \eta \in \mathcal{O},\ g \in G_\eta,\ \exists \zeta \in T_0^*P|_{P_0} \textit{ such that}$$
$$\exists s \textit{ with } \zeta = (G_{-s} \circ \Phi_t^Z)(\zeta)g,\ \langle Z^\omega, \zeta\rangle = -rE,\ \mu(\zeta g^{-1}) = r\eta\}$$

Proof. The main goal here is to compute the fiber of $C_6' \times_{diag} \Gamma_2'$ over a point $(\omega, r) \in \mathfrak{C}_E'$. By homogeneity of the fiber we can assume $r = 1$ and so the fiber is given by:

$$F_{(\omega,1)} := \{(x,0;\ g,\eta;\ t,E) \in T^*P_0 \times (T^*G \setminus 0) \times (T^*\mathbb{R} \setminus 0)\ :\ x \in P_0,\ \exists \zeta \in (T^*_0 P)_x$$
$$\text{such that } \exists s \text{ with } \zeta = (G_{-s} \circ \Phi^Z_t)(\zeta)g,\ \langle Z^\omega, \zeta \rangle = -E,\ \mu(\zeta g^{-1}) = \eta,$$
$$\chi_\eta(g)e^{itE} = \omega,\ \text{and } \eta \in \mathcal{O},\ g \in G_\eta\}$$

Since we chose $E > 0$ our constraint $\langle Z^\omega, \zeta \rangle = -E$ implies $\zeta \in T^*_+P|_{P_0} \subseteq T^*_0P|_{P_0}$ and therefore ζ corresponds to a unique null geodesic $\gamma \in \mathcal{N}$ with $\gamma(0) = x$, $H_Z(\gamma) = E$, $\mu(\gamma g^{-1}) = \eta$ and $\gamma = \Phi^Z_t(\gamma)g$. Therefore $(\gamma g^{-1}, \eta) \in \mu_{\mathcal{O}}^{-1}(0)$ and the image of this in the quotient is a periodic orbit in $\mathcal{N}_{\mathcal{O}}$ with period t and energy $\widetilde{H}_Z = E$.

Now, let's write $\pi : \mathbb{R} \times \mu_{\mathcal{O}}^{-1}(0) \to \mathbb{R} \times \mathcal{N}_{\mathcal{O}}$ for the projection map and recall that $\mathfrak{Y}_E \subseteq \mathbb{R} \times \mathcal{N}_{\mathcal{O}}$ is the set of periodic orbits for the reduced flow together with their periods. If we denote

$$\mathfrak{X} := \{(t,\gamma,\eta,g)\ :\ (t,\gamma,\eta) \in \pi^{-1}(\mathfrak{Y}_E)\ \ \text{and}\ \ g \in G_\eta\}$$

then $\dim \mathfrak{X} = \dim \pi^{-1}(\mathfrak{Y}_E) + \dim G - \dim \mathcal{O} = 2d + 2n - 1$ since $\mathfrak{Y}_E$ has dimension $2(n+\ell) - 1$ where $2\ell = \dim \mathcal{O}$. Note: the clean intersection hypothesis implies that $\mathfrak{Y}_E$ is a disjoint union of smooth manifolds with the clopen subset $\{0\} \times \widetilde{H}_Z^{-1}(E)$ having dimension $= 2(n+\ell)-1$ and the other components having dimension at most $2(n+\ell)-1$. Furthermore the map

$$\mathfrak{X} \to F_{(\omega,1)}$$
$$(t,\gamma,\eta,g) \mapsto (\gamma(0)g, g, \eta, t)$$

is a submersion. This, together with the fact that the holonomy map $\mathrm{Hol} : \mathfrak{Y}_E \to \mathrm{U}(1)$ is locally constant, implies that we have a clean composition of FIOs. Since the only part of the derivative $\gamma'(0)$ of γ captured in the image of our submersion $\mathfrak{X} \to F_{(\omega,1)}$ is $\eta = \mu(\gamma)$ it follows that the kernel of the above submersion at each point contains a $2d-2\ell$-dimension subspace of tangent vectors orthogonal to the tangent space $T_\eta\mathcal{O}$. The only other degeneracy comes the 1-dimensional space of vectors tangent to the curve γ itself and so we arrive at:

$$\dim F_{(\omega,1)} = 2(n+d) - 1 - 2(d-\ell) - 1 = 2(n+\ell) - 2$$

as desired. $\square$

All that remains now is the calculation of the principal symbol of $\Upsilon(\varphi)$. It's worth noticing, however, that from the expression for $T_{\varphi,E}$ we see that the actually wave front set $\mathrm{WF}'(\Upsilon(\varphi))$ will often be a proper subset of $\mathfrak{C}'_E$ depending on $\operatorname{supp}\widehat{\varphi}$. This is due to the varying dimensions of the components of $\mathfrak{Y}_E$ and the support of the principal symbol of $\Upsilon(\varphi)$ being constrained by $\operatorname{supp}\widehat{\varphi}$. We compute this principal symbol now.

Proof. The result concerning the wave front set will follow immediately from the calculation of the principal symbol since the constraint $T \in \operatorname{supp}\widehat{\varphi}$ comes from the support of the principal symbol.

We can compute a principal symbol for $T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})$ by composing their explicitly given Schwartz kernels. The Schwartz kernel for the composition is then a distribution on $S^1 \times G \times \mathbb{R}$ with Schwartz kernel

$$(\theta', g, t) \mapsto \sum_{m=1}^{\infty} (2\pi)^{-2} \widehat{\varphi}(t) e^{im(\theta' - tE)} \operatorname{Tr}(\kappa_m(g)).$$

Recalling that the principal symbol of $\mathcal{F}$ is given by

$$\pm \frac{1}{2} (2\pi)^{(d+3)/4} \langle \widehat{n}, - \rangle |d_{C_1}|^{1/2} \otimes |dt_1|^{1/2} \otimes |dg|^{1/2} \otimes |dt_2|^{1/2}$$

we have that the principal symbol of $\Upsilon(\varphi)$ at a point $(\omega, 1) \in \mathfrak{C}'_E$ is given by the integral over the fiber

$$F_{(\omega,1)} \cong \text{ quotient of holonomy } \omega \text{ clopen subset of } \mathfrak{Y}_E \text{ by the action of the flow } \widetilde{\Phi}^Z$$

of the product of the symbol of $\mathcal{F}$ and the symbol of

$$T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}}) \circ \Pi_* \circ \iota^* \circ \mathrm{diag}^*$$

restricted to the fiber. The symbol over a more general point $(\omega, r) \in \mathfrak{C}'_E$ is then obtained by homogeneity in r. Since $0 \in \operatorname{supp} \widehat{\varphi}$ and principal symbols are defined modulo symbols of lower order it suffices to compute this integral over the quotient of the clopen subset

$$\{0\} \times \widetilde{H}_Z^{-1}(E) \subseteq \mathfrak{Y}_E.$$

In this fibered product of symbols, the pairing of the g in the symbol for $T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes \mathrm{id}_{\mathbb{R}})$ and the g in the symbol for $\mathcal{F}$ amounts to replacing variables in the fibers of $T_0^* P|_{P_0} \cong \mathcal{N}$ with fiber variables in $\mathcal{N}_{\mathcal{O}}$ (here the "fibers" are diffeomorphic to $\mathcal{O}$). Since our fibered product is just over the quotient of $\{0\} \times \widetilde{H}_Z$ by the flow, the pairing of the t-variables in the symbol for $T_{\varphi,E} \circ (\mathcal{L}_{\mathcal{O}} \otimes_{\mathbb{R}})$ and the symbol for $\mathcal{F}$ simply amounts to setting $t = 0$ in both symbols and multiplying by $\widehat{\varphi}(0)$. The effect of restricting to P_0 along the diagonal $P_0 \hookrightarrow P \times P$ on the fibered product of symbols (aside from replacing the volume half-density on Γ with the one on $\widetilde{H}_Z^{-1}(E)$) is to divide by the function $\langle \widehat{n}, - \rangle$ and multiply by a dimensional constant, hence removing the function $\langle \widehat{n}, - \rangle$ from our symbol expression. Therefore, denoting by $C_{n,d}$ a dimensional constant and writing $\omega = e^{i\theta'}$, we obtain the symbol over the point $(\omega, 1)$ as:

$$
\begin{aligned}
C_{n,d}\omega^m \int_{\widetilde{H}_Z^{-1}(E)} \widehat{\varphi}(0) |d\omega \wedge dr|^{1/2} \frac{1}{|\nabla \widetilde{H}_Z|^2} \nabla \widetilde{H}_Z \llcorner dV_{\mathcal{N}_{\mathcal{O}}} \\
= C_{n,d}\omega^m \widehat{\varphi}(0) \operatorname{Vol}\left(\widetilde{H}_Z^{-1}(E)\right) |d\omega \wedge dr|^{1/2}.
\end{aligned}
$$

We can now recover the principal symbol over (ω, r) by scaling. Since the fibers are diffeomorphic to $\widetilde{H}_Z^{-1}(E)/\mathbb{R}$ where the $\mathbb{R}$-action is by the Hamiltonian flow of $\widetilde{H}_Z$ they have dimension $2(n+\ell) - 2$ and so the principal symbol over (ω, r) is given by:

$$
C_{n,d}\omega^m \widehat{\varphi}(0) \operatorname{Vol}\left(\widetilde{H}_Z^{-1}(E)\right) |r|^{(n+\ell)-1} |d\omega \wedge dr|^{1/2}
$$

where we note that $n + \ell - 1$ is half the dimension of our fiber. $\square$

Theorem 3.2.22. *Under the assumptions of 1.3.24 where the time* $\neq 0$ *part of the set* $\mathfrak{Y}_E$ *consists of finitely many isolated periodic orbits* $(T_1, \gamma_1), ..., (T_q, \gamma_q)$*, and assuming* $0 \notin \operatorname{supp} \widehat{\varphi}$ *we actually have* $\Upsilon(\varphi) \in I^{1/4}$ *with principal symbol at each* $(\operatorname{Hol}_{\mathcal{O}}(T_j, \gamma_j), r)$*,* $j = 1, ..., q$*, given by:*

$$
C_{n,d} \operatorname{Hol}_{\mathcal{O}}(T_j, \gamma_j)^m \frac{T_j^{\#}}{2\pi} \widehat{\varphi}(T_j) |\det(I - P_j)|^{-1/2} e^{i\pi \mathfrak{m}_j/4} |d\omega \wedge dr|^{1/2}
$$

where $T_j^{\#}$ is the primitive period of γ_j, P_j is the linearized Poincaré first return map of γ_j with respect to any local symplectic transversal and we have included the Maslov factor $e^{i\pi\mathfrak{m}_j/4}$ where $\mathfrak{m}_j$ is the Conley-Zehnder index of γ_j as in [34].

Proof. The proof is exactly the same as the previous one only instead of integrating over $\{0\} \times \widetilde{H}_Z^{-1}(E)$ with respect to its invariant measure we integrate over the respective periodic orbit γ_j with respect to the density from 1.3.24. □

These last three theorems respectively conclude the proofs of Theorems 3.2.1,3.2.2 and 3.2.3.

3.3. Index Theorems For Stationary Regions

Here we consider a slightly different class of spacetimes than those we have been studying so far. (M, g) is still assumed to be a globally hyperbolic spatially compact spacetime although now we assume the existence of two regions

$$(t_{j1}, t_{j2}) \times \Sigma_j \subseteq M, \; j = 1, 2$$

with each $\{t\} \times \Sigma_j$, $t \in (t_{j1}, t_{j2})$, $j = 1, 2$ a Cauchy hypersurface and the metric g in the standard form 1.1.11 in each of these two regions:

$$g|_{(t_{j1}, t_{j2}) \times \Sigma_j} = -(N_j^2 - |\eta_j|_h^2) dt^2 + \eta_j \otimes dt + dt \otimes \eta_j + h_j$$

It should be noted that despite writing t for the time coordinate in each of these stationary regions, we do not assume that the two foliations match-up in the region in-between. We also assume:

$$(t_{21}, t_{22}) \times \Sigma_2 \subseteq J^+((t_{11}, t_{12}) \times \Sigma_1)$$

Given this setup, we take $E \to M$ a complex vector bundle with non-degenerate sesquilinear fiber metric $\langle -, - \rangle$, compatible connection A, and $\langle -, - \rangle$-formally-self-adjoint Dirac-type operator $\not{D}$ such that A is the connection corresponding to the normally hyperbolic operator $\not{D}^2$. On each $(t_{j1}, t_{j2}) \times \Sigma_j$ we have a locally defined operator on sections of E:

$$D_j := -i\nabla^A_{\partial_t} \text{ over } (t_{j1}, t_{j2}) \times \Sigma_j$$

Lemma 3.3.1. *Let $\widehat{n}_j$ denote the unit forward normal to Σ_j and $\vec{\beta}_j$ the vector field on Σ h_j-dual to η_j. Write $\not{D} = -i\not{\nabla}^A + \Theta_A$ as usual. Over $(t_{j1}, t_{j2}) \times \Sigma_j$ we then have*

$$\not{D} = -N_j^{-1}\not{\widehat{n}}_j \left[D_j - iN_j h^{k\ell}\not{\widehat{n}}_j \not{\partial}_k \nabla^A_\ell + i\nabla^A_{\vec{\beta}_j} + N_j \not{\widehat{n}}_j \Theta_A \right]$$

Proof. This calculation follows from $\not{\nabla}^A = g^{k\ell}\not{\partial}_k \nabla^A_\ell$ together with our known formulas for g^{-1} on the stationary regions from 1.1:

$$g^{-1} = -N_j^{-2}\partial_t \otimes \partial_t + N_j^{-2}\vec{\beta}_j \otimes \partial_t + N_j^{-2}\partial_t \otimes \vec{\beta}_j + \widetilde{h}_j^{-1}$$

where

$$\widetilde{h}_j^{-1} = h_j^{-1} - N_j^{-2}\vec{\beta}_j \otimes \vec{\beta}_j$$

and also the fact that $\widehat{n}_j = N_j^{-1}(\partial_t - \vec{\beta}_j)$. $\square$

Motivated by the above we introduce the Clifford action on Σ_j.

Definition 3.3.2. For $j = 1, 2$ and $v \in T\Sigma_j$ we denote by $\gamma_j(v)$ the endomorphism of E given by

$$\gamma_j(v) := -i\vec{\hat{n}}_j\vec{v}$$

Since $\widehat{n}_j$ is normal to Σ_j it follows that the endomorphisms $\gamma_j(v), \gamma_j(w)$ for $v, w \in T\Sigma$ satisfy the Clifford relations:

$$\gamma_j(v)\gamma_j(w) + \gamma_j(w)\gamma_j(v) = -2h(v, w)\,\mathrm{id}_E$$

It's worth noting that the Clifford relations satisfied by $\vec{v}, \vec{w}$ for $v, w \in TM$ do not have a negative sign on the right-hand-side. We also set:

$$\nabla\!\!\!\!/^A_{\Sigma_j} := h^{k\ell}\gamma_j(\partial_k)\nabla^A_\ell$$

so that $\nabla\!\!\!\!/^A_{\Sigma_j}$ is a Riemannian Dirac-type operator on $E|_{\Sigma_j}$ and

$$D\!\!\!\!/ = -N_j^{-1}\vec{\hat{n}}_j\left[D_j + N_j\nabla\!\!\!\!/^A_{\Sigma_j} + i\nabla^A_{\vec{\beta}_j} + N_j\vec{\hat{n}}_j\Theta_A\right]$$

By making our usual assumptions necessary to perform spectral theory over each region $(t_{j1}, t_{j2}) \times \Sigma_j$ we can apply the results of 2.1 to obtain the following.

Lemma 3.3.3. *Assume that over each $(t_{j1}, t_{j2}) \times \Sigma_j$ we have $[D\!\!\!\!/, D_j] = 0$ and $\langle\vec{\hat{n}}_j(-), -\rangle$ is a positive-definite Hermitian fiber metric. Then via the Cauchy data isomorphisms,* $\ker D\!\!\!\!/ = \mathrm{FE}^0_D(\ker D\!\!\!\!/)$ *is isomorphic to the same finite-energy kernel of $D\!\!\!\!/$ over each of the induced standard stationary spacetimes $\mathbb{R} \times \Sigma_j$ obtained by extending $g|_{(t_{j1},t_{j2})\times\Sigma_j}$, A and*

$\langle -,- \rangle$ to be constant in t. Through these isomorphisms, D_j induces a $\langle \vec{\not{n}}_j(-), - \rangle$-self-adjoint operator on $\ker \not{D}$ for $j = 1, 2$ with discrete spectrum consisting of eigenvalues accumulating at $\pm\infty$ only.

As such we make the assumption that $[\not{D}, D_j] = 0$ for $j = 1, 2$ going forward. Furthermore, we assume that $E, \langle -,- \rangle$ and A are in temporal gauge over both regions $(t_{j1}, t_{j2}) \times \Sigma_j$. The energy estimates of 2.1 then imply the following.

Lemma 3.3.4. *For each $\psi \in \ker \not{D}$, the section over $(t_{j1}, t_{j2}) \times \Sigma_j$ obtained by restricting ψ and then applying D_j extends uniquely to a new element of $\ker \not{D}$ and its extension is equal to the element $D_j\psi$ obtained via the Cauchy-data isomorphism as in the previous lemma.*

Finally we have the following very explicit description of the operator D_j.

Lemma 3.3.5. *Via the standard form 3.3 for $\not{D}$ it follows that under the Cauchy data isomorphism $\ker \not{D} \cong L^2(\Sigma_j, E)$ the operator D_j is given by the $\langle \vec{\not{n}}_j(-), - \rangle$-self-adjoint elliptic differential operator*

$$\widehat{H}_j := -N_j \not{\nabla}^A_{\Sigma_j} - i\nabla^A_{\vec{\beta}_j} - N_j \vec{\not{n}}_j \Theta_A$$

While the above operator is indeed elliptic, the presence of the additional first order term $-i\nabla^A_{\vec{\beta}_j}$ and the N_j-factor prevent it from simply being a Riemannian signature Dirac-type operator. From this we obtain particularly nice expressions for the advanced and retarded fundamental solutions in these regions.

Theorem 3.3.6. *Let* $\psi \in C_c^\infty((t_{j1}, t_{j2}) \times \Sigma_j, E)$ *and let* $G^j_{adv/ret}$ *denote the advanced and retarded fundamental solutions for* $\not{D}$ *on the standard stationary spacetime* $\mathbb{R}_t \times \Sigma_j$ *obtained by extending* g, A *and* $\langle -, - \rangle$ *over* $(t_{j1}, t_{j2}) \times \Sigma_j$ *to be constant in* t*. Interpreting* ψ *as a function* $\mathbb{R} \to L^2(\Sigma_j, E)$ *we obtain*

$$(G_{ret,j}\psi)(t) = i \int_{\mathbb{R}} \mathbb{1}_{[0,\infty)}(t-s) e^{i(t-s)\widehat{H}_j}(N_j \widehat{\not{n}}_j \psi(s))ds$$
$$(G_{adv,j}\psi)(t) = -i \int_{\mathbb{R}} \mathbb{1}_{(-\infty,0]}(t-s) e^{i(t-s)\widehat{H}_j}(N_j \widehat{\not{n}}_j \psi(s))ds$$

where $\exp(i(t-s)\widehat{H}_j)$ *is defined via the functional calculus.*

Proof. From our above calculation we have $\not{D} = -N_j^{-1}\widehat{\not{n}}_j \left(D_j - \widehat{H}_j \right)$ in these regions and since we are assuming that $E, \langle -, - \rangle, A$ have been placed in temporal gauge we have $D_j = -i\partial_t$. But then $\partial_t \mathbb{1}_{[0,\infty)}(t) = \delta_0(t)$ and similarly for $\mathbb{1}_{(-\infty,0]}(t)$. Thus our above expressions are indeed fundamental solutions for $\not{D}$. Since they have the correct causal supports it follows from uniqueness of the advanced and retarded fundamental solutions that the above identities hold. □

The fundamental solution which arises in the index theorem is not the advanced or retarded fundamental solution, but a Feynman propagator. We now briefly explain what this is.

Recall that T_0^*M denotes the sub-cone-bundle of all non-zero null covectors, that

$$G_s : T_0^*M \to T_0^*M$$

denotes the geodesic flow, and

$$C := \{(\xi_2;\xi_1) \in T_0^*M \times T_0^*M \ : \ \exists s \in \mathbb{R} \text{ such that } \xi_2 = G_s(\xi_1)\}$$

Just as in the previous section 3.2 in the principal bundle case, it is well-known that we have a splitting into clopen subsets

$$C \setminus \operatorname{diag}(T_0^*M) = C^+ \sqcup C^-$$

where

$$C^+ = \{(x_2,\xi_2;x_1,\xi_1) \in C \ : \ x_2 \in J^\pm(x_1) \ \text{ if } \ \xi_2 \in T_\pm^*M\}$$

$$C^- = \{(x_2,\xi_2;x_1,\xi_1) \in C \ : \ x_2 \in J^\pm(x_1) \ \text{ if } \ \xi_2 \in T_\mp^*M\}$$

Let's think of these intuitively. C^+ is saying that x_2 is in the causal future of x_1 whenever the momentum ξ_2 of x_2 is future-directed. C^- is saying the opposite.

Since we're assuming that $\dim(M) \geq 3$ and that M is connected, the bundle T_0^*M has exactly two connected components and they are $T_\pm^*M$. As such, there are four ways to split $C \setminus \operatorname{diag}(T_0^*M)$ into a union of two clopen subsets $C^1 \sqcup C^2$ such that C^1, C^2 are inverse relations of one-another. We list the "positive" part of these relations here:

$$C^+_{ret} := \{(x_2,\xi_2;x_1,\xi_1) \in C \ : \ x_2 \in J^+(x_1)\}$$

$$C^+_{adv} := \{(x_2,\xi_2;x_1,\xi_1) \in C \ : \ x_2 \in J^-(x_1)\}$$

$$C^+_{Feyn} := \{(x_2,\xi_2;x_1,\xi_1) \in C \ : \ \exists s \in \mathbb{R}_{>0} \ \text{ such that } \ (x_2,\xi_2) = G_s(x_1,\xi_1)\}$$

A fundamental solution G for $\not{D}$ for which

$$\mathrm{WF}'(G) \subseteq \mathrm{diag}(T_0^*M) \cup C^+_{Feyn}$$

is called a **Feynman propagator** for $\not{D}$ [**30**]. We will need to introduce some spectral projectors in order to construct a Feynman propagator for $\not{D}$.

Definition 3.3.7. Via the functional calculus on $L^2(\Sigma_j, E)$ we define spectral projectors

$$p_0(\widehat{H}_j),\ p_>(\widehat{H}_j),\ p_<(\widehat{H}_j),\ p_\geq(\widehat{H}_j),\ p_\leq(\widehat{H}_j)$$

onto the kernel, positive, negative, non-negative and non-positive parts of the spectrum of $\widehat{H}_j$ respectively. When we want to interpret these as acting on $\ker \not{D}$ via the Cauchy data isomorphism we will replace $\widehat{H}_j$ with D_j in our notation.

Theorem 3.3.8. *On the standard stationary spacetime $\mathbb{R}_t \times \Sigma_j$ with induced $g, A, \langle -, - \rangle, \not{D}$ in temporal gauge we define*

$$(G_{Feyn,j}\psi)(t) := i \int_{\mathbb{R}} \left[\mathbb{1}_{[0,\infty)}(t-s) p_\geq(\widehat{H}_j) - \mathbb{1}_{(-\infty,0]}(t-s) p_<(\widehat{H}_j) \right] e^{i(t-s)\widehat{H}_j} (N_j \not{n}_j \psi(s)) ds$$

Then $G_{Feyn,j}$ is a Feynman propagator for $\not{D}$ on $\mathbb{R}_t \times \Sigma_j$.

Proof. The same computation as for $G_{adv/ret,j}$ shows that $G_{Feyn,j}$ is a fundamental solution for $\not{D}$. Write $M_j := \mathbb{R} \times \Sigma_j$ with our extended stationary metric, which we abuse notation to denote by g. Since the principal symbol of $\not{D}^2$ is the metric g it follows from propagation of singularities that $\mathrm{WF}'(G_{Feyn,j})$ is contained in $T_0^*M_j \times T_0^*M_j$, contains $\mathrm{diag}(T_0^*M_j)$, and the complement $\mathrm{WF}'(G_{Feyn,j}) \setminus \mathrm{diag}(T_0^*M_j)$ is invariant under the restriction of the geodesic flow to the complement of the diagonal in $T_0^*M_j \times T_0^*M_j$. Consider then an arbitrary point

$$(x_2, \xi_2; x_1, \xi_1) \in \mathrm{WF}'(G_{Feyn,j}) \text{ such that } x_1 \in J^+(x_2)$$

Then the t-coordinate of x_1 is greater than that of x_2 hence the kernel for $G_{Feyn,j}$ near the point (x_2, x_1) is given by the kernel of $ip_{\geq}(\widehat{H}_j)e^{it\widehat{H}_j} \circ N_j \not{n}_j$ and since we are projecting onto the non-negative part of the spectrum of $\widehat{H}_j$ we see that this operator extends to be holomorphic in $\Im(t) < 0$ and thus by the same argument as in [**4**] we must have both ξ_2, ξ_1 past-directed. If we instead assumed that $x_2 \in J^+(x_1)$ the same argument, but now applied to the negative spectral subspace, allows us to conclude that ξ_2, ξ_1 are both future-directed.

Suppose then for contradiction that there existed some $(x_2, \xi_2; x_1, \xi_1)$ in the primed wave front set whose $\mathbb{R} \times \mathbb{R}$-orbit under the geodesic flow separately in each variable does not intersect $\mathrm{diag}(T_0^*M_j)$. Then by propagation of singularities this entire $\mathbb{R} \times \mathbb{R}$-orbit must be in the wave front set since we never hit the diagonal hence by flowing x_1 along the geodesic flow sufficiently far both forwards and backwards we can reach both the causal future and causal past of x_2. Thus by our previous paragraph both covectors ξ_1, ξ_2 must be both future and past-directed, a contradiction. Thus there must always exist an $s \in \mathbb{R}$ such that $(x_2, \xi_2) = G_s(x_1, \xi_1)$ and by the previous paragraph we must therefore have $s > 0$, as desired. $\square$

The exact same proof as in [**4**] for normally hyperbolic operators applies to Dirac-type operators and allows us to conclude the following.

Proposition 3.3.9. *[4] The restrictions of the integrals kernels for the above $G_{adv/ret,j}$ and $G_{Feyn,j}$ to $(t_{j1}, t_{j2}) \times \Sigma_j$ respectively extend uniquely to the advanced and retarded fundamental solutions $G_{adv/ret}$ and to Feynman propagators for $G_{Feyn,j}$ for $\not{D}$ on all of M.*

At this point the rest of the proofs from **[4]** can be used almost verbatim. The only subtlety is that one must take care to avoid using Feynman propagators for $\not{D}^2$ as is frequently done in **[4]** since in the locally stationary, as opposed to locally ultrastatic, case our operators $\widehat{H}_j$ fail to anticommute with $N_j^{-1}\not{n}_j$ and so we only obtain formulae for the $\not{D}$-propagators and not the $\not{D}^2$-propagators.

Theorem 3.3.10. *Let (M, g) be a globally hyperbolic spatially compact spacetime together with two stationary regions*

$$(t_{j1}, t_{j2}) \times \Sigma_j \subseteq M, \;\; j = 1, 2$$

and a complex vector bundle $E \to M$ with non-degenerate sesquilinear form $\langle -, - \rangle$, compatible connection A and $\langle -, - \rangle$-self-adjoint Dirac-type operator $\not{D}$ such that A is the connection corresponding to $\not{D}^2$. Furthermore, assume that $\langle -, - \rangle$ and E are t-independent and A is in temporal gauge over each $(t_{j1}, t_{j2}) \times \Sigma_j$. Let $D_j := -i\nabla^A_{\partial_t}$ over $(t_{j1}, t_{j2}) \times \Sigma_j$ and assume that $[\not{D}, D_j] = 0$. Finally we assume that $\langle \not{n}_j(-), - \rangle$ is positive definite. Then

$$p_{\geq}(D_2) - p_{\geq}(D_1) \text{ is trace-class on } \ker \not{D} \cong L^2(\Sigma_2, E)$$

and for any Cauchy hypersurface $\Sigma \subseteq M$ we have

$$\operatorname{Tr}(p_{\geq}(D_2) - p_{\geq}(D_1)) = i \int_{\Sigma} \operatorname{tr}\left(\not{n}_{\Sigma}(G_{Feyn,2} - G_{Feyn,1})\right) dV_{\Sigma}$$

Proof. From our spectral descriptions of the kernels of $G_{Feyn,j}$ and $G_{adv/ret}$ we obtain:

$$\begin{aligned}
-i\left[G_{Feyn,j} - \frac{1}{2}G_{ret} - \frac{1}{2}G_{adv}\right] &= \Big[\mathbb{1}_{[0,\infty)}(t)p_{\geq}(D_j) - \mathbb{1}_{(-\infty,0]}(t)p_{<}(D_j) \\
&\qquad - \frac{1}{2}\mathbb{1}_{[0,\infty)}(t) + \frac{1}{2}\mathbb{1}_{(-\infty,0]}(t)\Big] e^{it\widehat{H}_j} N_j \not{n}_j \\
&= \Big[\mathbb{1}_{[0,\infty)}(t)p_{\geq}(D_j) - \mathbb{1}_{(-\infty,0]}(t)p_{<}(D_j) \\
&\qquad - \frac{1}{2}\mathbb{1}_{[0,\infty)}(t)(p_{\geq}(D_j) + p_{<}(D_j)) \\
&\qquad + \frac{1}{2}\mathbb{1}_{(-\infty,0]}(t)(p_{\geq}(D_j) + p_{<}(D_j))\Big] e^{it\widehat{H}_j} N_j \not{n}_j \\
&= \left(p_{\geq}(D_j) - p_{<}(D_j)\right) e^{it\widehat{H}_j} N_j \not{n}_j
\end{aligned}$$

and since restricting this kernel to $\Sigma_j \times \Sigma_j$ amounts to setting $t = 0$ it follows that

$$-i\left(G_{Feyn,j} - \frac{1}{2}G_{ret} - \frac{1}{2}G_{adv}\right)\Big|_{\Sigma_j \times \Sigma_j} = \left(p_{\geq}(D_j) - \frac{1}{2}\,\mathrm{id}\right) N_j \not{n}_j$$

Recalling our Cauchy data isomorphisms from the end of 2.1 we now denote

$$U_{21} := \mathrm{CD}_{\Sigma_2} \circ \mathrm{CD}_{\Sigma_1}^{-1}$$

for our Cauchy evolution operator. Given $\psi, \widetilde{\psi} \in C^{\infty}(\Sigma_1, E)$ we can extend $\psi, \widetilde{\psi}$ to solutions $\not{D}\psi = 0 = \not{D}\widetilde{\psi}$ of the Cauchy problem and apply divergence theorem to the integral of

$$0 = \langle \not{D}\psi, \widetilde{\psi}\rangle - \langle \psi, \not{D}\widetilde{\psi}\rangle \text{ over } J^+(\Sigma_1) \cap J^-(\Sigma_2)$$

to obtain

$$\int_{\Sigma_2} \langle \not{n}_2 U_{21}\psi, U_{21}\widetilde{\psi}\rangle dV_{\Sigma} = \int_{\Sigma_1} \langle \not{n}_1 \psi, \widetilde{\psi}\rangle dV_{\Sigma_1}$$

Since this holds for all such $\psi, \widetilde{\psi}$ and since $dV_g = N_j dt dV_{\Sigma_j}$ we see that at the level of kernels on M, U_{21} conjugates $N_1\not{n}_1$ to $N_2\not{n}_2$ and so

$$-i\left(G_{Feyn,1} - \frac{1}{2}G_{ret} - \frac{1}{2}G_{adv}\right)\Big|_{\Sigma_2\times\Sigma_2}$$

is given by the U_{21}-conjugate of $p_{\geq}(D_1) - \frac{1}{2}\,\mathrm{id}$ composed with $N_2\not{n}_2$. Thus we have the following identity of integral kernels:

$$p_{\geq}(D_2) - p_{\geq}(D_1) = -\left(\left(p_{\geq}(D_2) - \frac{1}{2}\,\mathrm{id}\right) N_2\not{n}_2 - U_{21}\left(p_{\geq}(D_1) - \frac{1}{2}\,\mathrm{id}\right) N_1\not{n}_1 U_{21}^*\right) N_2^{-1}\widehat{\not{n}}_2$$

Integrating the fiberwise trace over $\mathrm{diag}(\Sigma_2) \subseteq \Sigma_2 \times \Sigma_2$ and using Mercer's theorem we see that

$$\mathrm{Tr}(p_{\geq}(D_2) - p_{\geq}(D_1)) = \int_{\Sigma_2} i\,\mathrm{tr}\left((G_{Feyn,2} - G_{Feyn,1})N_2^{-1}\not{n}_2\right) N_2 dV_{\Sigma_2}$$

and we obtain our desired result over Σ_2 via cyclicity of the trace. To see that we can integrate over any Cauchy hypersurface, note that $G_{Feyn,2} - G_{Feyn,1}$ is C^1 (and actually smooth over the diagonal) as it is a difference of two Feynman propagators. As it is also a bisolution to $\not{D}\psi = 0$ it follows from 2.2 that the 1-form

$$J(v) := i\,\mathrm{tr}\left(\not{v}(G_{Feyn,2} - G_{Feyn,1})|_{\mathrm{diag}}\right)$$

is coclosed and so integrating over any Cauchy hypersurface Σ yields the same result by divergence theorem. $\square$

In the proof of the above result, we used the fact from [**30**] in the normally hyperbolic case (from [**19**] in the Dirac case) that Feynman propagators have unique local singularity structures on the diagonal. More precisely we fix once and for all decompositions

$$G_{Feyn,j} = G^{loc}_{Feyn,j} + G^{reg}_{Feyn,j}$$

with $G^{reg}_{Feyn,j} \in C^1(M \times M, E \boxtimes E^*)$ and remark that from [**30**], [**19**] the restriction of this decomposition to the diagonal is unique. Denoting

$$J_j(v) := i\,\mathrm{tr}\left(\not{v} G^{reg}_{Feyn,j}|_{\mathrm{diag}}\right)$$

we see that $J(v) = J_2(v) - J_1(v)$ and all three J, J_1, J_2 are C^1 1-forms on M.

We now conclude this section with an application of the above results to the case where $n + 1 = \dim(M)$ is even and so we have a chirality operator $\omega_{\mathbb{C}}$ acting on E, giving us a splitting

$$E = E_+ \oplus E_- \text{ into } \pm 1\text{-eigensubbundles}$$

In order to do this, we must make the unfortunate assumption that A is compatible with Clifford multiplication and hence, via the introduction to 2, we have

$$d(N^2 - |\eta|^2_h) = 0 = d\eta \text{ in particular.}$$

Then $\not{D}$ anticommutes with $\omega_{\mathbb{C}}$ and both $D_j = -i\nabla_0^A$ commute with $\omega_{\mathbb{C}}$. We therefore denote

$$\not{D}_{\pm} : C^\infty(M, E_\pm) \to C^\infty(M, E_\mp)$$

so that

$$\not{D} = \begin{pmatrix} 0 & \not{D}_+ \\ \not{D}_- & 0 \end{pmatrix}$$

Similarly by composing with the projections onto $E_\pm$ we obtain the $\pm$-parts of our Feynman propagators

$$G^{\pm}_{Feyn,j} : C_c^\infty(M, E_\pm) \to C^\infty(M, E_\mp)$$

Similarly we write $J_j^\pm$ for the above 1-forms J_j with $G_{Feyn,j}$ replaced by $G^\pm_{Feyn,j}$. But now that $d\theta = 0$ with $\theta = dt - (N^2 - |\eta|_h^2)^{-1}\eta$ from section 1.2 it follows that $\ker\theta$, the orthogonal complement of ∂_t, is an involutive distribution in TM. Thus the local arguments of [**4**] apply when we replace Σ_j with the integral submanifolds of $\ker\theta$. Hence

$$d^* J_1^- = \frac{\operatorname{tr}(V^-_{(n+1)/2}) - \operatorname{tr}(V^+_{(n+1)/2})}{(4\pi)^{(n+1)/2}\left(\frac{n+1}{2}\right)!}$$

where $V_k^\pm$ are the Hadamard coefficients (see [**2**]) for the normally hyperbolic operator $\not{D}_\mp \not{D}_\pm$ on $E_\pm$ and, in the case $E_\pm = S_\pm \otimes F$ is the twist of a spinor bundle with the Dirac operator induced by a connection B on F this is given by

$$(d^* J_1^-) dV_g = \text{ the degree } n+1 \text{ part of } -\widehat{A}(\nabla^g) \wedge \operatorname{ch}(\nabla^B)$$

with the usual c_1-correction in case $S_\pm$ is a Spin^c-bundle.

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